HWA CHONG INSTITUTION

2022 JC2 Preliminary Examination

Higher 2

MATHEMATICS

9758/02

16 September 2022

3 hours

Candidate Name	

CT Group	21

Centre Number	S				

Index Number				

Write here how many additional pieces of writing paper you have used (if any).	

Candidates answer on the Question Paper.

Additional materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Do not write anything on the List of Formulae (MF26).

Write in dark blue or black pen. You may use HB pencil for any diagrams or graphs. Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Write your answers in the spaces provided in the Question Paper.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part of question.

For Examiner's Use

Qn	Marks	Total	Remarks
1		4	
2		6	
3		7	
4		9	
5		14	
6		6	
7		8	
8		9	
9		11	
10		13	
11		13	
		100	

Remarks

- **a)** **INSTR**: Follow **instructions** as stated in Question (e.g. correct s.f , exact values, coordinates, similar form etc.)
- **b)** **NOT**: Correct Mathematical **Notations**
- **c)** **ACC**: **Accuracy** of Answers (e.g. affected by early rounding off, not writing +C for indefinite integrals etc.)

This document consists of **31** printed pages and **1** blank page.

Section A: Pure Mathematics [40 marks]

1 State a sequence of 3 transformations that will transform the curve with equation $y = x^2$ onto the curve with equation $y = -x^2 + 3x - 4$. [4]

2 A curve C has parametric equations

$$x = \sin t \tan t, \qquad y = \cos t, \qquad \text{where } 0 \le t < \frac{\pi}{2}.$$

(i) Sketch C, stating the equation(s) of any asymptote(s) and coordinates of any axial intercept(s). [2]

(ii) The region A is bounded by C, the y-axis and the lines $y = \frac{1}{2}$ and $y = \frac{1}{\sqrt{2}}$.

Find the exact area of A. [4]

3 The complex number z is given by $z = 2(\cos\beta + \mathrm{i}\sin\beta)$ where $0 < \beta < \frac{\pi}{2}$.

(i) Show that $\frac{z}{4 - z^2} = (k \operatorname{cosec} \beta)\mathrm{i}$, where k is a positive real constant to be determined. [3]

(ii) Given that the complex number $w = -\sqrt{3} + \mathrm{i}$, find the three smallest positive integer values of n such that $\left(\frac{z}{4 - z^2}\right)(w^*)^n$ is a real number. [4]

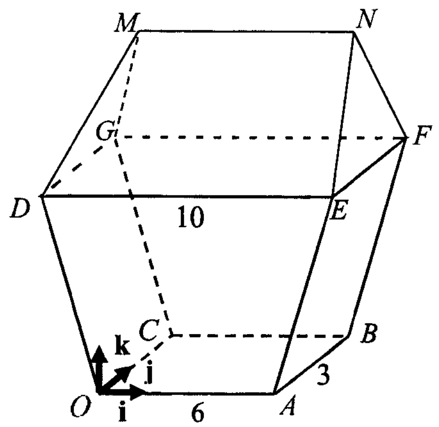
4 **(a)** **(i)** Find $\sum_{r=1}^{k}\left[\left(-\frac{1}{2}\right)^{r+1} + \ln(r+1)\right]$ in terms of k. Simplify your answer.

[4]

(ii) Hence determine if $\sum_{r=1}^{\infty}\left[\left(-\frac{1}{2}\right)^{r+1} + \ln(r+1)\right]$ exists. [1]

(b) The first term of an arithmetic series is positive. The sum of the first 6 terms of the series is 4.5, and the product of the first four terms of the series is 0. Find the 13th term of the series. [4]

5 The diagram below shows a 3-dimensional structure in which a pentahedron $DEFGMN$ lies on top of a trapezoidal prism $OABCDEFG$. Taking O as the origin, perpendicular vectors $\mathbf{i}$ and $\mathbf{j}$ are parallel to OA and OC respectively. The base of the structure sits on the horizontal x–y plane.



Planes $OABC$ and $DEFG$ are parallel to each other. It is given that OC, AB, DG and EF are parallel to one another where $OC = AB = DG = EF = 3$ units. It is also given that OA, CB, DE, GF and MN are parallel to one another where $OA = CB = MN = 6$ units and $DE = GF = 10$ units. The pentahedron $DEFGMN$ and the trapezoidal prism $OABCDEFG$ each has a height of 12 units.

(i) The point D has coordinates $(-2, 0, s)$. State the value of s. [1]

(ii) The line DM is parallel to the vector $\begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}$ and the plane $ABFE$ has equation $6x = 36 + z$. It is given that the line DM does not intersect with the plane $ABFE$. Find the value of t. [2]

(iii) Show that the equation of the plane DGM is given by $\mathbf{r} \cdot \begin{pmatrix} 6 \\ 0 \\ -1 \end{pmatrix} = k$, where k is a constant to be determined. [2]

(iv) Find the acute angle between the planes DGM and $DEFG$. [3]

(v) Find the coordinates of M and the exact shortest distance from M to the plane $ABFE$. [4]

(vi) Another plane Π has cartesian equation $x = c$, where c is a constant. If the three planes $OAED$, EFN and Π all intersect at the point E, find the value of c, showing your working clearly. [2]

Section B: Probability and Statistics [60 marks]

6 Toddler Roy is playing with a shape sorter toy set which has a box that has a triangle hole and a square hole. The toy set also comes with a triangle block and a square block. The box will light up if the triangle block is correctly placed into the triangle hole.

(a) Roy picks one of the two blocks and place it into the shape sorter box. There is a probability of 0.9 that Roy will correctly place the chosen block into the hole of the same shape. If he correctly places a chosen block into the shape sorter box, there is a probability of 0.4 that it is a square block. Find the probability that the shape sorter box will not light up. [3]

(b) Roy's mother took the two blocks that Roy has and adds 4 more identical triangle blocks and r more identical square blocks, where $r \geq 3$. All the blocks are given to his sister, Joy, to randomly select 5 blocks without replacement. Suppose the probability of Joy choosing exactly 2 square blocks is twice the probability of Joy choosing exactly 4 square blocks, find the value of r. [3]

7 A company uses a machine to produce chocolate bars. The machine is designed to produce chocolate bars with average fat content of 30 g.

After using the machine for many years, the manager wishes to test, at 5% level of significance, if the machine still maintains the average fat content in the chocolate bars at 30 g. He selects a sample of 40 chocolate bars for testing.

(i) State, giving a reason, whether it is necessary to assume that the fat content of the chocolate bars is normally distributed for the test to be valid. [1]

After some years of usage, the machine broke down. While waiting for the new machine to arrive, the company reverted to their traditional mode of producing handmade chocolate bars. The manager suspects that handmade chocolate bars have higher average fat content than those made by the machines. To verify his suspicion, the manager asked his staff to perform a hypothesis test at 5% significance level, on a random sample of 40 handmade chocolate bars.

The fat content, x grams, of the 40 handmade chocolate bars are summarised as follows:

$$\sum x = 1220, \quad \sum (x - 30.5)^2 = 50.$$

(ii) Calculate the unbiased estimates of the population mean and variance of the fat content in handmade chocolate bars. [2]

(iii) The sample mean fat content of the handmade chocolate bars is denoted by $\overline{x}$ grams. State the null and alternate hypotheses and calculate the range of values of $\overline{x}$ for which the null hypothesis would be rejected at 5% level of significance. Hence conclude if the manager's suspicion is valid at 5% significance level. [4]

(iv) The manager now knows the population variance of the fat content in handmade chocolate bars. It is given that the population variance is smaller than the unbiased estimate of the population variance calculated in part **(ii)**. Without carrying out another hypothesis test, explain with justification, if there will be a change in the manager's conclusion about his suspicion at 5% significance level. [1]

8 A school canteen committee consists of 4 parents, 2 student leaders and 4 teachers, chosen from 10 parents, 5 student leaders and 8 teachers.

(a) There is a married couple amongst the 10 parents. How many different canteen committees can be formed if the couple cannot serve on the committee together? [3]

The school canteen committee of 10 members has been formed.

(b) All members are to stand in a row to take a group photo with the Vice-Principal. Find the number of arrangements such that the Vice-Principal stands at the centre, both ends of the row are occupied by the students' leaders, and no two parents stand next to each other. [3]

(c) The committee members, together with the Vice-Principal, are seated at a round table with 11 chairs during lunch time. Find the probability that the parents are seated together and the teachers are separated. [3]

9 Abel has 1 white bag of marbles and 1 black bag of cards and he uses them to create a game. The white bag contains 2 red marbles, x blue marbles and $2x-1$ yellow marbles, where $x>1$. The black bag has 3 cards, and the cards are numbered with the number '0', '1' and '2' respectively.

In each round, Abel will first choose a marble randomly from the white bag. A red marble will give a score of 4. If a non-red marble is chosen, Abel will then choose a card randomly from his black bag. The score will then be twice of the number shown on the card drawn.

(i) Find the probability that Abel will get a score of 4 in a round of the game. Leave your answer in terms of x. [2]

(ii) Find the probability distribution of Abel's score in a round of the game in terms of x. [2]

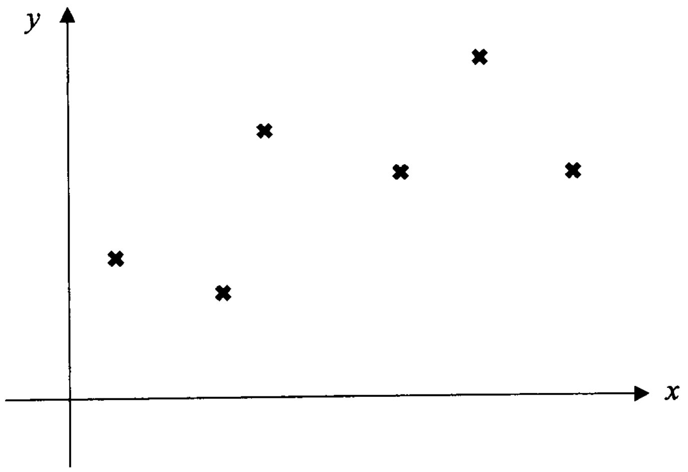
(iii) State the mode of Abel's score in a round of the game. [1]

(iv) Show that the variance of Abel's score in a round of the game is

$$\frac{60x+76}{3(3x+1)}-\frac{36(x+1)^2}{(3x+1)^2}.$$ [3]

(v) Given that $x=3$ and that Abel plays 100 rounds of the game, find the probability that the average score is not less than 2.5. [3]

10 **(a)** Draw the regression line of y on x and x on y on the diagram below, indicating the residual of one of the data points for each line. Hence describe the difference between the regression line of y on x and x on y. [3]



(b) Table A below shows the duration, t minutes, a diver can stay at different depths, d feet, below sea level.

Table A

d	50	60	70	80	90	100
t	80	55	45	35	25	22

(i) Sketch a scatter diagram of the data. [1]

(ii) Using the scatter diagram in part **(i)**, explain which of the following three models below is the most appropriate model for modelling the relationship between d and t.

(I) $t = ad + b$ where $a < 0$,

(II) $t = a\left(\frac{1}{d}\right) + b$ where $a > 0$, or

(III) $t = ae^{d} + b$ where $a < 0$.

State the equation of the regression line and the product moment correlation coefficient for the model, leaving your answers correct to 3 decimal places. [4]

(iii) Use your equation of the regression line from part **(ii)** to estimate the duration the diver can stay when he is 150 feet below sea level. Explain whether your estimate is reliable. [2]

(iv) A distance of 1 metre is equivalent to 3.28 feet. Re-write your equation from part **(ii)** so that it can be used to estimate the duration the diver can stay when he is at depth, D metres, below sea level. [1]

(v) A new data pair (d', t') is added to the data set given in Table A. If the product moment correlation coefficient found in part **(ii)** does not change with the addition of (d', t'), find a possible (d', t'). [2]

11 A cafeteria installed a vending machine which dispenses two types of coffee into disposable cups as follows:

(I) Black coffee, X ml, normally distributed with mean μ_1 ml and standard deviation 11.83 ml, or

(II) White coffee, by first releasing a quantity of black coffee, Y ml, normally distributed with mean μ_2 ml and standard deviation 11.83 ml and then adding milk, M ml, normally distributed with mean 35 ml and standard deviation 5.92 ml.

Given that $\mathrm{P}(X<175)=\mathrm{P}(Y>150)$, show that $\mu_1+\mu_2=325$. [2]

For the rest of this question, assume that $\mu_1=180$.

(i) Find the probability that the total volume of 2 randomly chosen cups of black coffee exceeds twice the volume of a randomly chosen cup of white coffee by less than 15 ml. State an assumption that is needed for your calculation to be valid. [4]

(ii) The black coffee is sold at $4 per cup while the white coffee is sold at $5 per cup. The cost of ingredients for the black coffee is 1 cent per ml and the cost of ingredients for the milk is 2 cents per ml. 100 cups of coffee are sold per day, n of which are black coffee and the rest are white coffee. Find the largest value of n such that the probability of the total profit earned per day exceeding $230 is at least 0.8. [4]

To boost the sales of coffee from the vending machine, if the volume of the coffee dispensed falls below a certain level, the customer receives the drink free of charge. It is given that $p\%$ of the customers who selected black coffee received the drink free of charge.

From a large number of customers who selected black coffee, three customers are chosen at random.

(iii) State, in terms of p, the probability that exactly one of the three customers receives the drink free of charge. [1]

(iv) Given that the probability of exactly one of the three customers receiving the drink free of charge is at most 0.1, find the range of values of p. [2]

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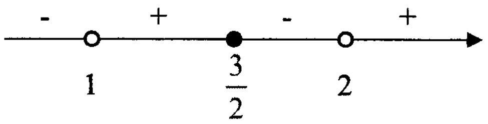
Qn	Suggested Solutions
1(i)	$\frac{x-1}{x-2} \le \frac{x-2}{x-1}$ $\frac{x-1}{x-2} - \frac{x-2}{x-1} \le 0$ $\frac{(x-1)^2-(x-2)^2}{(x-2)(x-1)} \le 0$ $\frac{(x-1-x+2)(x-1+x-2)}{(x-2)(x-1)} \le 0$ $\frac{(2x-3)}{(x-2)(x-1)} \le 0$  $\therefore x<1$ or $\frac{3}{2} \le x < 2$
(ii)	$f(x) = \frac{x-1}{x-2}$ $f\left(x-\frac{1}{2}\right) = \frac{\left(x-\frac{1}{2}\right)-1}{\left(x-\frac{1}{2}\right)-2}$ $= \frac{2x-3}{2x-5}$ OR $\frac{x-1}{x-2} = 1 + \frac{1}{x-2}$ $\downarrow$ Replace x with $x-\frac{1}{2}$ $\frac{2x-3}{2x-5} = 1 + \frac{2}{2x-5}$ $\therefore a = \frac{1}{2}$
(iii)	Replace x with $x-\frac{1}{2}$, $x-\frac{1}{2} < 1$ or $\frac{3}{2} \le x - \frac{1}{2} < 2$ $\therefore x < \frac{3}{2}$ or $2 \le x < \frac{5}{2}$

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Qn	Suggested Solutions
2(i)	Using cosine rule, $BD^2 = AD^2 + AB^2 - 2(AD)(AB)\cos\measuredangle BAD$ $= k^2 + 4k^2 - 4k^2\cos\left(x+\frac{\pi}{3}\right)$ $= 5k^2 - 4k^2\left(\cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3}\right)$ $= k^2\left(5 - 2\cos x + 2\sqrt{3}\sin x\right)$ (Shown).
(ii)	Since x is sufficiently small, then $\sin x \approx x$ and $\cos x \approx 1 - \frac{x^2}{2}$. $BD^2 \approx k^2\left[5 - 2\left(1-\frac{x^2}{2}\right) + 2\sqrt{3}(x)\right]$ $BD = k\left(3 + 2\sqrt{3}x + x^2\right)^{\frac{1}{2}}$ $= \sqrt{3}k\left[1+\left(\frac{2x}{\sqrt{3}}+\frac{x^2}{3}\right)\right]^{\frac{1}{2}}$ $= \sqrt{3}k\left[1+\frac{1}{2}\left(\frac{2x}{\sqrt{3}}+\frac{x^2}{3}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{2x}{\sqrt{3}}+\frac{x^2}{3}\right)^2+\ldots\right]$ $\approx \sqrt{3}k\left[1+\frac{x}{\sqrt{3}}+\frac{x^2}{6}-\frac{1}{8}\left(\frac{4x^2}{3}\right)\right]$ $= \sqrt{3}k + kx$
3(i)	$RHS = u_r - 2u_{r+1} + u_{r+2}$ $= \frac{1}{r!} - \frac{2}{(r+1)!} + \frac{1}{(r+2)!}$ $= \frac{(r+1)(r+2) - 2(r+2) + 1}{(r+2)!}$ $= \frac{r^2 + 3r + 2 - 2r - 4 + 1}{(r+2)!}$ $= \frac{r^2 + r - 1}{(r+2)!}$ (Shown)

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Qn	Suggested Solutions
(ii)	$\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}=\sum_{r=1}^{n}(u_r-2u_{r+1}+u_{r+2})$ $=u_1-2u_2+u_3$ $+u_2-2u_3+u_4$ $+u_3-2u_4+u_5$ $\vdots$ $+u_{n-2}-2u_{n-1}+u_n$ $+u_{n-1}-2u_n+u_{n+1}$ $+u_n-2u_{n+1}+u_{n+2}$ $=u_1-2u_2+u_2+u_{n+1}-2u_{n+1}+u_{n+2}$ $=1-1+\frac{1}{2}+\frac{1}{(n+1)!}-\frac{2}{(n+1)!}+\frac{1}{(n+2)!}$ $=\frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}$ As $n\to\infty$, $-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}\to 0$. $\therefore \sum_{r=1}^{\infty}\frac{r^2+r-1}{(r+2)!}=\frac{1}{2}$
(iii)	From MF26, see that $\mathrm{e}=1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{r!}+\ldots$ $=1+\sum_{r=1}^{\infty}\frac{1}{r!}$ $=1+\sum_{r=1}^{\infty}u_r$ $\therefore \sum_{r=1}^{\infty}\left(u_r-\frac{r^2+r-1}{(r+2)!}\right)$ $=\mathrm{e}-1-\frac{1}{2}$ $=\mathrm{e}-\frac{3}{2}$

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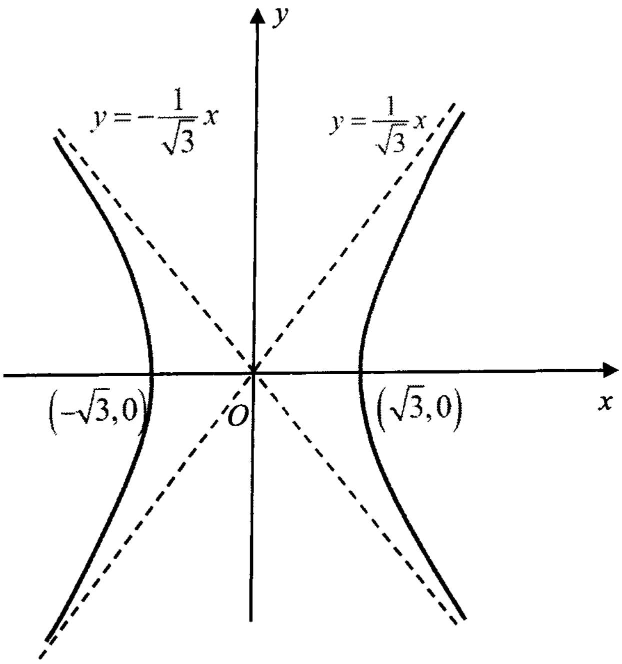
<table>
<tr><th>Qn</th><th>Suggested Solutions</th></tr>
<tr><td>4(i)</td><td>
</td></tr>
<tr><td>(ii)</td><td>$\text{Re}(z)\to\infty \Rightarrow x\to\infty$
As $x\to\infty$, $y\to\pm\frac{1}{\sqrt{3}}x$
i.e. $\arg(z)=\tan^{-1}\left(\frac{y}{x}\right)\to\tan^{-1}\left(\pm\frac{1}{\sqrt{3}}\right)=\pm\frac{\pi}{6}$
From the graph, z satisfies the right side of the hyperbola.
$\text{Re}(z)\to\infty$
When $y>0$,
$\arg(z)$ increases from 0 to $\frac{\pi}{6}$
When $y<0$,
$\arg(z)$ decreases from 0 to $-\frac{\pi}{6}$</td></tr>
<tr><td>5(a)</td><td>$\text{f}(x)=\frac{x^3}{x^2+k}=x-\frac{kx}{x^2+k}$</td></tr>
</table>

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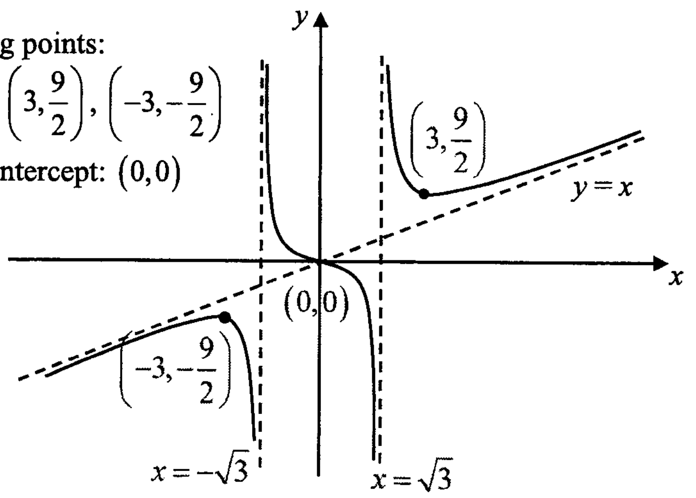
Qn	Suggested Solutions
	$f'(x)=1-\frac{k\left(x^2+k\right)-2x(kx)}{\left(x^2+k\right)^2}$ $=\frac{\left(x^2+k\right)^2-\left(k^2-kx^2\right)}{\left(x^2+k\right)^2}$ $=\frac{x^4+2kx^2+k^2-k^2+kx^2}{\left(x^2+k\right)^2}$ $=\frac{x^4+3kx^2}{\left(x^2+k\right)^2}$ $=\frac{x^2\left(x^2+3k\right)}{\left(x^2+k\right)^2}$ Since $k>0$, $x^2\left(x^2+3k\right)\geq 0$ and $\left(x^2+k\right)^2>0$, $f'(x)\geq 0$. Therefore f is an increasing function. (Shown)
(b)(i)	When $k=-3$, $f(x)=\frac{x^3}{x^2-3}=x-\frac{3x}{x^2-3}$ Vertical asymptotes: $x=\pm\sqrt{3}$ Oblique asymptote: $y=x$ When $f'(x)=0$, $x^4-9x^2=0$ $x^2\left(x^2-9\right)=0$ $x=0$ or $x=\pm 3$ Turning points: $(0,0)$, $\left(3,\frac{9}{2}\right)$, $\left(-3,-\frac{9}{2}\right)$ Axial intercept: $(0,0)$ 

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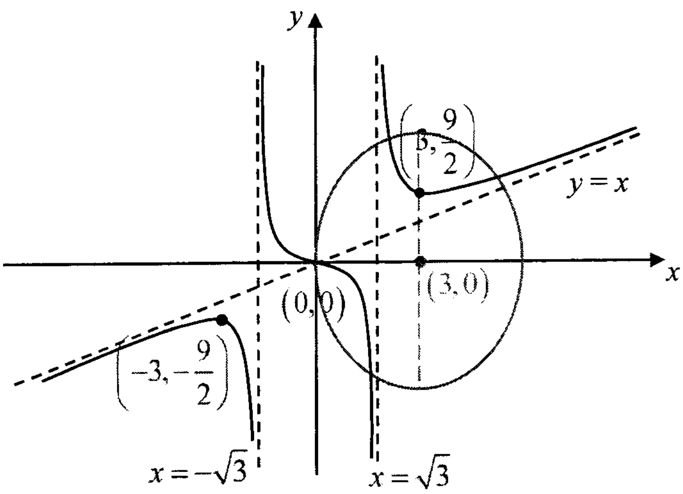
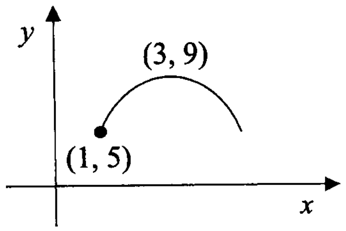
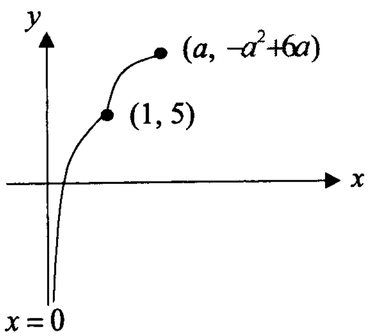
Qn	Suggested Solutions
(iii)	 For equation to have more than 3 real roots, $a > \frac{81}{4}$.
6(i)	Complete the square $-x^2+6x=-[x^2-6x]=-(x-3)^2+9$  For f to be 1-1 on $[1, a]$, $a \le 3$ $\therefore$ the range of values of a is $1 \le a \le 3$
(ii)	

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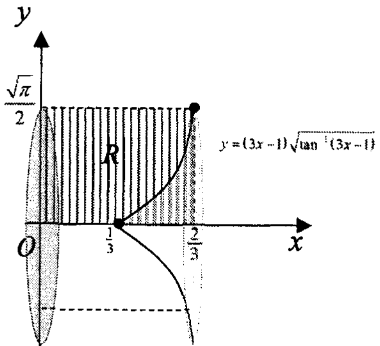
Qn	Suggested Solutions
(iii)	Let $y=\mathrm{f}(x) \Leftrightarrow x=\mathrm{f}^{-1}(y)$ For $0<x<1$ $5+\ln x=y$ $\ln x=y-5$ $x=\mathrm{e}^{y-5},\ y<5$ For $0\le x\le a\le 3$ $y=-(x-3)^2+9$ $(x-3)^2=9-y$ $x=3\pm\sqrt{9-y}$ Since $x\le 3$ $x=3-\sqrt{9-y},\ 5\le y\le -a^2+6a$ $\therefore \mathrm{f}^{-1}: x \mapsto \begin{cases} \mathrm{e}^{x-5}, & x<5 \\ 3-\sqrt{9-x}, & 5\le x\le -a^2+6a \end{cases}$
7(i)	$\int w^2 \tan^{-1} w \, \mathrm{d}w$ $=\frac{w^3}{3}\tan^{-1} w-\frac{1}{3}\int \frac{w^3}{1+w^2}\,\mathrm{d}w$ $=\frac{w^3}{3}\tan^{-1} w-\frac{1}{3}\int w-\frac{w}{w^2+1}\,\mathrm{d}w$ $=\frac{w^3}{3}\tan^{-1} w-\frac{w^2}{6}+\frac{1}{6}\ln(w^2+1)+C$
(ii)	 Volume generated by R $=(\pi)\left(\frac{\sqrt{\pi}}{2}\right)^2\left(\frac{2}{3}\right)-\pi\int_{\frac{1}{3}}^{\frac{2}{3}}(3x-1)^2\tan^{-1}(3x-1)\,\mathrm{d}x$

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Qn	Suggested Solutions
	From (i), $\int w^2 \tan^{-1} w \, \mathrm{d}w = \frac{w^3}{3}\tan^{-1} w - \frac{w^2}{6} + \frac{1}{6}\ln(w^2+1) + C$ Let $w = 3x-1 \;\Rightarrow\; \frac{\mathrm{d}w}{\mathrm{d}x} = 3$ $\int_{\frac{1}{3}}^{\frac{2}{3}} (3x-1)^2 \tan^{-1}(3x-1)\, \mathrm{d}x$ $= \frac{1}{3}\int_0^1 (w)^2 \tan^{-1}(w)\, \mathrm{d}w$ $= \frac{1}{3}\left[\frac{w^3}{3}\tan^{-1} w - \frac{w^2}{6} + \frac{1}{6}\ln(w^2+1)\right]_0^1$ $= \frac{1}{3}\left[\frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln 2\right]$ Volume generated by R $= (\pi)\left(\frac{\sqrt{\pi}}{2}\right)^2\left(\frac{2}{3}\right) - \pi\int_{\frac{1}{3}}^{\frac{2}{3}} (3x-1)^2 \tan^{-1}(3x-1)\, \mathrm{d}x$ $= \frac{\pi^2}{6} - \frac{\pi}{3}\left[\frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln 2\right]$ $= \frac{5\pi^2}{36} + \frac{\pi}{18} - \frac{\pi}{18}\ln 2$ unit^3
8(i)	Since the coefficients of the polynomial are all real and $a+\mathrm{i}$ is a root, then $a-\mathrm{i}$ is also a root. $z^2 - 2\sqrt{2}z + b = [z-(a+\mathrm{i})][z-(a-\mathrm{i})]$ $= z^2 - 2az + (a^2+1)$ By comparing z term, $-2\sqrt{2} = -2a$ $a = \sqrt{2}$. By comparing constant term, $b = a^2 + 1 = (\sqrt{2})^2 + 1 = 3$.
	Alternative Method $(a+\mathrm{i})^2 - 2\sqrt{2}(a+\mathrm{i}) + b = 0$ $(a^2 - 1 - 2\sqrt{2}a + b) + (2a - 2\sqrt{2})\mathrm{i} = 0$ By comparing real and imaginary parts,

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Qn	Suggested Solutions
	$\left(2a-2\sqrt{2}\right)=0$ $a=\sqrt{2}$ $a^2-1-2\sqrt{2}a+b=0$ $2-1-4+b=0$ $b=3$
(ii)	$f(-z)=(-z)^6-3(-z)^4+11(-z)^2-9$ $=z^6-3z^4+11z^2-9$ $=f(z)$
(iii)	Since the coefficients of $f(z)$ are all real and $\sqrt{2}+i$ is a root, then $\sqrt{2}-i$ is also a root. Thus $z^2-2\sqrt{2}z+3$ is a quadratic factor of $f(z)$. Since $f(z)=f(-z)$, then $-\sqrt{2}+i$ and $-\sqrt{2}-i$ are also roots of $f(z)=0$. $\left[z-\left(-\sqrt{2}-i\right)\right]\left[z-\left(-\sqrt{2}+i\right)\right]=\left(z+\sqrt{2}\right)^2-i^2$ $=z^2+2\sqrt{2}+3$ Thus $z^2+2\sqrt{2}z+3$ is a quadratic factor of $f(z)$. $f(z)=z^6-3z^4+11z^2-9$ $=\left(z^2-2\sqrt{2}z+3\right)\left(z^2+2\sqrt{2}z+3\right)\left(z^2+Cz+D\right)$ By comparing constant term, $-9=9D\Rightarrow D=-1$ $f(z)=\left(z^2-2\sqrt{2}z+3\right)\left(z^2+2\sqrt{2}z+3\right)\left(z^2+Cz-1\right)$ $=\left[\left(z^2+3\right)^2-\left(2\sqrt{2}z\right)^2\right]\left(z^2+Cz-1\right)$ $=\left(z^4+14z^2+9\right)\left(z^2+Cz-1\right)$ By comparing coefficient of z term, then $0=9Cz\Rightarrow C=0$. Therefore, $f(z)=\left(z^2-2\sqrt{2}z+3\right)\left(z^2+2\sqrt{2}z+3\right)\left(z^2-1\right)$.

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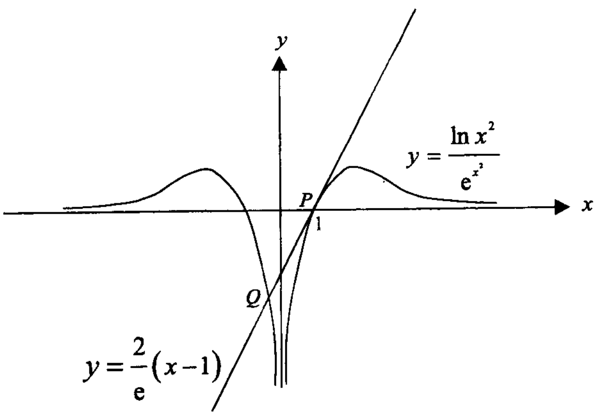
Qn	Suggested Solutions
9(i)	$e^{x^2}y=\ln x^2$ Differentiating wrt x both sides, $2xe^{x^2}y+e^{x^2}\frac{dy}{dx}=\frac{1}{x^2}(2x)$ $2x\ln x^2+e^{x^2}\frac{dy}{dx}=\frac{2}{x}$ (Shown) $\left(\because e^{x^2}y=\ln x^2\right)$
(ii)	When $x=1$, $y=\frac{\ln 1^2}{e^{1^2}}=0$ $\frac{dy}{dx}=\frac{\frac{2}{1}-2(1)\ln 1^2}{e^{1^2}}=\frac{2}{e}$ (from (i)) Hence equation of tangent at P: $y=\frac{2}{e}(x-1)$ **Method 1** $\frac{2}{e}(x-1)=\frac{\ln x^2}{e^{x^2}}$ Using GC, another point of intersection between the tangent $y=\frac{2}{e}(x-1)$ and C: $y=\frac{\ln x^2}{e^{x^2}}$ is $Q\,(-0.49516,-1.1001)$ (to 5 sf).  **Method 2** Substitute $y=\frac{2}{e}(x-1)$ into the equation for C:

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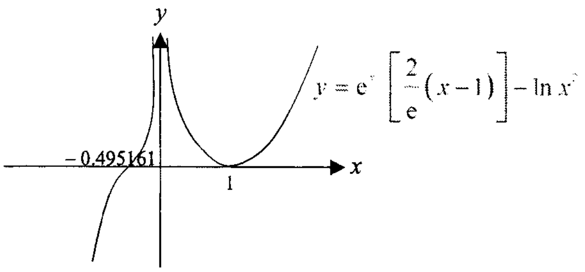
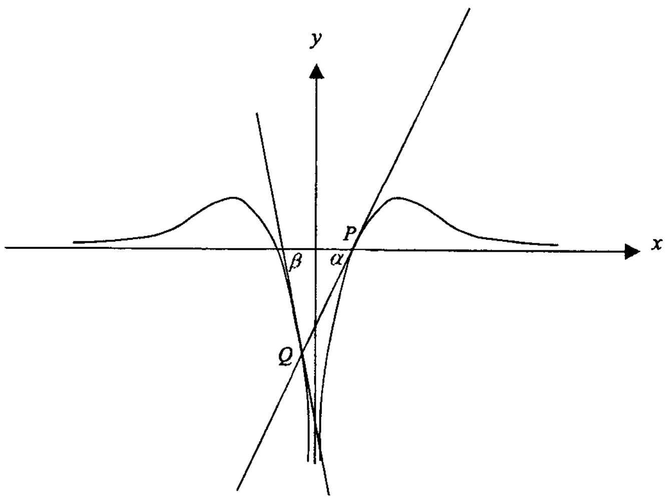
Qn	Suggested Solutions
	$e^{x^2}\left[\frac{2}{e}(x-1)\right]=\ln x^2$ ..... (1) $e^{x^2}\left[\frac{2}{e}(x-1)\right]-\ln x^2=0$  At Q, $x=-0.495161$ $y=\frac{2}{e}(-0.495161-1)=-1.100078256$ Hence, $Q\ (-0.49516,-1.1001)$ (to 5 sf).
(iii)	At Q, $\frac{d}{dx}\left[\frac{\ln(x^2)}{e^{x^2}}\right]\Bigg\|_{x=-0.495161}=-4.2502646$ (from GC) (Alternatively, the above value can be obtained by substituting $x=-0.495161$ into equation in (i))  $\alpha=\tan^{-1}\left(\frac{2}{e}\right)=36.3441°$ $\beta=\tan^{-1}(4.2502646)=76.7603°$ Required acute angle between the 2 tangents

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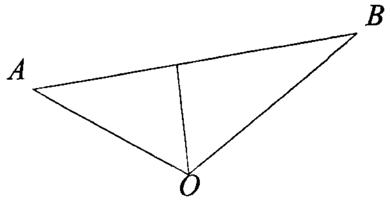
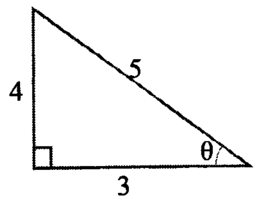
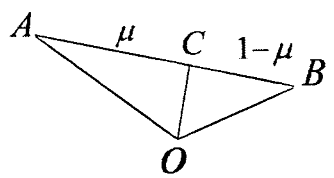
Qn	Suggested Solutions
	$=180°-36.3441°-76.7603°$ $=66.9°$ (1 d.p.)
10(i)	$\frac{1}{2}\lvert\underset{\sim}{a}\times\underset{\sim}{b}\rvert=16$ $=>\lvert\underset{\sim}{a}\times\underset{\sim}{b}\rvert=32$  Since $\hat{\underset{\sim}{b}}=\frac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}$ $\therefore \underset{\sim}{b}=\lvert\underset{\sim}{b}\rvert\hat{\underset{\sim}{b}}=5\underset{\sim}{d}$ $=>\lvert\underset{\sim}{a}\times 5\underset{\sim}{d}\rvert=32$ $\lvert\underset{\sim}{a}\times\underset{\sim}{d}\rvert=\frac{32}{5}$ $\lvert\underset{\sim}{a}\times\underset{\sim}{d}\rvert$ is the shortest distance from A to OB OR the perpendicular height of the triangle OAB with OB as the base.
(ii)	Let θ be the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ $\lvert\underset{\sim}{a}\times\underset{\sim}{b}\rvert=32$ $\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\sin\theta=32$ $\sin\theta=\frac{4}{5}$  Since θ is obtuse, $\cos\theta=-\frac{3}{5}$ $\underset{\sim}{a}\cdot\underset{\sim}{b}=\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\theta$ $\underset{\sim}{a}\cdot\underset{\sim}{b}=(8\times5)\left(-\frac{3}{5}\right)=-24$
(iii)	 By ratio theorem, $\overrightarrow{OC}=\ \mu\underset{\sim}{b}+(1-\mu)\underset{\sim}{a}$

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Qn	Suggested Solutions
	$\lvert\overrightarrow{OC}\rvert = \lvert\mu \underset{\sim}{b} + (1-\mu)\underset{\sim}{a}\rvert$ $\lvert\overrightarrow{OC}\rvert^2 = \lvert\mu \underset{\sim}{b} + (1-\mu)\underset{\sim}{a}\rvert^2$ $\frac{401}{49} = \left[\mu \underset{\sim}{b} + (1-\mu)\underset{\sim}{a}\right]\bullet\left[\mu \underset{\sim}{b} + (1-\mu)\underset{\sim}{a}\right]$ $\frac{401}{49} = \mu^2 \lvert\underset{\sim}{b}\rvert^2 + 2\mu(1-\mu)\underset{\sim}{a}\bullet\underset{\sim}{b} + (1-\mu)^2\lvert\underset{\sim}{a}\rvert^2$ $\frac{401}{49} = 25\mu^2 - 48\mu(1-\mu) + 64(1-\mu)^2$ $401 = 1225\mu^2 - 2352\mu(1-\mu) + 3136(1-2\mu+\mu^2)$ $6713\mu^2 - 8624\mu + 2735 = 0$ $\mu = \frac{5}{7}, \frac{547}{959}$ $\overrightarrow{OC} = \frac{2}{7}\underset{\sim}{a} + \frac{5}{7}\underset{\sim}{b}$, $\overrightarrow{OC} = \frac{412}{959}\underset{\sim}{a} + \frac{547}{959}\underset{\sim}{b}$

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Qn	Suggested Solutions
11(a) (i)	Let the point where the diagonals meet be M. $AM=\sqrt{a^2-x^2}$ $MC=\sqrt{b^2-x^2}$ **Method 1**: Area of kite $ABCD$ = Area of $\triangle ABD$ + Area of $\triangle BDC$ $=\frac{1}{2}(2x)\sqrt{a^2-x^2}+\frac{1}{2}(2x)\sqrt{b^2-x^2}$ $=(x)\sqrt{a^2-x^2}+(x)\sqrt{b^2-x^2}$ $\therefore K=x\sqrt{a^2-x^2}+x\sqrt{b^2-x^2}$ **Method 2:** $AC=\sqrt{a^2-x^2}+\sqrt{b^2-x^2}$ Area of kite $ABCD$ $=\frac{1}{2}(AC)(BD)$ $=\frac{1}{2}\left(\sqrt{a^2-x^2}+\sqrt{b^2-x^2}\right)(2x)$ $\therefore K=x\sqrt{a^2-x^2}+x\sqrt{b^2-x^2}$
(a) (ii)	$\frac{dK}{dx}=\frac{x}{2}(-2x)(a^2-x^2)^{-\frac{1}{2}}+(a^2-x^2)^{\frac{1}{2}}+\frac{x}{2}(-2x)(b^2-x^2)^{-\frac{1}{2}}+(b^2-x^2)^{\frac{1}{2}}$ $\frac{dK}{dx}=(a^2-x^2)^{\frac{1}{2}}+(b^2-x^2)^{\frac{1}{2}}-x^2\left[\frac{1}{\sqrt{a^2-x^2}}+\frac{1}{\sqrt{b^2-x^2}}\right]$ $=\sqrt{a^2-x^2}+\sqrt{b^2-x^2}-x^2\left[\frac{\sqrt{a^2-x^2}+\sqrt{b^2-x^2}}{\sqrt{a^2-x^2}\sqrt{b^2-x^2}}\right]$ $=\left(\sqrt{a^2-x^2}+\sqrt{b^2-x^2}\right)\left[1-\frac{x^2}{\sqrt{a^2-x^2}\sqrt{b^2-x^2}}\right]$ For stationary value of K, $\frac{dK}{dx}=0$ $\frac{dK}{dx}=\frac{\left(\sqrt{a^2-x^2}+\sqrt{b^2-x^2}\right)\left[\sqrt{a^2-x^2}\sqrt{b^2-x^2}-x^2\right]}{\sqrt{a^2-x^2}\sqrt{b^2-x^2}}$ ----(*) $\sqrt{a^2-x^2}>0,\quad \sqrt{b^2-x^2}>0$ $\sqrt{a^2-x^2}+\sqrt{b^2-x^2}>0$

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Qn	Suggested Solutions
	$\sqrt{a^2-x^2}\sqrt{b^2-x^2}-x^2=0$ $x^2=\sqrt{a^2-x^2}\sqrt{b^2-x^2}$ $x^4=(a^2-x^2)(b^2-x^2)$ $x^4=a^2b^2-x^2(a^2+b^2)+x^4$ $x^2(a^2+b^2)=a^2b^2$ $x^2=\dfrac{a^2b^2}{a^2+b^2}$ $x=\dfrac{ab}{\sqrt{a^2+b^2}},\quad x>0$
	Alternative Method: $\dfrac{dK}{dx}=\dfrac{a^2-2x^2}{\sqrt{a^2-x^2}}+\dfrac{b^2-2x^2}{\sqrt{b^2-x^2}}=0$ $\dfrac{a^2-2x^2}{\sqrt{a^2-x^2}}=\dfrac{2x^2-b^2}{\sqrt{b^2-x^2}}$ $(a^2-2x^2)\sqrt{b^2-x^2}=(2x^2-b^2)\sqrt{a^2-x^2}$ $(a^2-2x^2)^2(b^2-x^2)=(2x^2-b^2)^2(a^2-x^2)$ $(a^4-4a^2x^2+4x^4)(b^2-x^2)=(4x^4-4b^2x^2+b^4)(a^2-x^2)$ $x^2(a^4-b^4)=a^4b^2-a^2b^4$ $x^2=\dfrac{a^2b^2(a^2-b^2)}{(a^2-b^2)(a^2+b^2)}$ $x^2=\dfrac{a^2b^2}{(a^2+b^2)}$
(a) (iii)	$AC=\sqrt{a^2-\dfrac{a^2b^2}{a^2+b^2}}+\sqrt{b^2-\dfrac{a^2b^2}{a^2+b^2}}$ $=\sqrt{\dfrac{a^4+a^2b^2-a^2b^2}{a^2+b^2}}+\sqrt{\dfrac{b^4+a^2b^2-a^2b^2}{a^2+b^2}}$ $=\sqrt{\dfrac{a^4}{a^2+b^2}}+\sqrt{\dfrac{b^4}{a^2+b^2}}$ $=\dfrac{a^2+b^2}{\sqrt{a^2+b^2}}$ $\therefore AC=\sqrt{a^2+b^2}$
(a) (iv)	From (iii) Since $AC^2=AB^2+BC^2$ $\therefore \measuredangle ABC=90°$ or $\dfrac{\pi}{2}$ radians

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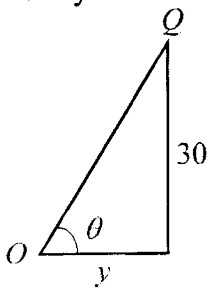
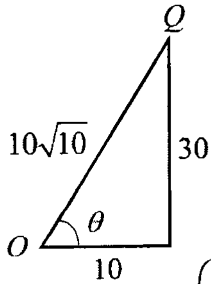
Qn	Suggested Solutions
(b)	Let the horizontal distance travelled by the kite be y m.  $\tan\theta = \dfrac{30}{y}$ $\theta = \tan^{-1}\left(\dfrac{30}{y}\right)$ $\dfrac{\mathrm{d}\theta}{\mathrm{d}y} = \dfrac{1}{1+\left(\dfrac{30}{y}\right)^2}\left(-\dfrac{30}{y^2}\right)$ $= -\dfrac{30}{y^2+30^2}$ When $y = \sqrt{\left(10\sqrt{10}\right)^2 - 30^2} = 10$ and $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 2.4$, $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \dfrac{\mathrm{d}\theta}{\mathrm{d}y} \times \dfrac{\mathrm{d}y}{\mathrm{d}t}$ $= -\dfrac{30}{10^2+30^2}(2.4)$ $= -\dfrac{9}{125}$ rad/s The angle is decreasing at $\dfrac{9}{125}$ or 0.072 rad/s **Alternatively,**  $\tan\theta = \dfrac{30}{y}$ diff wrt t, $\sec^2\theta \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -\dfrac{30}{y^2}\dfrac{\mathrm{d}y}{\mathrm{d}t}$ When $OQ = 10\sqrt{10}$, $y = 10$ and $\sec^2\theta = \left(\dfrac{10\sqrt{10}}{10}\right)^2 = 10$ $\therefore 10\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -\dfrac{30}{100}\dfrac{\mathrm{d}y}{\mathrm{d}t}$ $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -\dfrac{3}{100}(2.4)$ $= -\dfrac{9}{125}$ rad/s The angle is decreasing at $\dfrac{9}{125}$ or 0.072 rad/s.
12(i)	$v = \dfrac{\mathrm{d}x}{\mathrm{d}t}$

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Qn	Suggested Solutions
(ii)	Speed $v=\frac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ Given $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\alpha\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2=10$ $\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t}+\alpha(v)^2=10$ $\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t}=10-\alpha v^2$
(iii)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\alpha\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2=10.$ Given that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=4.375$ when $\frac{\mathrm{d}x}{\mathrm{d}t}=1.5$. $4.375+\alpha(1.5)^2=10$ $\Rightarrow \alpha=\frac{5}{2}$ $\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t}=10-\frac{5}{2}v^2=\frac{5}{2}(4-v^2)$ $\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}v}=\frac{2}{5(4-v^2)}$ $\Rightarrow \int\frac{5}{2}\,\mathrm{d}t=\int\frac{1}{4-v^2}\,\mathrm{d}v$ $\Rightarrow \frac{5}{2}t+C=\frac{1}{4}\ln\left\|\frac{2+v}{2-v}\right\|$ $\Rightarrow 10t+C'=\ln\left\|\frac{2+v}{2-v}\right\|$ $\Rightarrow e^{10t+C'}=\left\|\frac{2+v}{2-v}\right\|$ $\Rightarrow Ae^{10t}=\frac{2+v}{2-v}$, where $A=\pm e^{C'}$ When $t=0, x=0, v=0$ $\Rightarrow Ae^{0}=\frac{2+(0)}{2-(0)}$ $\Rightarrow A=1$ $\Rightarrow e^{10t}=\frac{2+v}{2-v}$ $\Rightarrow e^{10t}(2-v)=2+v$ $\Rightarrow 2e^{10t}-ve^{10t}=2+v$ $\Rightarrow 2e^{10t}-2=v+ve^{10t}$ $\Rightarrow v=\frac{2e^{10t}-2}{e^{10t}+1}$

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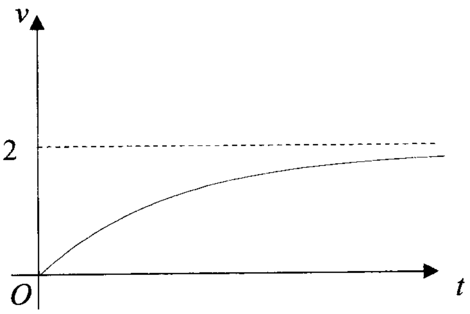
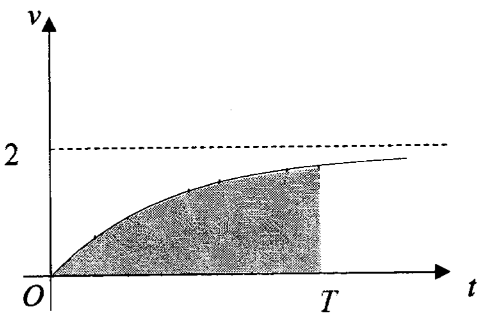
Qn	Suggested Solutions
	$\Rightarrow v=\dfrac{2-2\mathrm{e}^{-10t}}{1+\mathrm{e}^{-10t}}$ where $k=2, m=1$
(iv)	 As $t\to\infty, \mathrm{e}^{-10t}\to 0, v\to 2$ and $\dfrac{\mathrm{d}v}{\mathrm{d}t}\to 0$ according to graph. Thus $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ will **increase** and approach 2 m/s. $\therefore\ \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}=\dfrac{\mathrm{d}v}{\mathrm{d}t}$ will **decrease** and approach $0\ \mathrm{m/s^2}$.
(v)	
	Area under the graph $\int_0^T \dfrac{2-2\mathrm{e}^{-10t}}{1+\mathrm{e}^{-10t}}\,\mathrm{d}t$ $=2\int_0^T \dfrac{\mathrm{e}^{5t}-\mathrm{e}^{-5t}}{\mathrm{e}^{5t}+\mathrm{e}^{-5t}}\,\mathrm{d}t$ $=\dfrac{2}{5}\int_0^T \dfrac{5\mathrm{e}^{5t}-5\mathrm{e}^{-5t}}{\mathrm{e}^{5t}+\mathrm{e}^{-5t}}\,\mathrm{d}t$ $=\dfrac{2}{5}\left[\ln\left(\mathrm{e}^{5t}+\mathrm{e}^{-5t}\right)\right]_0^T$ $=\dfrac{2}{5}\left[\ln\left(\mathrm{e}^{5T}+\mathrm{e}^{-5T}\right)-\ln\left(\mathrm{e}^{0}+\mathrm{e}^{0}\right)\right]$ $=\dfrac{2}{5}\ln\left(\dfrac{\mathrm{e}^{5T}+\mathrm{e}^{-5T}}{2}\right)$ where $\beta=5$
	Alternative Method:

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Qn	Suggested Solutions
	$\int \frac{2e^{10t}-2}{e^{10t}+1}\,dt$ $=\int_0^T \int \frac{2e^{10t}}{e^{10t}+1}-\frac{-2e^{-10t}}{1+e^{-10t}}\,dt$ $=\frac{1}{5}\int_0^T \frac{10e^{10t}}{e^{10t}+1}\,dt+\int_0^T \frac{-10e^{-10t}}{1+e^{-10t}}\,dt$ $=\left[\frac{1}{5}\ln(e^{10t}+1)+\frac{1}{5}\ln(e^{-10t}+1)\right]_o^T$ $\frac{1}{5}\ln\left[\frac{(e^{10T}+1)(e^{-10T}+1)}{4}\right]$ $=\frac{1}{5}\ln\left[\frac{(e^{5T}+e^{-5T})^2}{4}\right]$
	Alternative Method $\int_0^T 2-\frac{4e^{-10t}}{e^{-10t}+1}\,dt$ $=2T+\frac{4}{10}\int_0^T \frac{-10e^{-10t}}{e^{-10t}+1}\,dt$ $=2T+\frac{2}{5}\ln(1+e^{-10T})-\frac{2}{5}\ln 2$ $=\frac{2}{5}e^{5T}+\frac{2}{5}\ln\left(\frac{1+e^{-10t}}{2}\right)$ $=\frac{2}{5}\ln\left(\frac{e^{5T}+e^{-5T}}{2}\right)$
(v)	$\frac{2}{5}\ln\left(\frac{e^{5T}+e^{-5T}}{2}\right)$ represents the distance the First Aid Kit dropped from the cargo drone in T seconds

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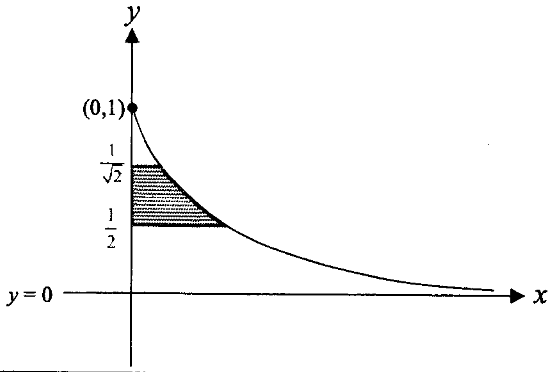
Qn	Suggested Solutions
1	$y=-\left(x-\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2-4$ $=-\left(x-\frac{3}{2}\right)^2-\frac{7}{4}$ 1. Translate $\frac{3}{2}$ units in the positive x direction. 2. Reflect about the x axis 3. Translate $\frac{7}{4}$ units in the negative y direction
	1. Translate $\frac{3}{2}$ units in the positive x direction. 2. Translate $\frac{7}{4}$ units in the positive y direction 3. Reflect about the x axis
2(i)	
(ii)	$\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} x \, \mathrm{d}y$ $=\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \sin t \tan t(-\sin t) \, \mathrm{d}t$ $=\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} -\sin^2 t \tan t \, \mathrm{d}t$ $=\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} (\cos^2 t-1)\tan t \, \mathrm{d}t$ $\frac{\mathrm{d}y}{\mathrm{d}t}=-\sin t$ when $y=\frac{1}{2}, t=\frac{\pi}{3}$ when $y=\frac{1}{\sqrt{2}}, t=\frac{\pi}{4}$

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Qn	Suggested Solutions
	$=\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \sin t\cos t-\tan t\,\mathrm{d}t \quad \left(=\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{\sin 2t}{2}-\tan t\,\mathrm{d}t\right)$ $=\left[\frac{\sin^2 t}{2}+\ln(\cos t)\right]_{\frac{\pi}{3}}^{\frac{\pi}{4}}$ or $\left[-\frac{1}{4}\cos 2t+\ln(\cos t)\right]_{\frac{\pi}{3}}^{\frac{\pi}{4}}$ $=\left[\frac{1}{4}+\ln\left(\frac{1}{\sqrt{2}}\right)-\frac{3}{8}-\ln\left(\frac{1}{2}\right)\right]$ $=\ln\sqrt{2}-\frac{1}{8}$ units^2
3(i)	**Method 1** $z=2(\cos\beta+\mathrm{i}\sin\beta)=2\mathrm{e}^{\mathrm{i}\beta}$ $\frac{z}{4-z^2}=\frac{2\mathrm{e}^{\mathrm{i}\beta}}{4-4\mathrm{e}^{\mathrm{i}(2\beta)}}$ $=\frac{\mathrm{e}^{\mathrm{i}\beta}}{2\mathrm{e}^{\mathrm{i}\beta}\left(\mathrm{e}^{-\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\beta}\right)}$ $=-\frac{1}{2\left(\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{-\mathrm{i}\beta}\right)}$ $=-\frac{1}{2(2\sin\beta)\mathrm{i}} \quad \left(\because \mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{-\mathrm{i}\beta}=2\,\mathrm{Im}\left(\mathrm{e}^{\mathrm{i}\beta}\right)\mathrm{i}\right)$ $=\left(\frac{1}{4}\operatorname{cosec}\beta\right)\mathrm{i}$ where $k=\frac{1}{4}$ $\therefore k=\frac{1}{4}$
	Method 2 $\frac{z}{4-z^2}=\frac{2\cos\beta+\mathrm{i}(2\sin\beta)}{4-4\cos^2\beta-\mathrm{i}(8\sin\beta\cos\beta)+4\sin^2\beta}$ $=\frac{2\cos\beta+\mathrm{i}(2\sin\beta)}{4-4\left(1-\sin^2\beta\right)-\mathrm{i}(8\sin\beta\cos\beta)+4\sin^2\beta}$ $=\frac{2\cos\beta+\mathrm{i}(2\sin\beta)}{8\sin^2\beta-\mathrm{i}(8\sin\beta\cos\beta)}$ $=\frac{\mathrm{i}(2\sin\beta-2\cos\beta\mathrm{i})}{4\sin\beta(2\sin\beta-2\cos\beta\mathrm{i})}$ $=\left(\frac{1}{4}\operatorname{cosec}\beta\right)\mathrm{i}$ where $k=\frac{1}{4}$

HCI C2 H2 Mathematics Preliminary Examinations 2022 Mark Scheme (Paper 2)

Qn	Suggested Solutions
(ii)	$\arg(w)=\arg\left(-\sqrt{3}+\mathrm{i}\right)=\frac{5\pi}{6}$ $\Rightarrow \arg\left(w^*\right)=-\frac{5\pi}{6}$ $\arg\left(\frac{z}{4-z^2}\right)=\arg\left[\left(\frac{1}{4}\operatorname{cosec}\beta\right)\mathrm{i}\right]=\frac{\pi}{2}$, since for $0<\beta<\frac{\pi}{2}$, $\frac{1}{4}\operatorname{cosec}\beta>0$. $\arg\left(\left(\frac{z}{4-z^2}\right)\left(w^*\right)^n\right)=\arg\left(\frac{z}{4-z^2}\right)+\arg\left(\left(w^*\right)^n\right)$ $=\arg\left(\frac{z}{4-z^2}\right)+n\arg\left(w^*\right)$ $=\frac{\pi}{2}+n\left(-\frac{5\pi}{6}\right)$ $=\frac{\pi}{2}-\frac{5n\pi}{6}$ For $\left(\frac{z}{4-z^2}\right)\left(w^*\right)^n$ to be a real number, $\arg\left(\left(\frac{z}{4-z^2}\right)\left(w^*\right)^n\right)=k\pi$, where k is an integer. Therefore $\frac{\pi}{2}-\frac{5n\pi}{6}=k\pi$ $\Rightarrow n=\frac{3-6k}{5}$ Hence using GC, the three smallest positive integers are $n=3$ (when $k=-2$), $n=9$ (when $k=-7$), and $n=15$ (when $k=-12$).
	Method 2: $\arg\left(w^*\right)^n=n\arg\left(w^*\right)$ $=n\left(-\frac{5\pi}{6}\right)$ For $\left(\frac{z}{4-z^2}\right)\left(w^*\right)^n$ to be a real number, $\arg\left(w^*\right)^n=\frac{\pi}{2}+k\pi$, where k is an integer.

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Qn	Suggested Solutions
	$-\frac{5n\pi}{6}=\frac{\pi}{2}+k\pi$ $n=-\frac{3}{5}-\frac{6k}{5}$ using GC, the three smallest positive integers are $n=3$ (when $k=-3$), $n=9$ (when $k=-8$), and $n=15$ (when $k=-13$).
4(a) (i)	$\sum_{r=1}^{k}\left[\left(-\frac{1}{2}\right)^{r+1}+\ln(r+1)\right]$ $=\sum_{r=1}^{k}\left(-\frac{1}{2}\right)^{r+1}+\sum_{r=1}^{k}\ln(r+1)$ $=\sum_{r=1}^{k}\left[\left(-\frac{1}{2}\right)^{r+1}\right]+\ln 2+\ln 3+\ln 4+...+\ln(k+1)$ $=\left(\frac{1}{4}\right)\left[\frac{1-\left(-\frac{1}{2}\right)^{k}}{1-\left(-\frac{1}{2}\right)}\right]+\ln\left[(k+1)!\right]$ $=\frac{1}{6}\left[1-\left(-\frac{1}{2}\right)^{k}\right]+\ln\left[(k+1)!\right]$
4 (a) (ii)	As $k\to\infty$, $\lim_{k\to\infty}\left\{\frac{1}{6}\left[1-\left(-\frac{1}{2}\right)^{k}\right]\right\}=\frac{1}{6}$ $\lim_{k\to\infty}\left\{\ln\left[(k+1)!\right]\right\}=\infty$ Therefore, the sum to infinity of the series does not exist.
(b)	Let a be the first term of AP and d be the common difference. $S_6=4.5$ $\Rightarrow\frac{6}{2}(2a+5d)=4.5$ $\Rightarrow 2a+5d=1.5$

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Qn	Suggested Solutions
	$u_1u_2u_3u_4=0$ $\Rightarrow a(a+d)(a+2d)(a+3d)=0$ $\therefore d=-a$ or $-\frac{a}{2}$ or $-\frac{a}{3}$ When $d=-a$, $\Rightarrow 2a+5(-a)=1.5$ $\Rightarrow a=-\frac{1}{2}$ (rej. $\because a>0$) When $d=-\frac{a}{2}$, $\Rightarrow 2a+5\left(\frac{-a}{2}\right)=1.5$ $\Rightarrow a=-3$ (rej. $\because a>0$) When $d=-\frac{a}{3}$, $\Rightarrow 2a+5\left(\frac{-a}{3}\right)=1.5$ $\Rightarrow a=4.5$ $\therefore T_{13}=4.5+12(-1.5)=-13.5$
5(i)	$s=12$
5(ii)	Given l_{DM} is parallel to $\begin{pmatrix}1\\1\\t\end{pmatrix}$ Plane *ABFE* : $6x-z=36 \Rightarrow \underset{\sim}{r}\bullet\begin{pmatrix}6\\0\\-1\end{pmatrix}=36$ If *DM* doesn't intersect with *ABFE*, then *DM* must be parallel to *ABFE*, i.e. perpendicular to the normal vector of *ABFE*. $\begin{pmatrix}1\\1\\t\end{pmatrix}\bullet\begin{pmatrix}6\\0\\-1\end{pmatrix}=0 \Rightarrow 6-t=0$ $\therefore t=6$
(iii)	From part (i), $t=6$,

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Qn	Suggested Solutions
	$\overrightarrow{DM}$ is parallel to $\begin{pmatrix}1\\1\\6\end{pmatrix}$ $\overrightarrow{DG}//\begin{pmatrix}0\\1\\0\end{pmatrix}$ Normal $=\begin{pmatrix}1\\1\\6\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-6\\0\\1\end{pmatrix}=-\begin{pmatrix}6\\0\\-1\end{pmatrix}$ Equation of plane: $\underset{\sim}{r}\bullet\begin{pmatrix}6\\0\\-1\end{pmatrix}=\begin{pmatrix}-2\\0\\12\end{pmatrix}\bullet\begin{pmatrix}6\\0\\-1\end{pmatrix}=-24$ (shown) $\therefore k=-24$
(iv)	Normal vector of plane $DGM=\begin{pmatrix}6\\0\\-1\end{pmatrix}$ Normal vector of plane $DEFG=\begin{pmatrix}0\\0\\1\end{pmatrix}$ Angle between the 2 planes DGM and $DENM=\cos^{-1}\left\lvert\dfrac{\begin{pmatrix}6\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{37}}\right\rvert$ $=80.53768°=80.5°$ (1 d.p.)
(v)	$\overrightarrow{OM}=\begin{pmatrix}-2+\lambda\\\lambda\\12+6\lambda\end{pmatrix}$ Since height of the structure is 26 units, $12+6\lambda=24$ $\lambda=2$ $\overrightarrow{OM}=\begin{pmatrix}0\\2\\24\end{pmatrix}$

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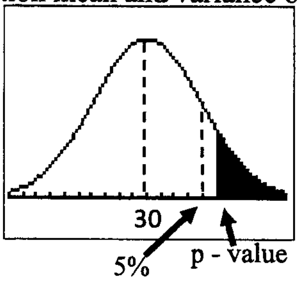
Qn	Suggested Solutions
	Coordinates of $M(0,\ 2,\ 24)$
	$\overrightarrow{AM}=\begin{pmatrix}0\\2\\24\end{pmatrix}-\begin{pmatrix}6\\0\\0\end{pmatrix}=\begin{pmatrix}-6\\2\\24\end{pmatrix}$ Shortest distance $=\dfrac{\left\|\overrightarrow{AM}\cdot\begin{pmatrix}6\\0\\-1\end{pmatrix}\right\|}{\sqrt{37}}$ $=\dfrac{\left\|\begin{pmatrix}-6\\2\\24\end{pmatrix}\cdot\begin{pmatrix}6\\0\\-1\end{pmatrix}\right\|}{\sqrt{37}}=\dfrac{\|-36-24\|}{\sqrt{37}}=\dfrac{60}{\sqrt{37}}$ Alternative Method Note that plane AGM parallel to plane $ABFE$ plane AGM: $\underset{\sim}{r}\cdot\dfrac{\begin{pmatrix}6\\0\\-1\end{pmatrix}}{\sqrt{37}}=\dfrac{-26}{\sqrt{37}}$ plane $ABFE$: $\underset{\sim}{r}\cdot\dfrac{\begin{pmatrix}6\\0\\-1\end{pmatrix}}{\sqrt{37}}=\dfrac{36}{\sqrt{37}}$ Distance between the 2 planes $=\dfrac{36-(-24)}{\sqrt{37}}=\dfrac{60}{\sqrt{37}}$ units
(vi)	$\Pi: x=c \Rightarrow \underset{\sim}{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=c$ which is a y-z plane $D(-2,\ 0,\ 12)$ Since DE =10 units $D(8,\ 0,\ 12)$ Since they intersect at E, $x=c=8$
6(a)	Let T and S denote the event that a triangle and square block is chosen respectively. Let C denote the event that a block is correctly placed into the shaper sorter toy.

HCI C2 H2 Mathematics Preliminary Examinations 2022 Mark Scheme (Paper 2)

<table>
<tr><th>Qn</th><th>Suggested Solutions</th></tr>
<tr><td></td><td><u>Method 1</u>
$\mathrm{P}(C)=0.9$
$\mathrm{P}(S \mid C)=0.4 \Rightarrow \mathrm{P}(T \mid C)=0.6$
$\Rightarrow \dfrac{\mathrm{P}(T \cap C)}{\mathrm{P}(C)}=0.6$
$\Rightarrow \mathrm{P}(T \cap C)=(0.6)(0.9)=0.54$
Required probability $=1-0.54=0.46$

<u>Method 2</u>
$\mathrm{P}(C)=0.9$
$\Rightarrow \mathrm{P}(C')=0.1$
$\mathrm{P}(S \mid C)=0.4$
$\Rightarrow \dfrac{\mathrm{P}(S \cap C)}{\mathrm{P}(C)}=0.4$
$\Rightarrow \mathrm{P}(S \cap C)=(0.4)(0.9)=0.36$
Required probability $=\mathrm{P}(C')+\mathrm{P}(S \cap C)$
$=0.1+0.36=0.46$</td></tr>
<tr><td>(b)</td><td>$\mathrm{P}(S=2)=2\mathrm{P}(S=4)$
$\dfrac{{}^{r+1}C_2\,{}^5C_3}{{}^{r+6}C_5}=2\left(\dfrac{{}^{r+1}C_4\,{}^5C_1}{{}^{r+6}C_5}\right)$
$\dfrac{(r+1)!(10)}{(r-1)!2!}=2\dfrac{(r+1)!(5)}{(r-3)!4!}$
$\dfrac{1}{2(r-1)!}=\dfrac{1}{(r-3)!4!}$
$12(r-3)!=(r-1)!$
$12(r-3)!=(r-3)!(r-2)(r-1)$
$12=(r-2)(r-1)$ since $r \geq 3$
$r^2-3r-10=0$
$(r+2)(r-5)=0$
$r=-2$ (rej) or $r=5$</td></tr>
<tr><td>7(i)</td><td>It is not necessary for the fat content of the chocolate bars to be normally distributed. As the sample size (number of chocolate bars = 40) used is large, by Central Limit Theorem, the sample mean fat content of the chocolate bars is approximately normally distributed for the test to be valid.</td></tr>
</table>

HCI C2 H2 Mathematics Preliminary Examinations 2022 Mark Scheme (Paper 2)

Qn	Suggested Solutions
(ii)	$\bar{x}=\dfrac{1220}{40}=30.5$ $s^2=\dfrac{1}{n-1}\sum(x-\bar{x})^2$ $=\dfrac{50}{39}$ ≈ 1.28205 ≈ 1.28 (3 s.f.)
(iii)	Let X be the fat content in a chocolate bar in g. Let μ and σ be the population mean and variance of X. $H_0: \mu = 30$ $H_1: \mu > 30$  Under H_0, since n is large, by Central Limit Theorem, $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ approximately. Test statistic, $Z=\dfrac{\bar{X}-\mu}{S/\sqrt{n}} \sim N(0,1)$ approximately. For test to be rejected at 5% level of significance, $z=\dfrac{\bar{x}-\mu}{s/\sqrt{n}} \geq 1.64485$ $\dfrac{\bar{x}-30}{\sqrt{\dfrac{50}{(39)(40)}}} \geq 1.64485$ $\bar{x} \geq 30.294475$ $\bar{x} \geq 30.3$ (3 s.f.) Since $\bar{x}=\dfrac{1220}{40}=30.5 \geq 30.2945$, we reject H_0 at the 5% level of significance and conclude that the manager's suspicion is valid.
(iv)	From part **(iii)**, it was concluded that the manager's suspicion is valid, i.e. reject H_0 $\Rightarrow$ Test statistic is in the critical region . $\Rightarrow$ Test statistic ≥ 1.64486

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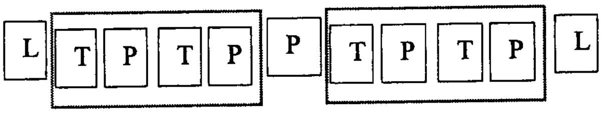
Qn	Suggested Solutions
	Now with a smaller population variance, the new test statistic will be larger. $\Rightarrow$ New Test statistic value >Old Test statistic value i.e $\dfrac{\bar{x}-\mu}{\sqrt{\dfrac{\sigma^2}{n}}} > \dfrac{\bar{x}-\mu}{\sqrt{\dfrac{s^2}{n}}}$ H_0 is still rejected at 5% level of significance. Thus the conclusion will be the same, i.e. conclude that the manager's suspicion is valid.
8(a)	<u>**Method 1**</u> Total number of committees formed $=\binom{5}{2}\times\binom{10}{4}\times\binom{8}{4}$ $=147000$ Number of committees with the couple serve together $=\binom{5}{2}\times\binom{8}{2}\times\binom{8}{4}$ $=19600$ Required number of committees formed $=147000-19600$ $=127400$
	<u>**Method 2**</u> <u>Case 1</u>: Wife is in and husband is out No. of committees $=\binom{5}{2}\times\binom{8}{3}\times\binom{8}{4}=39200$ <u>Case 2</u>: Wife us out and husband is in No. of committees $=\binom{5}{2}\times\binom{8}{3}\times\binom{8}{4}=39200$ <u>Case 3</u>: The couple is out No. of committees $=\binom{5}{2}\times\binom{8}{4}\times\binom{8}{4}=49000$ Required number of committees formed $=39200+39200+49000=127400$
(b)	 Number of arrangements if no two parents are to stand next to each other $=2!\times4!\times4!\times3\times3$ =10368
(c)	No. of circular arrangements if all parents are together and teachers are separated

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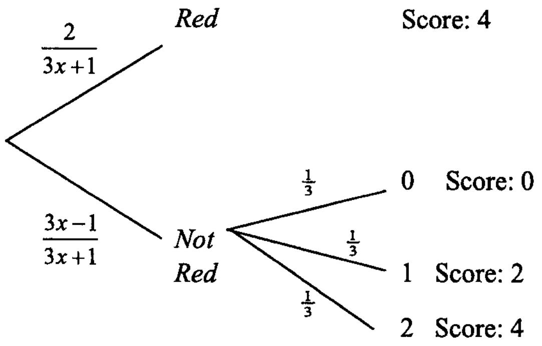
Qn	Suggested Solutions
	$=3!\times 4!\times 4!$ $=3456$ Required probability $=\dfrac{3!4!4!}{10!}=\dfrac{1}{1050}$ $=9.52\times 10^{-4}$ (3 s.f.)
9(i)	Let S be the score of Abel's game. $\mathrm{P}(S=4)$ $=\mathrm{P}(\{\text{Red}\},\{\text{Non-red, black 2}\})$ $=\dfrac{2}{3x+1}+\dfrac{3x-1}{3x+1}.\dfrac{1}{3}$ $=\dfrac{6+3x-1}{3(3x+1)}$ $=\dfrac{3x+5}{3(3x+1)}$ 
(ii)	Let S be Abel's score in a round. $\mathrm{P}(S=0)$ $=\mathrm{P}(\{\text{Non-red, Card zero}\})$ $=\dfrac{3x-1}{3x+1}.\dfrac{1}{3}$ $=\dfrac{3x-1}{3(3x+1)}$ $\mathrm{P}(S=2)$ $=\mathrm{P}(\{\text{Non-red, Card 1}\})$

HCI C2 H2 Mathematics Preliminary Examinations 2022 Mark Scheme (Paper 2)

Qn	Suggested Solutions
	s: 0, 2, 4; $\mathrm{P}(S=s)$: $\frac{3x-1}{3(3x+1)}$, $\frac{3x-1}{3(3x+1)}$, $\frac{3x+5}{3(3x+1)}$
(iii)	Mode = 4
(iv)	$\mathrm{E}(S)$ $=0\cdot\mathrm{P}(S=0)+2\cdot\mathrm{P}(S=2)+4\mathrm{P}(S=4)$ $=2\left(\frac{3x-1}{3(3x+1)}\right)+4\left(\frac{3x+5}{3(3x+1)}\right)$ $=\frac{6(x+1)}{(3x+1)}$ $\mathrm{E}(S^2)$ $=0^2\cdot\mathrm{P}(S=0)+2^2\cdot\mathrm{P}(S=2)+4^2\cdot\mathrm{P}(S=4)$ $=2^2\left(\frac{3x-1}{3(3x+1)}\right)+4^2\left(\frac{3x+5}{3(3x+1)}\right)$ $=\frac{12x-4+48x+80}{3(3x+1)}$ $=\frac{60x+76}{3(3x+1)}$ $\therefore\ \mathrm{Var}(S)=\mathrm{E}(S^2)-[\mathrm{E}(S)]^2$ $=\frac{60x+76}{3(3x+1)}-\frac{36(x+1)^2}{(3x+1)^2}$
(v)	$x=3$ $\mathrm{E}(S)=\frac{18(3)+18}{3(10)}=2.4$ $\mathrm{Var}(S)=\mathrm{E}(S^2)-[\mathrm{E}(S)]^2$ $=\frac{60(3)+76}{3(3(3)+1)}-(2.4)^2$ $=\frac{256}{3(10)}-(2.4)^2=\frac{208}{75}$ $n=100$ is large, by Cental Limit Theorem,

s	0	2	4
$\mathrm{P}(S=s)$	$\frac{3x-1}{3(3x+1)}$	$\frac{3x-1}{3(3x+1)}$	$\frac{3x+5}{3(3x+1)}$

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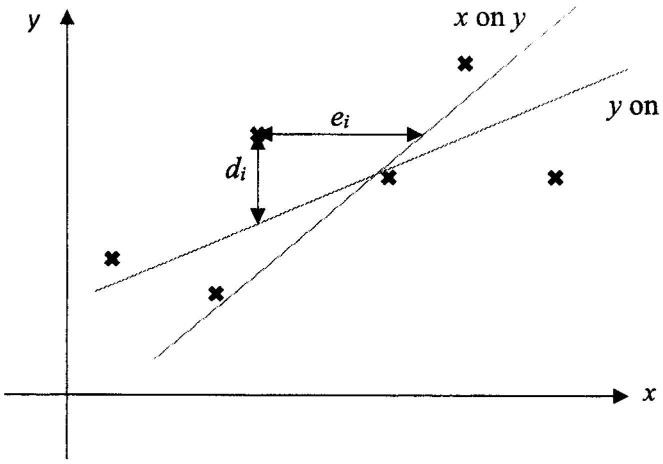
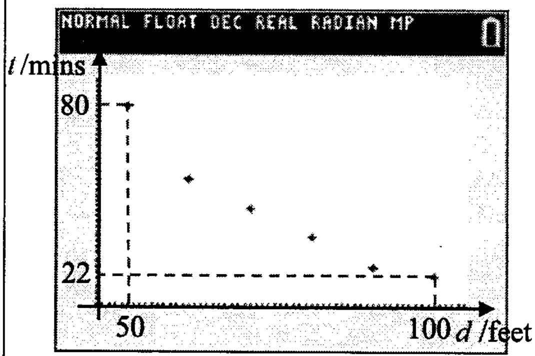
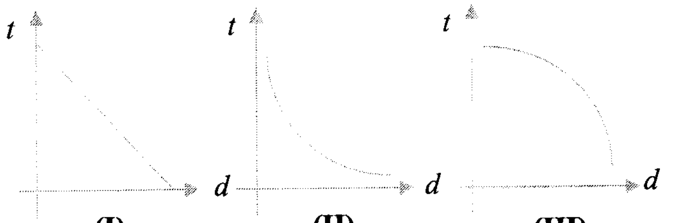
Qn	Suggested Solutions
	$\overline{S} \sim \mathrm{N}\left(2.4, \dfrac{208}{75(100)}\right)$ approximately. $\mathrm{P}\left(\overline{S} \geq 2.5\right) = 0.2740929 = 0.274$ (3 s.f.)
10(a)	 The regression line of y on x is the line which minimizes the sum of squares of the residuals in the vertical direction, d_i, i.e. $\sum d_i^2$, while the regression line of x on y is the line which minimizes the sum of squares of the residuals in the horizontal direction, e_i, i.e. $\sum e_i^2$
(b) **(i)**	

HCI C2 H2 Mathematics Preliminary Examinations 2022 Mark Scheme (Paper 2)

Qn	Suggested Solutions
(ii)	The scatter diagram shows a decreasing, concave upward trend. Hence model (II) is a better fit for the data.  (I) $t = ad + b$ (II) $t = a\left(\frac{1}{d}\right) + b$ (III) $t = ae^{d} + b$ From GC, $t = 5767.446342\left(\frac{1}{d}\right) - 37.61923382$ $\therefore t = 5767.446\left(\frac{1}{d}\right) - 37.619$ (3 d.p.) $r = 0.995$ (3 d.p.)
	(I) Decreasing, linear relationship **(II)** Non-linear, Decreasing and concave upwards trend **(III)** Non-linear, Decreasing and concavce downwards trend Since scatter diagram in **(b)(i)** shows a non-linear relationship between the 2 variables and the characteristics of the scatter plot shows that a decreasing and concave upwards curve like in Model (II) will best model the relationship for the 2 variables.
(iii)	When $d = 150$, $t = 5767.446342\left(\frac{1}{150}\right) - 37.61923382$ $t = 0.830$ (3 s.f.) The estimate is not reliable because $d = 150$ is outside the data range $[50, 100]$.
(iv)	$1 \text{ m} = 3.28 \text{ ft} \Rightarrow D \text{ m} = 3.28D \text{ ft} = d$ $\therefore d = 3.28D$ $t = 5767.446342\left(\frac{1}{3.28D}\right) - 37.61923382$ $t = \frac{1760}{D} - 37.6$ (3 s.f.)
(v)	From GC, $\left(\overline{\frac{1}{d}}, \bar{t}\right) = (0.0141, 43.7)$ (3 s.f.)

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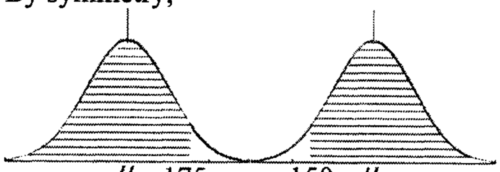
Qn	Suggested Solutions
	$\overline{\frac{1}{d}} = 0.0140939153$ $\bar{d} = \frac{1}{0.0140939153}$ $\bar{d} = 70.9526 = 71.0$ (3 s.f.)
11	$X \sim N(\mu_1, 11.83^2)$, $Y \sim N(\mu_2, 11.83^2)$ Given that $P(X < 175) = P(Y > 150)$, $P\left(Z < \frac{175-\mu_1}{11.83}\right) = P\left(Z > \frac{150-\mu_2}{11.83}\right)$ $\frac{175-\mu_1}{11.83} = -\frac{150-\mu_2}{11.83}$ $175 - \mu_1 = -150 + \mu_2$ $\therefore \mu_1 + \mu_2 = 175 + 150$ $= 325$
	By symmetry,  $175 - \mu_1 = \mu_2 - 150$ $\therefore \mu_1 + \mu_2 = 175 + 150$ $= 325$
(i)	$X_1 + X_2 - 2(Y+M) \sim N(2(180) - 2(180),\ 2(11.83^2) + 2^2(174.9953))$ i.e. $X_1 + X_2 - 2(Y+M) \sim N(0,\ 979.879)$ $P(0 < X_1 + X_2 - 2Y < 15) = 0.184$ (3 s.f.) Assume that the volumes of each cup of black coffee and milk added from the vending machine are (i.e. X_1 , X_2 , Y and M) independent of one another.
(ii)	$X \sim N(180, 11.83^2)$, $Y \sim N(145, 11.83^2)$, $M \sim N(35, 5.92^2)$ B = Cost Price of 1 cup of Black Coffee = $0.01X$ W = Cost Price of 1 cup of White Coffee = $0.01Y + 0.02M$ $B \sim N(1.8, 0.1399489)$, $W = 0.01Y + 0.02M \sim N(2.15, 0.02801345)$ Since n cups of black coffee are sold per day, $(100-n)$ cups of white coffee are sold per day.

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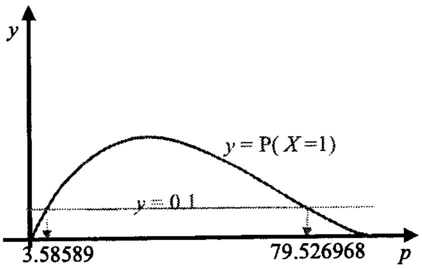
Qn	Suggested Solutions
	Let P_B be profit for black coffee, and P_W be profit for white coffee and T be the total profit per day. $P_B = 4n - (B_1 + B_2 + ... + B_n)$ $E(P_B) = 2.2n$ $Var(P_B) = 0.01^2(11.83^2)n$ $= 0.01399489n$ $E(P_W) = 5(100-n) - 2.15(100-n) = 285 - 2.85n$ $Var(P_W) = (100-n)0.02801345$ $T \sim N(285 - 0.65n, 2.801345 - 0.01401856n)$ $P(T > 230) \geq 0.8$ n : $P(T>230)$ 81 : 0.9657 82 : 0.907 83 : 0.794 Therefore, largest number of cups of black coffee sold per day is 82.
(iii)	Let F be the number of customers selecting regular black coffee receives the drink free of charge. $F \sim B\left(3, \frac{p}{100}\right)$ $P(F=1) = 3\left(\frac{p}{100}\right)\left(1-\frac{p}{100}\right)^2$
(iv)	$3\left(\frac{p}{100}\right)\left(1-\frac{p}{100}\right)^2 \leq 0.1$  $0 \leq p \leq 3.58$ or $79.6 \leq p \leq 100$ (3 s.f.)

BP~257

NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2
Higher 2

NAME

CLASS 2ma2

REGISTRATION NUMBER

MATHEMATICS

9758

Preliminary Examination

30 August 2022

Paper 1

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, class and registration number in the boxes above.
Please write clearly and use capital letters.

Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing and/or scientific calculator is expected, where appropriate.
All relevant working, statements and reasons must be shown in order to obtain full credit for your solution.

You are reminded of the need for clear presentation in your answers.
Up to 2 marks may be deducted for improper presentation.

The number of marks is given in the brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Question Number	Marks Possible	Marks Obtained
1	5	
2	6	
3	6	
4	7	
5	7	
6	8	
7	8	
8	9	
9	9	
10	10	
11	12	
12	13	
Presentation Deduction		−1 / −2
TOTAL	100	

This document consists of **7** printed pages.

1 **(i)** Given that $I_n = \int_1^{e} x(\ln x)^n \, dx$ for $n \in \mathbb{Z}$, $n \geq 0$, show that

$$I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$$

for all $n \in \mathbb{Z}^+$. [2]

(ii) Find the exact volume of the solid generated when the region bounded by the curve $y = \sqrt{x} \ln x$, the x-axis and the line $x = e$ is rotated completely about the x-axis. [3]

2 In this question, p is a constant such that $p > 1$.

(i) Using an algebraic method, solve $\dfrac{px^2 - 1}{x^2 + (1-p)x - p} \geq 1$. Express your answer in terms of p. [4]

(ii) Hence, or otherwise, solve $\dfrac{px^2 - 1}{x^2 + (p-1)|x| - p} \geq 1$. [2]

3 Relative to the origin O, points A, B and C have position vectors **a**, **b** and 3**a** – 4**b** respectively, where **a** and **b** are non-zero and non-parallel.

(i) Given that OAB is an equilateral triangle, show that $\mathbf{a} \cdot \mathbf{b} = k|\mathbf{a}|^2$, where k is a constant to be determined. [2]

(ii) Deduce whether OA and AC are perpendicular. [2]

(iii) Interpret geometrically the vector equation $\mathbf{r} \times (\mathbf{b} - \mathbf{a}) = \mathbf{a} \times (\mathbf{b} - \mathbf{a})$. [2]

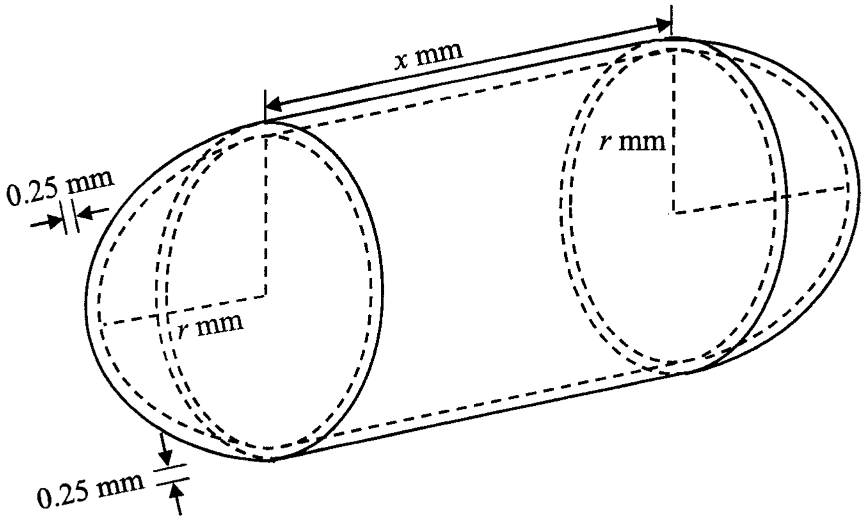
4 **(i)** Two complex numbers w and u satisfy the equations

$$w^* - 2iu = 8 \quad \text{and} \quad (2i - 1)w + 2u^* = 4.$$

Find w and u, giving your answers in the form $x + iy$, where x and y are real. [4]

(ii) Given that both w and u are roots of the equation $(z^2 - 4z + c)(z^2 - dz + 10) = 0$, find the values of the real numbers c and d. [3]

5



The diagram above shows a capsule consisting of two identical hollow hemispherical caps at the two ends and a hollow cylindrical body in the middle. It has a capacity of 600 mm^3 to contain medicinal substance, which is surrounded by a hard gelatin shell that has a thickness of 0.25 mm. The internal radius and internal length of the cylindrical body are r mm and x mm respectively. The internal radius of each hemispherical cap is r mm. The volume of gelatin used in making the hard shell is V mm^3.

(i) Show that $V=\left(r+\frac{1}{4}\right)^2\left(\frac{\pi}{3}+\frac{600}{r^2}\right)-600.$ [3]

(ii) Using differentiation, find the value of r that would result in the least amount of gelatin used to make the hard shell. [4]

6 **(i)** By considering u_k-u_{k+1}, where $u_k=\frac{1}{k!}$, find

$$\frac{3}{4!}+\frac{4}{5!}+\frac{5}{6!}+\cdots+\frac{3n+2}{(3n+3)!}$$

in terms of n. [4]

(ii) Find $\sum_{r=5}^{3n+3}\frac{r-1}{r!}$. Hence show that $\sum_{r=5}^{3n+3}\frac{3}{r!}<\frac{1}{24}$. [4]

7 Amy cuts off pieces of ribbon from a long roll of ribbon. The first piece of ribbon she cuts off is 160 cm long and each successive piece is 8 cm shorter than the preceding piece.

(a) What is the maximum number of pieces of ribbon that Amy can cut off? [2]

After Amy has cut off the number of pieces found in part **(a)**, her friend, Bala, continues to cut off more pieces from the same remaining long roll of ribbon, such that the first piece is 160 cm long and each successive piece is p times as long as the preceding piece, where p is a constant such that $0 < p < 1$.

(b) Given that the total length of ribbon that Bala cuts off can never be greater than 10 m regardless of the number of pieces he cuts off, find the largest value of p. Taking p to be this value and by solving an appropriate inequality, find the maximum number of pieces that can be cut off before the total length of ribbon that Bala cuts off exceeds 9.5 m. [4]

(c) After Amy has cut off the number of pieces found in part **(a)** and Bala has cut off the number of pieces found in part **(b)**, there is 4.5 cm of ribbon left in the roll of ribbon. Determine the length of the original long roll of ribbon, leaving your answer to the nearest cm. [2]

8 Two complex numbers are $z = 2\left(\cos\frac{\pi}{4} - \mathrm{i}\sin\frac{\pi}{4}\right)$ and $w = \left(-\mathrm{i}\sqrt{3}\right)z$.

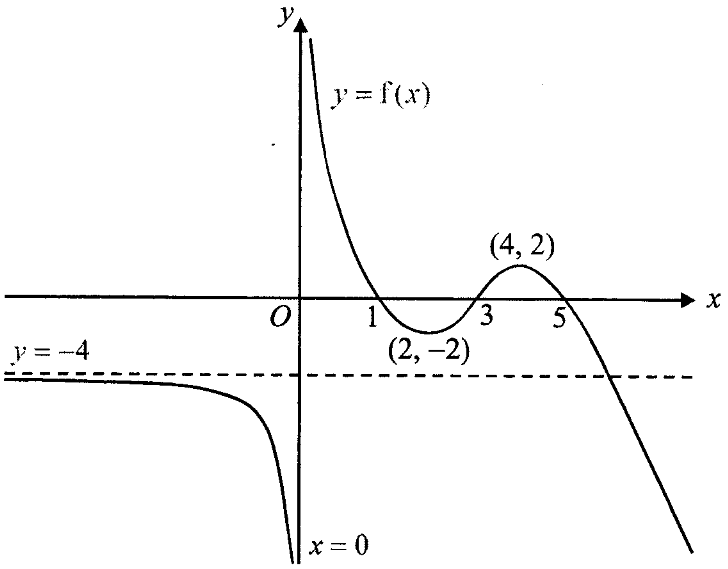
(i) Show that $z + w^* = re^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$ for some positive constant r to be determined exactly. [3]

(ii) Hence find the values of n such that $(z + w^*)^n$ is purely imaginary. [2]

It is given that $v = \frac{z + w^*}{z^* w}$.

(iii) By finding $\arg(v)$ or otherwise, find an equation relating $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$. Also, find $|v|$ exactly. [4]

9 The graph of $y = \mathrm{f}(x)$ has two stationary points at $(2, -2)$ and $(4, 2)$ and intersects the x-axis at $x = 1$, $x = 3$ and $x = 5$ as shown in the diagram below. There is a vertical asymptote $x = 0$ and a horizontal asymptote $y = -4$.



On separate clearly labeled diagrams, sketch the following graphs. Label clearly, where possible, the asymptotes, stationary points and points of intersection of the curves with the axes in your diagrams.

(a) $y = \dfrac{1}{\mathrm{f}(x)}$ [4]

(b) $y = -\mathrm{f}'(x)$ [3]

The curve Q has parametric equations $x = t + 4$ and $y = pt^2 + q$, where p and q are real positive constants. State a cartesian equation of Q and the range of possible values of q such that the curve Q meets the curve $y = \mathrm{f}(x)$ exactly once. [2]

10 It is given that $f(x)=(a+x)^n$, where a and n are non-zero real constants such that $a>0$, and x is non-zero. The first, third and fifth terms in the Maclaurin series of $f(x)$ are the first, second and third terms of an infinite geometric series G respectively.

(i) Find the possible values of n, showing all working clearly. [4]

Assume instead for the remainder of this question that $n=-1$.

(ii) Show that the range of values of x for the Maclaurin series of $f(x)$ to converge is equal to the range of values of x for G to converge. [2]

In the case where G converges, the sum to infinity of G is denoted by S.

(iii) Find S in terms of a and x. [2]

(iv) Find, in terms of a, the range of values of S as x varies. Show your working clearly. [2]

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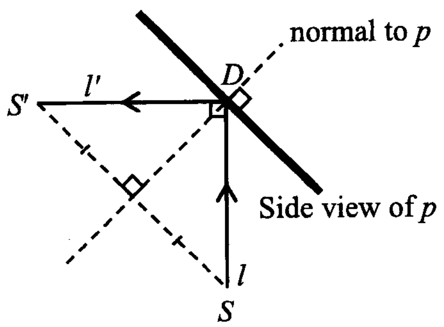


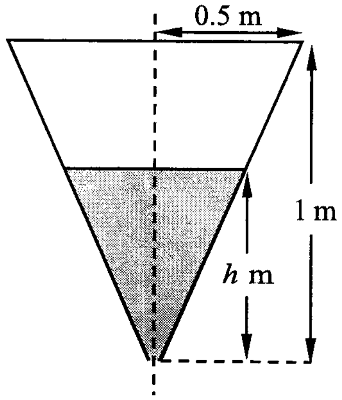
The diagram above shows a glass prism. The surface of the prism is part of the plane p with equation $by-z=4$, where b is an integer constant. A ray of light l passes through a point S with coordinates $(5,1,3)$, and travels in a direction parallel to the unit vector $\frac{2}{3}\mathbf{i}+c\mathbf{j}+\frac{2}{3}\mathbf{k}$, where c is a negative constant, until it hits the surface of the prism at point D. The light ray is reflected by the surface of the prism. The reflected ray of light, l', passes through the point S', where S' is the image of point S. It is given that the lines l and l' are perpendicular and they lie on the same plane as the normal to p passing through point D. Also, the acute angle between l and the normal to p is equal to the acute angle between l' and the normal to p.

(i) Find the exact value of c and show that $b=1$. [4]

(ii) Find the coordinates of point D. [3]

(iii) Show that the coordinates of S' are $(m,-1,1)$ for some integer constant m to be determined and find a vector equation of l' in exact form. [5]

12 Water is flowing out from a conical funnel from its tip at the bottom. The top radius of the funnel is 0.5 m and the depth of the funnel is 1 m. The funnel is initially filled to the brim and no water is added to the funnel thereafter. The side view of the funnel and the water inside it are shown in the diagram below.



It can be assumed that the vertical length of the tip cut off at the bottom (to produce the funnel) is insignificant compared to the depth of the conical funnel. It is also known that the velocity v ms^{-1} of the water flowing out from the funnel at its tip can be modelled by *Torricelli's law*,

$$v^2 = 2gh,$$

where g ms^{-2} is the constant acceleration due to gravity and h m is the depth of the water in the cone at time t s.

(i) Express the volume W m^3 of the water in the funnel at time t s in terms of h. [2]

It is given that the radius of the tip cut off at the bottom of the funnel is a m. Also, it is known that the volume of water flowing out from the funnel per second is equal to the product of the cross-sectional area of the hole at the bottom of the funnel and the velocity v ms^{-1} of the water flowing out from the funnel.

(ii) Use the above information and Torricelli's law to express $\frac{\mathrm{d}W}{\mathrm{d}t}$ in terms of a, g and h. Hence, form a differential equation relating h and t. [4]

(iii) Show that $h = \left[1-\left(ka^2\sqrt{2g}\right)t\right]^p$ for some rational constants k and p to be determined. [3]

(iv) Find the time T s taken for the funnel to become empty. Express your answer in terms of a and g. [2]

(v) Sketch the graph of h against $\frac{t}{T}$. [2]