RAFFLES INSTITUTION
2022 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 22

MATHEMATICS **9758/01**
Paper 1 **3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper. **You may use the blank page on page 2 if necessary and you are reminded to indicate the question number(s) clearly.**
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

For Examiner's Use Only						
Q1	Q2	Q3	Q4	Q5	Q6	Q7
/ 6	/ 6	/ 7	/ 7	/ 8	/ 9	/ 9
Q8	Q9	Q10	Q11		TOTAL	
/ 10	/ 11	/ 13	/ 14		/ 100	

This document consists of **23** printed pages and **1** blank page.

[Turn over

BLANK PAGE

You may continue your working on this page if necessary, indicating the question number(s) clearly.

1 **(i)** On the same axes, sketch the graphs of $y=\dfrac{x}{x-a}$ and $y=2|x-a|+1$, where a is a positive constant. [3]

(ii) Hence solve the inequality $\dfrac{x}{x-a}>2|x-a|+1$, leaving your answer in terms of a. [3]

2 A function is defined as $f(x)=\dfrac{1+2x-x^2}{2(x^2-2x)}$, $x\in\mathbb{R}, x\neq 0,2$.

(i) Show that $f(x)$ can be written in the form $q\left(\dfrac{2-(x+p)^2}{(x+p)^2-1}\right)$, where p and q are constants to be found. [2]

(ii) Hence describe a sequence of transformations that will transform the graph of $y=\dfrac{2-x^2}{x^2-1}$ onto the graph of $y=f(x)$. [2]

(iii) Determine, with the help of a sketch or otherwise, the set of values of k for which the equation $\dfrac{2-x^2}{x^2-1}=k$ has no real roots. [2]

3 A curve has parametric equations

$$x = a\left(1+\frac{1}{t}\right), \quad y = a\left(t-\frac{1}{t^2}\right),$$

where a is a constant and $t \neq 0$.

(i) Find the equations of the tangent and the normal to the curve at the point P where $t = -\frac{1}{2}$. [5]

(ii) The tangent at P meets the y-axis at Q and the normal at P meets the y-axis at R. Show that the area of triangle PQR is $\frac{241}{120}a^2$. [2]

4 **(a)** The point R has position vector $\mathbf{r}$. Given that $\mathbf{r} = \begin{pmatrix} a \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$, where a is a real number, describe geometrically the set of all possible positions of the point R, as a varies. [2]

(b) **(i)** The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively. Show that the point F, the foot of perpendicular from the origin O to the line passing through P and Q, has position vector $(1-\lambda)\mathbf{p} + \lambda\mathbf{q}$, where

$$\lambda = \frac{|\mathbf{p}|^2 - \mathbf{p}.\mathbf{q}}{|\mathbf{q}-\mathbf{p}|^2}.$$ [4]

(ii) Write down an inequality satisfied by λ for F to lie within the line segment PQ. [1]

5 **Do not use a calculator in answering this question.**

The complex numbers z and w are given by

$$z = \sin\left(\frac{\pi}{6}\right) + \mathrm{i}\cos\left(\frac{\pi}{6}\right) \text{ and } w = \sqrt{2}\left[\sin\left(\frac{\pi}{6}\right) + \mathrm{i}\cos\left(\frac{\pi}{3}\right)\right].$$

(i) Find $|z|$ and $\arg(z)$. Hence find the value of z^3. [3]

(ii) By considering some suitable form of w or otherwise, find w^4. [3]

(iii) Hence find the value of $z^{2022} - w^{2020}$. [2]

6 **(a)** A sequence is such that $u_1 = p$, where p is a constant, and $u_{n+1} = \frac{4}{u_n}$, for $n > 0$.

Describe how the sequence behaves when

(i) $p = 2$, [1]

(ii) $p = 3$. [1]

(b) Another sequence $v_1, v_2, v_3, \ldots$ is such that $v_n = v_{n-1} + n$, where $n \geq 2$, and $v_1 = A$.

Find, in terms of A and n, an expression for

(i) v_n, [4]

(ii) $\sum_{r=1}^{n} v_r$. [3]

(You need not simplify your answer. You may use the result $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.)

7 It is given that $y=\sqrt{2+\cos^2 x}$.

(a) Show that

(i) $2y\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin 2x$, [1]

(ii) $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-\cos 2x$. [1]

(b) Hence find the Maclaurin series of y, up to and including the term in x^4. [5]

(c) By substituting $x = \frac{\pi}{6}$, show that $\sqrt{11} \approx 2\sqrt{3}\left(1 - \frac{\pi^2}{216} + \frac{\pi^4}{31104}\right)$. [2]

8 A curve C is defined by $y = \dfrac{(\ln x)^4}{\sqrt{x}}$ where $0 < x < 10$.

(i) Find the exact volume generated when the area bounded by C, the x-axis and the lines $x = 1$ and $x = \mathrm{e}$ is rotated about the x-axis through 360°. [4]

(ii) Find the area enclosed by C and the lines $y = 1$ and $y = \mathrm{e}$. [6]

9 The functions f, g and h are defined as follows:

$$\mathrm{f}: x \mapsto x^3 - 1, \quad x \in \mathbb{R},$$
$$\mathrm{g}: x \mapsto \mathrm{e}^{-3x}, \quad x \in \mathbb{R}, x < 0,$$
$$\mathrm{h}: x \mapsto \frac{x+1}{x-1}, \quad x \in \mathbb{R}, x \neq 1.$$

(i) Define in a similar form, the inverse functions g^{-1} and h^{-1}. [4]

(ii) Write down the rule of the composite function ff. [1]

(iii) It is known that the equation $\mathrm{ff}(x) = 0$ has only one real root α.

Find the exact value of α and hence show that $\mathrm{g}^{-1}(\alpha) = k \ln 2$, where k is a constant to be found. [2]

(iv) Show that the composite function hg exists and write down the range of this function. [2]

(v) Denoting composite functions hh as h^2, hhh as h^3 and so on, find the value of x for which $\mathrm{h}^m(x) = \mathrm{h}^{-1}(-1)$, where m is a positive even integer. [2]

10 Glucose in the blood stream is reduced at a rate proportional to the amount of glucose present in the blood stream.

Let G milligrams (mg) be the amount of glucose in 1 decilitre (dL) of blood stream at time t minutes and let λ denote the positive constant of proportionality.

(i) Write down a differential equation relating G, λ and t. Solve this differential equation to find an expression of G in terms of λ and t. [3]

Through extensive research, doctors recommended that a healthy glucose level in the blood stream of an adult should be between 70 mg/dL to 100 mg/dL.

An adult patient, Neo, has 80 mg/dL of glucose in his blood stream at time $t = 0$ minutes. Due to a medical condition, he is not able to extract glucose from the food he eats. As he is not able to replenish the glucose in his blood stream, he is thus losing his glucose in the blood stream at a rate (in mg/dL per minute) corresponding to $\lambda = 0.005$ per minute.

(ii) Find the approximate time it would take for Neo's glucose level in the blood stream to fall below the healthy range if there is no intake of glucose. [2]

In order to maintain Neo's glucose level in the blood stream in the healthy range, glucose is injected into his blood stream intravenously at a constant rate of μ mg/dL per minute.

(iii) Write down a differential equation relating G, μ and t to model Neo's situation. Solve this differential equation to find an expression of G in terms of μ and t. [5]

(iv) Explain whether it is recommended to keep injecting glucose at a rate of 0.7 mg/dL per minute into Neo's blood stream. [2]

(v) Find the maximum range of values of μ so that Neo's glucose level in the blood stream will always be in the healthy range. [1]

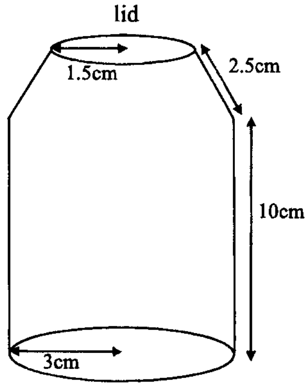
11 [It is given that 1 ml = 1 cm^3 and that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]

Globally, 2 trillion drink cans are produced every year. Drink cans constitute part of the packaging cost for beverage companies and using the appropriate material in appropriate amounts will enable them to save costs. A new beverage company wants to produce drink cans using a particular material for the top and bottom of a cylindrical shaped drink can, and another material for the curved body. For a fixed thickness, the material for the top and bottom costs \$1.20/m^2 and the material for the curved side costs \$0.90/m^2.

(i) For a fixed volume of 100π ml per can, show, using differentiation, that the radius r of the most economical can is approximately 3.35 cm and evaluate the corresponding height h. [7]

(ii) Hence, find the cost of the most economical can, giving your answer correct to the nearest 0.1 cent. [1]

In order to make the can more attractive, the company redesigns the can such that the base radius of the can is 3 cm and the height of the cylindrical part is 10 cm. At the top of each can, there is a tapering in before being covered by a circular lid of radius 1.5 cm, with the dimensions of the can as shown below.



(iii) The beverage is pumped into the can at a rate of 90π ml/s. Find the rate at which the liquid level in the can is increasing when it is 1 cm from the lid of the can. [6]

11 [Continued]

RAFFLES INSTITUTION
2022 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 22

MATHEMATICS **9758/02**

Paper 2 **3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper. **You may use the blank pages on page 2, 23 and 24 if necessary and you are reminded to indicate the question number(s) clearly.**
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

For Examiner's Use Only

Section A: Pure Mathematics	Q1	Q2	Section B: Probability & Statistics	Q5	Q6	Q7	Q8
	/7	/8		/7	/8	/10	/10
	Q3	Q4		Q9	Q10	TOTAL	
	/12	/13		/12	/13	/100	

This document consists of **21** printed pages and **3** blank pages.

RAFFLES INSTITUTION
Mathematics Department

[Turn over

BLANK PAGE

You may continue your working on this page if necessary, indicating the question number(s) clearly.

Section A: Pure Mathematics [40 marks]

1 (i) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cos^2 x}\right)=\frac{k\sin x}{\cos^n x}$, where the values of the constants k and n are to be determined. [2]

(ii) Hence use integration by parts to evaluate $\int_0^{\frac{\pi}{4}} \sin^2 x \sec^3 x \, \mathrm{d}x$, leaving your answer in the form $a + b\ln c$, where a, b and c are exact constants. [5]

2 **Do not use a calculator in answering this question.**

(a) Let $\mathrm{f}(z)$ be a polynomial in z of degree 4 with real coefficients. The equation $\mathrm{f}(z)=0$ has four roots, namely α, β, γ and δ such that they satisfy the following two conditions:

$$\alpha\beta\gamma\delta<0 \text{ and } \alpha^2+\beta^2+\gamma^2+\delta^2<0.$$

Based on the two conditions, a student concludes that the equation $\mathrm{f}(z)=0$ has one positive real root, one negative real root and a pair of complex conjugate roots.

State, with reasons, whether the student's claim is true. [3]

(b) It is given that $g(z) = z^4 + z^3 - 2z^2 + 4z - 24$.

Verify that $z = 2i$ is a root of the equation $g(z) = 0$. Hence find the other roots of the equation. [5]

3 The line L_1 has equation

$$1-y=\frac{z-1}{2},\ x=2,$$

and meets the xy-plane at point P. The point A has position vector $\begin{pmatrix}3\\-1\\2\end{pmatrix}$ with reference to the origin O.

(i) Find a vector equation of the line L_2 which passes through O and P. [3]

(ii) Find an equation of the plane π containing both L_1 and L_2, in the scalar product form. [2]

(iii) The points A and C are on different sides of π such that AC is perpendicular to π. The distance of C from π is t times the distance of A from π. Find, in terms of t, the position vector of C. [5]

(iv) Find the value of t such that the line OC is parallel to the plane with equation $\mathbf{r}.\begin{pmatrix}2\\0\\1\end{pmatrix}=2$. [2]

4 **(a)** The first 3 terms of a geometric progression are a, b, 2 and the first 3 terms of an arithmetic progression are 2, a, b, with non-zero common difference. Find the values of a and b. [4]

(b) $u_1, u_2, u_3, u_4, \ldots, u_{2n}$ is a sequence of $2n$ positive terms, with $n > 1$. The odd-numbered terms form an arithmetic progression with common difference p and the even-numbered terms form a geometric progression with common ratio $\frac{6}{5}$. Given that $u_1 = u_2 = p$ and the sum of the odd-numbered terms is less than the sum of the even-numbered terms, find the least value of n. [4]

4 **[Continued]**

(c) A geometric progression of n terms has first term q and common ratio r, where q is non-zero and $r \neq 1$. For $k \leq n$, find the difference between the sum of the last k terms and the sum of the first k terms, simplifying your answer. [5]

Section B: Probability and Statistics [60 marks]

5 A club in a school has 5 members from Class P, 4 members from Class Q and 3 members from Class R. Five members are to be chosen for an upcoming competition.

(i) Find the number of ways the team of five can be chosen so that it has exactly two members from each of Class P and Class Q. [1]

(ii) Find the number of ways the team of five can be chosen so that it has at least two members from Class R. [2]

(iii) All the 12 members of the club go to a cinema. Find the number of ways they can sit in a row so that no more than two members from Class P are next to each other. [4]

6 A car insurance company collected the following data about the percentage occurrence of accident-involved vehicles, p% for vehicles of different weight, w tons.

w (tons)	2.2	1.9	1.7	1.5	1.3	1.1	1.0	0.9
p (%)	2.6	3.2	3.8	4.3	5.4	5.3	7.4	8.6

(i) Calculate the value of the product moment correlation coefficient between w and p, and explain whether your answer suggests that a linear model is appropriate. [2]

(ii) Draw a scatter diagram of the data. [1]

One of the values of p appears to be incorrect.

(iii) Indicate the corresponding point on your diagram by labelling it R, and explain why the scatter diagram for the remaining points may be consistent with a model of the form $\ln p = a + bw$. [2]

(iv) Omitting R, calculate least squares estimates of a and b for the model $\ln p = a + bw$. [2]

(v) Assume that the value of w at R is correct. Estimate the value of p for this value of w. [1]

7 On average, 11% of the students in school A have been infected before by a contagious disease. Every class has 20 students. The number of students in a class who has been infected by the disease before is denoted by X.

(i) State, in the context of this question, two assumptions needed for X to be well modelled by a binomial distribution. Explain why your assumptions may not be met. [4]

Assume now that the context above is well-modelled by a binomial distribution.

(ii) A class is chosen at random. Find the probability that at least 1 but fewer than 10 students in the class has been infected by the disease before. [2]

Fatihah is a student reporter tasked to ask students in a randomly chosen class, one by one, if they have been infected by the disease before.

(iii) Find the probability that the 20^{th} student she asks is the fifth student who has been infected by the disease before. [2]

(iv) Without doing any further calculation, is the probability found in **(iii)** higher or lower than the probability that there are exactly 5 students in the class who have been infected by the disease before? Explain your answer. [2]

8 A game is played by throwing a fair coin and two fair four-sided dice. The dice are coloured red or blue, and have faces numbered from 1 to 4. If the coin shows a head, then the score is the number shown on the red die. Otherwise, the score is the sum of the numbers shown on the two dice.

(i) Show that the probability that a game results in a score of 4 is $\frac{7}{32}$. [2]

(ii) Find the expectation and variance of the score. [5]

(iii) The game is played 35 times. Estimate the probability that the average score of the 35 games is at least 4, given that the first and second games result in a score of 3 and 4 respectively. [3]

9 A large company claims that its employees work an average of 41 hours a week. After receiving feedback from some employees that they work longer than claimed, a survey involving 200 randomly chosen employees is conducted. The amount of time they spend at work per week, x thousand hours, are summarised below.

$$\sum x = 8.71 \qquad \sum x^2 = 0.505$$

(i) Calculate unbiased estimates of the population mean and variance of the amount of time spent by an employee at work per week. [2]

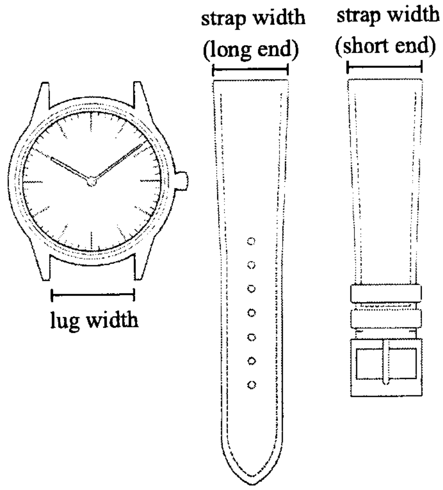
(ii) Test, at the 5% significance level, whether the average working hours of an employee per week is more than 41. State hypotheses for the test, defining any symbols you use. [5]

(iii) After the company restructures its operations, it claims that the average working hours a week is now 40. The human resource manager takes a random sample of 12 employees and finds that they spend an average of 40.1 hours per week at work. Suppose now that the population variance is k hours^2.

(a) Stating a necessary assumption, find the range of possible values of k if the manager concludes that there is insufficient evidence to reject the company's claim that the average working hours is 40, at the 5% significance level. [4]

(b) Explain why the Central Limit Theorem does not apply in this context. [1]

10 **In this question you should state the parameters of any normal distributions you use.**



A leather craftsman company handcrafts customised leather watch straps according to the lug width of their customers' watches. Over a period of time it is found that the lug widths of their customers' watches are normally distributed; 85% of the watches have lug widths less than 21 mm, and 15% of the watches have lug widths less than 19 mm.

(a) Find the mean and the standard deviation of the lug width of their customers' watches. [3]

The widths of the straps made by the company follow the normal distribution with mean 19.6 mm and standard deviation 1.1 mm.

(b) Find the expected number of straps of width more than 20.2 mm in a randomly chosen batch of 40 straps. [3]

The straps are handcrafted in pairs. Each pair consists of a long end strap and a short end strap. The strap widths of both the long end straps and the short end straps follow the same normal distribution, and are independent of each other. In order for the strap to fit into the lug of the watch, the strap width needs to be shorter than the lug width of the watch. If the strap width is less than 0.2 mm shorter than the lug width of the watch, it is considered a good fit. Otherwise, the strap is a bad fit. A pair of strap is only usable for the watch if the long end strap and the short end strap are both good fits.

(c) A customer walks in with a watch with lug width of 20 mm. He randomly chooses a pair of straps. Show that the probability the pair of straps is usable for his watch is 0.00487, correct to 3 significant figures. [3]

10 [Continued]

(d) Another customer walks in with two watches of lug widths 18.5 mm and 20 mm respectively. He randomly chooses 2 pairs of straps of different designs. Find the probability that at least 1 pair of straps are usable for any of his watches, giving your answer correct to 5 decimal places. [4]

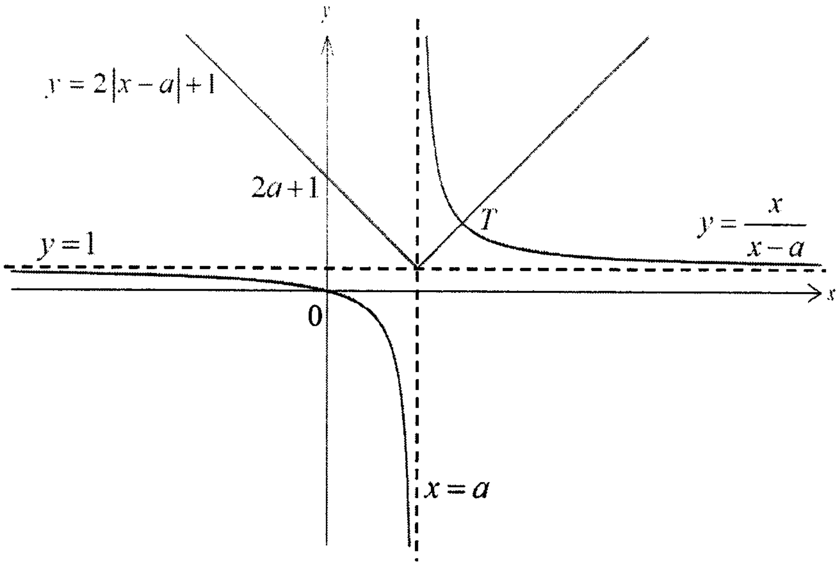
BLANK PAGE

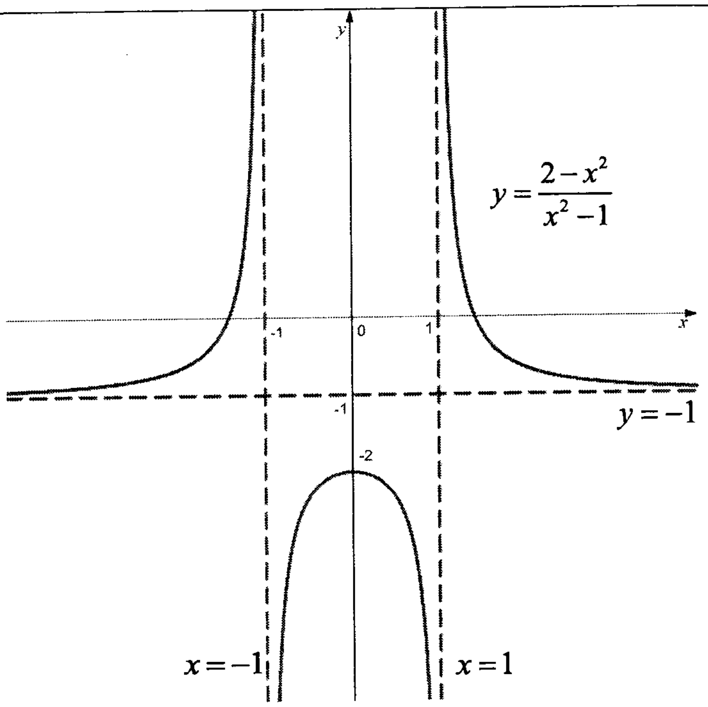
You may continue your working on this page if necessary, indicating the question number(s) clearly.

BLANK PAGE

You may continue your working on this page if necessary, indicating the question number(s) clearly.

RAFFLES INSTITUTION
2022 Year 6 H2 Mathematics Preliminary Examination Paper 1
Solutions

1(i)	
(ii)	Let the 2 graphs intersect at the point *T*. To solve for *T*, consider $\frac{x}{x-a}=2(x-a)+1$ $x=2(x-a)^2+x-a$ $(x-a)^2=\frac{a}{2}$ $x=a\pm\sqrt{\frac{a}{2}}$ Since $x>a$ at the point *T*, $x=a+\frac{\sqrt{2a}}{2}$. Thus the solution is $a<x<a+\frac{\sqrt{2a}}{2}$.

2(i)	$\dfrac{1+2x-x^2}{2(x^2-2x)}$ $=\dfrac{1}{2}\left[\dfrac{2-(x^2-2x+1)}{(x^2-2x+1)-1}\right]$ $=\dfrac{1}{2}\left[\dfrac{2-(x-1)^2}{(x-1)^2-1}\right]$ Thus $p=-1$ and $q=\dfrac{1}{2}$.
(ii)	The 2 transformations (in either order) are 1. A translation of 1 unit in the positive x-direction 2. A scaling parallel to the y-axis by a factor of $\dfrac{1}{2}$.
(iii)	 The equation $\dfrac{2-x^2}{x^2-1}=k$ has no real roots for values of k in the set $(-2, -1]$.
	Alternative Solution: $\dfrac{2-x^2}{x^2-1}=k$ $2-x^2=kx^2-k$ $(k+1)x^2-(k+2)=0$ If $k=-1$, $(k+1)x^2-(k+2)=0$ becomes $-1=0$, so there is no solution. If $k\neq-1$, the quadratic equation will have no real roots if $0-4[-(k+2)(k+1)]<0$ $(k+2)(k+1)<0$ $-2<k<-1$ So $-2<k\leq-1$ and the required solution set is $(-2, -1]$.

3(i)	$\frac{dx}{dt}=-\frac{a}{t^2}$; $\frac{dy}{dt}=a\left(1+\frac{2}{t^3}\right)$ $\frac{dy}{dx}=\frac{a\left(1+\frac{2}{t^3}\right)}{-\frac{a}{t^2}}=-\frac{t^3+2}{t}$ When $t=-\frac{1}{2}$, $\frac{dy}{dx}=\frac{15}{4}$, $x=-a$, $y=-\frac{9}{2}a$ Equation of tangent at P: $y+\frac{9}{2}a=\frac{15}{4}(x+a)$ $y=\frac{15}{4}x-\frac{3}{4}a$ Gradient of normal at $P=-\frac{4}{15}$ Equation of normal at P: $y+\frac{9}{2}a=-\frac{4}{15}(x+a)$ $y=-\frac{4}{15}x-\frac{143}{30}a$
(ii)	At point Q, $x=0$, $y=-\frac{3}{4}a$ At point R, $x=0$, $y=-\frac{143}{30}a$ Area of triangle PQR $=\frac{1}{2}\lvert a\rvert\left(\frac{143}{30}\lvert a\rvert-\frac{3}{4}\lvert a\rvert\right)$ $=\frac{1}{2}\left(\frac{241}{60}\right)\lvert a\rvert^2$ $=\frac{241}{120}a^2$ (shown)

4(a)

$$\mathbf{r}=\begin{pmatrix}a\\2\\3\end{pmatrix}\times\begin{pmatrix}1\\4\\5\end{pmatrix}=\begin{pmatrix}-2\\-5a+3\\4a-2\end{pmatrix}=\begin{pmatrix}-2\\3\\-2\end{pmatrix}+a\begin{pmatrix}0\\-5\\4\end{pmatrix},\ a\in\mathbb{R}$$

R lies on the line passing through the point with coordinates $(-2,3,-2)$, and the line is parallel to the vector $-5\mathbf{j}+4\mathbf{k}$.

(b)

Since F lies on the line PQ, then

$$\overrightarrow{OF}=\mathbf{p}+\lambda(\mathbf{q}-\mathbf{p}),\ \text{for some } \lambda\in\mathbb{R}.$$

Since $\overrightarrow{OF}$ is perpendicular to the line PQ, then

$$(\mathbf{p}+\lambda(\mathbf{q}-\mathbf{p}))\cdot(\mathbf{q}-\mathbf{p})=0$$

$$\mathbf{p}\cdot(\mathbf{q}-\mathbf{p})+\lambda|\mathbf{q}-\mathbf{p}|^2=0$$

$$\mathbf{p}\cdot\mathbf{q}-|\mathbf{p}|^2+\lambda|\mathbf{q}-\mathbf{p}|^2=0$$

$$\lambda=\frac{|\mathbf{p}|^2-\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}-\mathbf{p}|^2}$$

Substitute value of λ into (1) :

$$\overrightarrow{OF}=\mathbf{p}+\frac{|\mathbf{p}|^2-\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}-\mathbf{p}|^2}(\mathbf{q}-\mathbf{p})$$

$$=\left(1-\frac{|\mathbf{p}|^2-\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}-\mathbf{p}|^2}\right)\mathbf{p}+\frac{|\mathbf{p}|^2-\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}-\mathbf{p}|^2}\mathbf{q}$$

$$=(1-\lambda)\mathbf{p}+\lambda\mathbf{q}$$

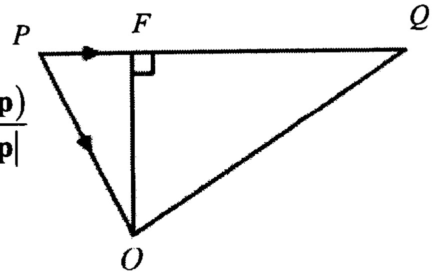
Alternative Method:

Consider $\overrightarrow{PF}$ as the projection vector of $\overrightarrow{PO}$ onto $\overrightarrow{PQ}$,

$$\overrightarrow{PF}=\left(\overrightarrow{PO}\cdot\overrightarrow{PQ}\right)\overrightarrow{PQ}$$

$$=\frac{-\mathbf{p}\cdot(\mathbf{q}-\mathbf{p})}{|\mathbf{q}-\mathbf{p}|}\times\frac{(\mathbf{q}-\mathbf{p})}{|\mathbf{q}-\mathbf{p}|}$$

$$=\frac{|\mathbf{p}|^2-\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}-\mathbf{p}|^2}(\mathbf{q}-\mathbf{p})$$



Hence we have

$$\overrightarrow{OF}=\overrightarrow{OP}+\overrightarrow{PF}$$

$$=\mathbf{p}+\frac{|\mathbf{p}|^2-\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}-\mathbf{p}|^2}(\mathbf{q}-\mathbf{p})$$

$$=\left(1-\frac{|\mathbf{p}|^2-\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}-\mathbf{p}|^2}\right)\mathbf{p}+\left(\frac{|\mathbf{p}|^2-\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}-\mathbf{p}|^2}\right)\mathbf{q}=(1-\lambda)\mathbf{p}+\lambda\mathbf{q}$$

where $\lambda=\dfrac{|\mathbf{p}|^2-\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}-\mathbf{p}|^2}$ (shown)

(b)(ii)

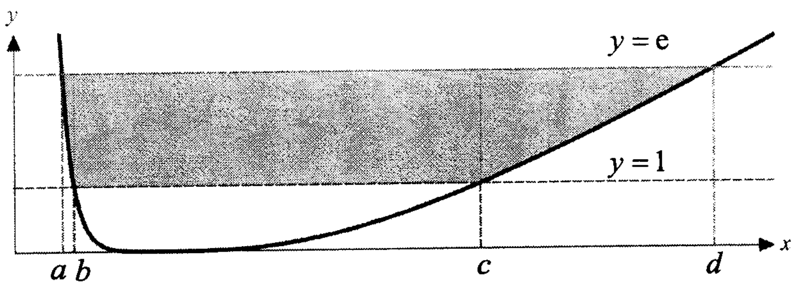
For F to lie within the line segment PQ,

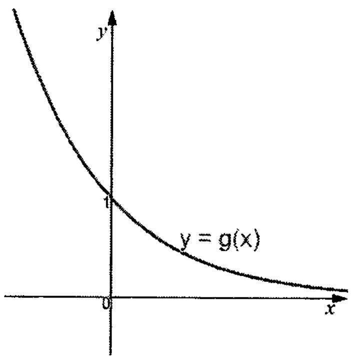
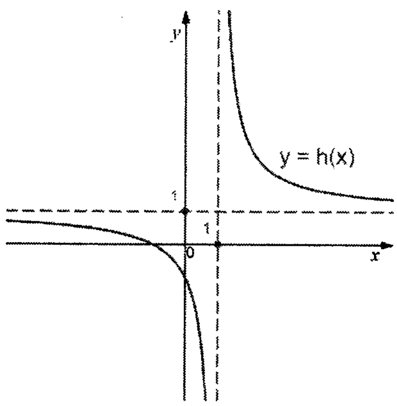
$$0\le\lambda\le1$$

5(i)	$z=\sin\left(\frac{\pi}{6}\right)+\mathrm{i}\cos\left(\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{2}-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$ $=\cos\left(\frac{\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{3}\right)=\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}.$ $\therefore \lvert z\rvert=1$ and $\arg(z)=\frac{\pi}{3}.$ Hence $z^3=\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)^3=\mathrm{e}^{\mathrm{i}\pi}=-1.$ **Alternatively**: $z=\sin\left(\frac{\pi}{6}\right)+\mathrm{i}\cos\left(\frac{\pi}{6}\right)=\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}.$ $\lvert z\rvert=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1,\ \arg(z)=\tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)=\frac{\pi}{3}$
(ii)	$w=\sqrt{2}\left[\sin\left(\frac{\pi}{6}\right)+\mathrm{i}\cos\left(\frac{\pi}{3}\right)\right]=\sqrt{2}\left(\frac{1}{2}+\mathrm{i}\frac{1}{2}\right)=\frac{\sqrt{2}}{2}+\mathrm{i}\frac{\sqrt{2}}{2}.$ Thus $w^4=\left(\frac{\sqrt{2}}{2}+\mathrm{i}\frac{\sqrt{2}}{2}\right)^4=\left(\frac{1}{2}+\mathrm{i}-\frac{1}{2}\right)^2=\mathrm{i}^2=-1$ **Alternatively:** $\lvert w\rvert=\sqrt{\left(\sqrt{2}\sin\frac{\pi}{6}\right)^2+\left(\sqrt{2}\cos\frac{\pi}{3}\right)^2}=\sqrt{2\left(\frac{1}{2}\right)^2+2\left(\frac{1}{2}\right)^2}=1.$ $\arg(w)=\tan^{-1}\left(\frac{\sqrt{2}\cos\frac{\pi}{3}}{\sqrt{2}\sin\frac{\pi}{6}}\right)=\tan^{-1}(1)=\frac{\pi}{4}.$ Thus $w^4=\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)^4=\mathrm{e}^{\mathrm{i}\pi}=-1.$
(iii)	Hence $z^{2022}-w^{2020}=\left(z^3\right)^{674}-\left(w^4\right)^{505}$ $=(-1)^{674}-(-1)^{505}$ $=1-(-1)$ $=2$ OR $\mathrm{e}^{674\pi\mathrm{i}}-\mathrm{e}^{505\pi\mathrm{i}}$ $=\mathrm{e}^{0\pi\mathrm{i}}-\mathrm{e}^{\pi\mathrm{i}}$ $=1-(-1)$ $=2$

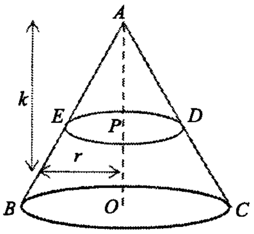
6(a)(i)	$u_1 = 2$ $u_2 = \frac{4}{u_1} = \frac{4}{2} = 2$ $u_3 = \frac{4}{u_2} = \frac{4}{2} = 2$ The sequence is a constant sequence of 2.
(a)(ii)	$u_1 = 3$ $u_2 = \frac{4}{u_1} = \frac{4}{3}$ $u_3 = \frac{4}{u_2} = \frac{4}{\frac{4}{3}} = 3$ The sequence alternates between 3 and $\frac{4}{3}$.
(b)(i)	$v_2 = v_1 + 2$ $v_3 = v_2 + 3$ $\quad = v_1 + 2 + 3$ $v_4 = v_3 + 4$ $\quad = v_1 + 2 + 3 + 4$ $v_n = v_1 + 2 + 3 + 4 + ... + n$ $\quad = v_1 + \frac{n-1}{2}(2+n)$ $\quad = v_1 + \frac{(n-1)(n+2)}{2}$ $\therefore v_n = A + \frac{(n-1)(n+2)}{2}$
(b)(ii)	$\sum_{r=1}^{n} v_r = \sum_{r=1}^{n}\left(A + \frac{(r-1)(r+2)}{2}\right)$ $= \sum_{r=1}^{n}\left(A + \frac{r^2+r-2}{2}\right)$ $= \sum_{r=1}^{n} A + \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$ $= \sum_{r=1}^{n}(A-1) + \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{r=1}^{n} r$ $= n(A-1) + \frac{1}{12}n(n+1)(2n+1) + \frac{1}{4}(n)(n+1)$

7(i)	$y=\sqrt{2+\cos^2 x} \quad \Rightarrow \quad y^2=2+\cos^2 x$ Differentiating w.r.t x, $2y\frac{\mathrm{d}y}{\mathrm{d}x}=2\cos x(-\sin x)=-\sin 2x$ (shown) ------ (1)
(ii)	Differentiating (1) w.r.t x, $2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-2\cos 2x$ $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-\cos 2x$ (shown) ------ (2)
	Differentiating (2) w.r.t x, $y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\sin 2x$ $y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=2\sin 2x$ ------- (3) Differentiating (3) w.r.t x, $y\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)+3\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)+3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=4\cos 2x$ $y\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+4\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)+3\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2=4\cos 2x$ ------ (4) When $x=0, y=\sqrt{3}, \frac{\mathrm{d}y}{\mathrm{d}x}=0, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{\sqrt{3}}{3}, \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0, \frac{\mathrm{d}^4y}{\mathrm{d}x^4}=\sqrt{3}$ By Maclaurin's Theorem, $y\approx\sqrt{3}-\frac{\sqrt{3}}{6}x^2+\frac{\sqrt{3}}{24}x^4$
	$\sqrt{2+\cos^2\left(\frac{\pi}{6}\right)}\approx\sqrt{3}-\frac{\sqrt{3}}{6}\left(\frac{\pi}{6}\right)^2+\frac{\sqrt{3}}{24}\left(\frac{\pi}{6}\right)^4$ $\sqrt{2+\left(\frac{\sqrt{3}}{2}\right)^2}\approx\sqrt{3}-\frac{\sqrt{3}}{216}\pi^2+\frac{\sqrt{3}}{31104}\pi^4$ $\frac{\sqrt{11}}{2}\approx\sqrt{3}\left(1-\frac{\pi^2}{216}+\frac{\pi^4}{31104}\right)$ $\sqrt{11}\approx2\sqrt{3}\left(1-\frac{\pi^2}{216}+\frac{\pi^4}{31104}\right)$ (shown)

8(i)	Required volume $= \pi \int_1^e \left(\frac{(\ln x)^4}{\sqrt{x}} \right)^2 dx$ $= \pi \int_1^e \frac{1}{x} (\ln x)^8 \, dx$ $= \pi \left[\frac{(\ln x)^9}{9} \right]_1^e$ $= \frac{\pi}{9}$
(ii)	 $\frac{(\ln x)^4}{\sqrt{x}} = e \quad \Rightarrow \quad x = 0.32735, 4.76172$ $\frac{(\ln x)^4}{\sqrt{x}} = 1 \quad \Rightarrow \quad x = 0.40892, 3.17520$ Let $a = 0.32735$, $b = 0.40892$, $c = 3.1752$ and $d = 4.7617$ (5 sf) <u>Method 1</u> Required area $= (d-a)e - \int_a^b \frac{(\ln x)^4}{\sqrt{x}} dx - (c-b) - \int_c^d \frac{(\ln x)^4}{\sqrt{x}} dx$ = 6.2482 (5 sf) = 6.25 (3 sf). <u>Method 2</u> Required area $= \int_a^b \left(e - \frac{(\ln x)^4}{\sqrt{x}} \right) dx + (c-b)(e-1) + \int_c^d \left(e - \frac{(\ln x)^4}{\sqrt{x}} \right) dx.$ = 6.2482 (5 sf) = 6.25 (3 sf). <u>Method 3</u> Required area $= \int_a^d \left(e - \frac{(\ln x)^4}{\sqrt{x}} \right) dx - \int_b^c \left(1 - \frac{(\ln x)^4}{\sqrt{x}} \right) dx.$ = 6.2482 (5 sf) = 6.25 (3 sf).

9(i)	Let $y = e^{-3x}$. Then $-3x = \ln y \Rightarrow x = -\frac{1}{3}\ln y$. So $g^{-1} : x \mapsto -\frac{1}{3}\ln x, \quad x \in \mathbf{R}, \quad x > 1$. Let $y = \frac{x+1}{x-1}$. Then $yx - y = x + 1 \Rightarrow yx - x = y + 1$ $\Rightarrow x(y-1) = y + 1$ $\Rightarrow x = \frac{y+1}{y-1}$. So $h^{-1} : x \mapsto \frac{x+1}{x-1}, \quad x \in \mathbf{R}, x \neq 1$.
(ii)	$ff(x) = (x^3 - 1)^3 - 1$.
(iii)	Given that $ff(\alpha) = 0$ means $(\alpha^3 - 1)^3 - 1 = 0 \Rightarrow (\alpha^3 - 1)^3 = 1$ $\Rightarrow \alpha^3 - 1 = 1$ $\Rightarrow \alpha^3 = 2$ $\Rightarrow \alpha = \sqrt[3]{2}$. So $g^{-1}(\alpha) = -\frac{1}{3}\ln \alpha = -\frac{1}{3}\ln 2^{\frac{1}{3}} = -\frac{1}{9}\ln 2$ (Shown). Thus $k = -\frac{1}{9}$.
(iv)	Since $R_g = (1, \infty) \subseteq D_h = (-\infty, 1) \cup (1, \infty)$, hg exists (Shown). $(-\infty, 0) \xrightarrow{g} (1, \infty) \xrightarrow{h} (1, \infty)$ The range of the function is $(1, \infty)$.  
(v)	Since $h^2(x) = x$ (i.e. h is a self-inverse function), and $h^m(x) = x$ if m is an even integer. Thus $h^m(x) = h^{-1}(-1)$ becomes $x = h^{-1}(-1) = 0$.

10(i)	$\frac{dG}{dt} = -\lambda G$ $\int \frac{1}{G}\,dG = \int -\lambda\ dt$ $\ln G = -\lambda t + C$, where $C \in$ □ $G = Ae^{-\lambda t}$, where $A = e^{C}$.
(ii)	When $t = 0$, $G = 80 \Rightarrow A = 80$ $G = 80e^{-0.005t}$ For $G \le 70$, $e^{-0.005t} \le 0.875$ $-0.005t \le \ln 0.875$ $t \ge 26.706$ The approximate time is 26.7 minutes.
(iii)	$\frac{dG}{dt} = \mu - 0.005G = -0.005(G - 200\mu)$ $\int \frac{1}{G-200\mu}\,dG = \int -0.005\ dt$ $\ln\lvert G-200\mu\rvert = -0.005t + D$, where $D \in$ □ $G - 200\mu = Be^{-0.005t}$, where $B = \pm e^{D}$. When $t = 0$, $G = 80 \Rightarrow B = 80 - 200\mu$ $\therefore G = 200\mu + (80 - 200\mu)e^{-0.005t}$
(iv)	When $\mu = 0.7$, $G = 140 - 60e^{-0.005t}$ Observe that when $t \to \infty$, $G \to 140$. This means it is **<u>not recommended</u>** as after a long period of time, the glucose level in the blood stream will exceed 100 mg/dL. <u>Alternatively:</u> For $G \ge 100$, we have $140 - 60e^{-0.005t} \ge 100$ $\Rightarrow 60e^{-0.005t} \le 40$ $\Rightarrow t \ge 81.093$ This means it is **<u>not recommended</u>** as after approximately 81 minutes the glucose level in the blood stream will exceed 100 mg/dL.
(v)	$70 < 200\mu < 100 \Rightarrow 0.35 < \mu < 0.5$

11(i)	Let C be the material cost of a can $\pi r^2 h = 100\pi$ $h = \dfrac{100}{r^2}$ ------ (1) $C = 2\pi rh(0.9\times10^{-4}) + 2\pi r^2(1.2\times10^{-4})$ ------ (2) Sub (1) into (2), $C = 2\pi r\dfrac{100}{r^2}(0.9\times10^{-4}) + 2\pi r^2(1.2\times10^{-4})$ $= \dfrac{0.018\pi}{r} + 0.00024\pi r^2$ $\dfrac{dC}{dr} = -\dfrac{0.018\pi}{r^2} + 0.00048\pi r$ when $\dfrac{dC}{dr} = 0,\ r^3 = \dfrac{0.018}{0.00048}$ $r = 3.3472 = 3.35$ (3sf) (shown) $h = 8.9258$ or 8.9107 (if use $r = 3.35$) $\dfrac{d^2C}{dr^2} = \dfrac{0.036\pi}{r^3} + 0.00048\pi$, for $r > 0$, $\dfrac{d^2C}{dr^2} > 0$, so C is minimum Thus, the most economical can has a radius of 3.35 cm (3sf) and height 8.93 cm (3sf)
(ii)	The cost of the can, $C = \dfrac{0.018\pi}{3.35} + 0.00024\pi(3.35)^2 = 0.0253$ (3sf) The most economical can costs 2.5 cents each. (1dp)
(iii)	 Refer to the diagram above. Let BC be the diameter of the top of the cylindrical part of can with O as the centre and let ED be the diameter of the lid with P as the centre. The lines BE and CD are extended to meet at A as shown in the diagram. The points A, B, C, D and E lie in the same plane and ABC and AED form two right cones. Let $OB = 3$ cm, $PE = 1.5$ cm, then $BE = EA = 2.5$ cm. Hence $BA = 5$ cm, $OA = 4$ cm. Let r be the radius of the liquid surface, and k be the vertical distance from the liquid surface to A. Using similar triangles, $\dfrac{r}{k} = \dfrac{3}{4}$

	Volume of liquid above level BC, $V=\frac{1}{3}\pi(3^2)(4)-\frac{1}{3}\pi r^2 k$ $=\frac{1}{3}\pi(3^2)(4)-\frac{3}{16}\pi k^3$ $\Rightarrow \frac{dV}{dk}=-\frac{9}{16}\pi k^2$ When the liquid level is 1cm from the lid of the can, $k=3$ $\frac{dk}{dt}=\frac{dV}{dt}\times\frac{dk}{dV}$ $=90\pi\times\left(-\frac{16}{9\pi(3)^2}\right)$ $=-\frac{160}{9}$ Since k is decreasing at $\frac{160}{9}$ cm/s, it follows that the liquid level in the can is increasing at $\frac{160}{9}$ cm/s when it is 1cm from the top of the can.

RAFFLES INSTITUTION
2022 Year 6 H2 Mathematics Preliminary Examination Paper 2 Solutions

1(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cos^2 x}\right)$ $=\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)^{-2}$ $=-2(\cos x)^{-3}(-\sin x)$ $=\frac{2\sin x}{\cos^3 x}$ (i.e. $k=2$ and $n=3$)
(ii)	$\int_0^{\frac{\pi}{4}} \sin^2 x \sec^3 x \,\mathrm{d}x=\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^3 x}\,\mathrm{d}x$ $=\int_0^{\frac{\pi}{4}} \frac{2\sin x}{\cos^3 x}\cdot\frac{\sin x}{2}\,\mathrm{d}x$ $=\left[\frac{1}{\cos^2 x}\cdot\frac{\sin x}{2}\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2 x}\cdot\frac{\cos x}{2}\,\mathrm{d}x$ $=\frac{1}{2}\left[\frac{\sin x}{\cos^2 x}\right]_0^{\frac{\pi}{4}}-\frac{1}{2}\int_0^{\frac{\pi}{4}}\sec x\,\mathrm{d}x$ $=\frac{1}{2}\left[\frac{\sin\frac{\pi}{4}}{\cos^2\frac{\pi}{4}}-0\right]-\frac{1}{2}\left[\ln(\sec x+\tan x)\right]_0^{\frac{\pi}{4}}$ $=\frac{1}{2}\left[\frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}}\right]-\frac{1}{2}\left[\ln(\sec\frac{\pi}{4}+\tan\frac{\pi}{4})-\ln(1+0)\right]$ $=\frac{1}{\sqrt{2}}-\frac{1}{2}\ln(\sqrt{2}+1)$

2(a)	The student's claim is true. Reasons: 1. From $\alpha^2+\beta^2+\gamma^2+\delta^2<0$, it shows that at least one of the roots is complex. 2. As the coefficients of $\mathrm{f}(z)$ are real, we know that complex roots exist in conjugate pairs, so there is at least one pair of complex conjugate roots. 3. If there are 2 pairs of complex conjugate roots, then $\alpha\beta\gamma\delta>0$. However, given that $\alpha\beta\gamma\delta<0$, then there is only a pair of complex conjugate roots and 2 real roots of opposite signs. So with 1, 2 and 3, we can conclude that the equation $\mathrm{f}(z)=0$ has one positive real root, one negative real root and a pair of complex conjugate roots.
(b)	Since $\mathrm{f}(2\mathrm{i})=(2\mathrm{i})^4+(2\mathrm{i})^3-2(2\mathrm{i})^2+4(2\mathrm{i})-24$ $=16-8\mathrm{i}+8+8\mathrm{i}-24=0$, so $z=2\mathrm{i}$ is a root of the equation $\mathrm{f}(z)=0$ (verified). As complex roots occur in conjugate pair, so $z=-2\mathrm{i}$ is the other complex root. Now $(z-2\mathrm{i})(z+2\mathrm{i})=z^2+4$. Hence $z^4+z^3-2z^2+4z-24=(z^2+4)(z^2+az-6)$. Comparing the coefficient of z^3: $a=1$. Thus $z^4+z^3-2z^2+4z-24=(z^2+4)(z^2+z-6)$ $=(z^2+4)(z+3)(z-2)$ So the other 3 roots of the equation $\mathrm{f}(z)=0$ are $-2\mathrm{i}$, 2 and -3.

3(i)	**Method 1** The z-coordinate of P is 0, since P lies on the xy-plane. Thus, putting $z=0$ into L_1, $1-y=\frac{0-1}{2} \quad \Rightarrow \quad y=\frac{3}{2}$ The position vector of P is $\overrightarrow{OP}=\begin{pmatrix}2\\ \frac{3}{2}\\ 0\end{pmatrix}$. Hence, equation of L_2 is $\mathbf{r}=\lambda\begin{pmatrix}4\\3\\0\end{pmatrix},\ \lambda\in\mathbb{R}$ **Method 2** Vector equation of line L_1: $\mathbf{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}0\\-1\\2\end{pmatrix},\ \mu\in\mathbb{R}$ Equation of xy-plane : $\mathbf{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ To find P, consider $\left(\begin{pmatrix}2\\1\\1\end{pmatrix}+\mu\begin{pmatrix}0\\-1\\2\end{pmatrix}\right)\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ $1+2\mu=0$ $\mu=-\frac{1}{2}$ Hence, $\overrightarrow{OP}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\left(-\frac{1}{2}\right)\begin{pmatrix}0\\-1\\2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4\\3\\0\end{pmatrix}$. Hence, equation of L_2 is $\mathbf{r}=\lambda\begin{pmatrix}4\\3\\0\end{pmatrix},\ \lambda\in\mathbb{R}$
(ii)	$\begin{pmatrix}4\\3\\0\end{pmatrix}\times\begin{pmatrix}0\\-1\\2\end{pmatrix}=\begin{pmatrix}6\\-8\\-4\end{pmatrix}=-2\begin{pmatrix}-3\\4\\2\end{pmatrix}$ Hence, equation of π is $\mathbf{r}\cdot\begin{pmatrix}-3\\4\\2\end{pmatrix}=0$
(iii)	Let F be the foot of perpendicular from A to π.

Equation of the line AF is $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$.

Since F lies on the line AF, then

$$\overrightarrow{OF} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}, \text{ for some } \lambda \in \square .$$

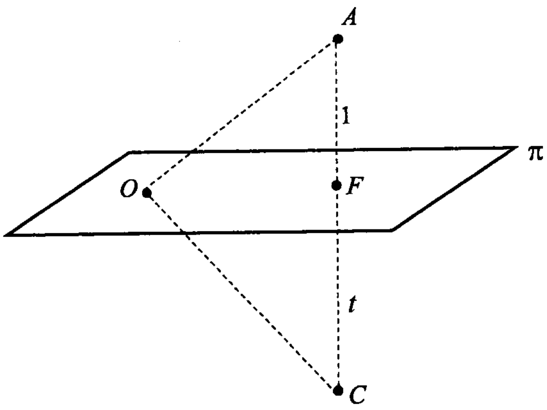
Since F is also on π, then

$$\left[\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right] \square \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = 0$$

$$(-9-4+4) + \lambda(9+16+4) = 0$$

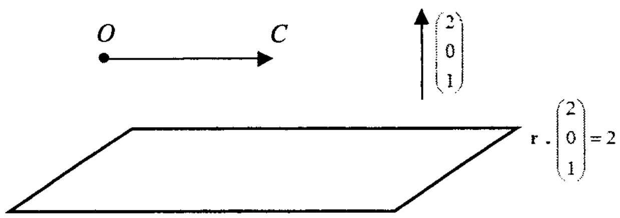
$$\lambda = \frac{9}{29}$$

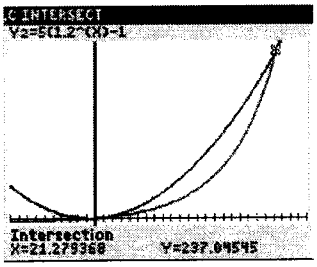
Hence, $\overrightarrow{OF} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \frac{9}{29} \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = \frac{1}{29} \begin{pmatrix} 60 \\ 7 \\ 76 \end{pmatrix}$.



By Ratio Theorem, $\overrightarrow{OF} = \dfrac{t\overrightarrow{OA} + \overrightarrow{OC}}{(t+1)}$

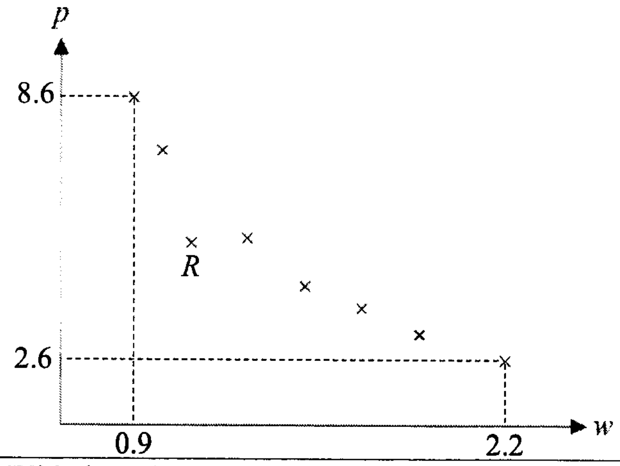
$$\overrightarrow{OC} = (t+1)\overrightarrow{OF} - t\overrightarrow{OA} = \frac{(t+1)}{29} \begin{pmatrix} 60 \\ 7 \\ 76 \end{pmatrix} - t \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{29} \begin{pmatrix} 60-27t \\ 7+36t \\ 76+18t \end{pmatrix}$$

(iv)	 OC must be perpendicular to the normal vector of the plane. $\overrightarrow{OC}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}=0$ $\frac{1}{29}\begin{pmatrix}60-27t\\7+36t\\76+18t\end{pmatrix}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}=0$ $2(60-27t)+(76+18t)=0$ $-36t+196=0$ $t=\frac{49}{9}$

4(a)	Given $a, b, 2$ is GP: $\frac{b}{a}=\frac{2}{b} \Rightarrow \quad b^2=2a \quad$ - (1) Given $2, a, b$ is AP: $a-2=b-a \quad \Rightarrow \quad 2a-b=2 \quad$ - (2) Solving the 2 equations: $b^2-b-2=0$ $(b-2)(b+1)=0$ $b=2$ or $b=-1$ If $b=2$, then $a=2$, $\Rightarrow$ common diff, $b-a=0$ (reject) If $b=-1$, then $a=\frac{1}{2}$, $\Rightarrow$ common diff, $b-a\neq 0$ $\therefore a=\frac{1}{2},\ b=-1$
(b)	$p, p, p+d, pr, p+2d, pr^2..., p+(2n-1)d, pr^{2n-1}$ Sum of odd terms $=\frac{n}{2}\left[2p+(n-1)p\right]=\frac{pn}{2}(n+1)$ Sum of even terms $=\frac{p\left(1.2^n-1\right)}{1.2-1}=5p\left(1.2^n-1\right)$ Given $\frac{pn}{2}(n+1)<5p\left(1.2^n-1\right)\Rightarrow\frac{n}{2}(n+1)<5\left(1.2^n-1\right)$ **(since $p>0$)** Using GC, $n>21.3$ Least value of n is 22.  **Alternatively**, From G.C., $n=21,\ \frac{n}{2}(n+1)-5\left(1.2^n-1\right)=5.9744>0$ $n=22,\ \frac{n}{2}(n+1)-5\left(1.2^n-1\right)=-18.03<0$ Hence, the least value of n is 22.
(c)	Sum of 1st k terms $=\frac{q\left(r^k-1\right)}{r-1}$ Sum of last k terms $=\frac{q\left(r^n-1\right)}{r-1}-\frac{q\left(r^{n-k}-1\right)}{r-1}$ $=\frac{q}{r-1}\left(r^n-1-r^{n-k}+1\right)$ $=\frac{q}{r-1}\left(r^n-r^{n-k}\right)=\frac{q}{r-1}r^{n-k}\left(r^k-1\right)$

	Difference required $= \left\lvert \frac{q\left(r^k-1\right)}{r-1} - \frac{q}{r-1} r^{n-k}\left(r^k-1\right) \right\rvert$ $= \left\lvert \frac{q}{r-1}\left[\left(r^k-1\right) - r^{n-k}\left(r^k-1\right)\right] \right\rvert$ $= \left\lvert \frac{q}{r-1}\left(r^k-1\right)\left(1-r^{n-k}\right) \right\rvert$

5(i)	2 from each of P and Q, 1 from R: ${}^5C_2 \times {}^4C_2 \times {}^3C_1 = 180$
(ii)	2 from Class R & 3 from P or Q: ${}^3C_2 \times {}^9C_3 = 252$ 3 from Class R & 2 from P or Q: ${}^3C_3 \times {}^9C_2 = 36$ Total number of ways is 288 **Alternatively,** Consider complement cases of none from R or one from R: ${}^{12}C_5 - {}^9C_5 - {}^3C_1 \times {}^9C_4 = 288$
(iii)	Class P students are all separated : $7! \times {}^8C_5 \times 5! = 33868800$ separated in groups of 2, 1, 1, 1 : $7! \times {}^8C_1 \times {}^7C_3 \times 5! = 169344000$ (or $7! \times {}^8C_4 \times 4 \times 5!$) separated in groups of 2, 2, 1 : $7! \times {}^8C_2 \times {}^6C_1 \times 5! = 101606400$ (or $7! \times {}^8C_3 \times 3 \times 5!$) Total number of ways is 304 819 200 **Alternatively,** consider the complement method: separated in groups of 3, 1, 1 : $7! \times {}^8C_3 \times 3 \times 5! = 101\ 606\ 400$ separated in groups of 3, 2 : $7! \times {}^8C_2 \times 2! \times 5! = 33\ 868\ 800$ separated in groups of 4, 1 : $7! \times {}^8C_2 \times 2! \times 5! = 33\ 868\ 800$ 5 members of P are seated in a group : $8! \times 5! = 4\ 838\ 400$ Total number of ways = $12! - 101606400 - 33868800 - 33868800 - 4838400 = 304819200$

6(i)	Since $r = -0.92821 = -0.928$ (3.s.f) is close to -1, it suggests a strong negative linear correlation, a linear model seems appropriate.
(ii)	
(iii)	With the point R removed, the values of p decreases as w increases, but by decreasing amounts. Hence it is consistent with a model of the form $\ln p = a + bw$.
(iv)	From GC, $\ln p = 2.910569 - 0.916387w$ … (1) $a = 2.91$ (3 s.f.) $b = -0.916$ (3 s.f.)`
(v)	Substitute $w = 1.1$ into (1) : $\ln p = 2.910569 - 0.916387(1.1)$ $p = 6.70$ (to 3 s.f.)

7(i)	<u>Condition 1</u> The event that a student infected by the disease is independent of any other student infected by the disease. This might not be true. Since the disease is contagious, it can easily be passed from one student to another. <u>Condition 2</u> The probability of a student infected by the disease is constant. This might not be true as the probability will depend on each individual's lifestyle or exposure, some will have a higher chance of catching the disease.
(ii)	$X \sim \text{B}(20, 0.11)$ $\text{P}(1 \le X < 10) = \text{P}(1 \le X \le 9)$ $= \text{P}(X \le 9) - \text{P}(X = 0)$ $= 0.999983 - 0.097230$ $= 0.902753$ $= 0.903$
(iii)	Let Y denote the number of students who has once been infected by the disease before out of 19. $Y \sim \text{B}(19.0.11)$ Required probability $= \text{P}(Y = 4) \times 0.11 = 0.0109$ (to 3s.f.) <u>Alternative solution</u> Required probability $= \frac{^{19}C_4}{^{20}C_5} \text{P}(X = 5) = 0.0109$ (to 3 s.f.)
(iv)	Lower. The event described in part **(iii)** is a subset of having exactly 5 students in the class who have been infected by the disease before. For example, it is possible that the first 5 students approached are the only 5 students in the class who has once been infected. This event would have been considered under "exactly 5 students" in the class have once been infected, but not considered in **(iii)**.

8(i) Let X be the score obtained in one game.

$$\mathrm{P}(X=4)=\mathrm{P}((\mathrm{H},4),(\mathrm{T},1+3),(\mathrm{T},2+2))$$
$$=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot 3=\frac{7}{32}$$

(ii)

$$\mathrm{P}(X=1)=\mathrm{P}(\mathrm{H},1)=\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$$
$$\mathrm{P}(X=2)=\mathrm{P}((\mathrm{H},2),(\mathrm{T},1+1))=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{5}{32}$$
$$\mathrm{P}(X=3)=\mathrm{P}((\mathrm{H},3),(\mathrm{T},1+2))=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot 2=\frac{3}{16}$$
$$\mathrm{P}(X=5)=\mathrm{P}((\mathrm{T},1+4),(\mathrm{T},2+3))=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot 4=\frac{1}{8}$$
$$\mathrm{P}(X=6)=\mathrm{P}((\mathrm{T},2+4),(\mathrm{T},3+3))=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot 3=\frac{3}{32}$$
$$\mathrm{P}(X=7)=\mathrm{P}(\mathrm{T},3+4)=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot 2=\frac{1}{16}$$
$$\mathrm{P}(X=8)=\mathrm{P}(\mathrm{T},4+4)=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{32}$$
$$\mathrm{E}(X)=\sum_{r=1}^{8} r\,\mathrm{P}(X=r)=\frac{15}{4}\quad\text{(or 3.75)}$$
$$\mathrm{E}(X^2)=\sum_{r=1}^{8} r^2\,\mathrm{P}(X=r)=\frac{35}{2}\quad\text{(or 17.5)}$$
$$\mathrm{Var}\,X=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2=\frac{35}{2}-\left(\frac{15}{4}\right)^2=\frac{55}{16}$$

(iii)

$$\mathrm{P}\left(\frac{X_1+\cdots+X_{35}}{35}\geq 4 \,\middle|\, X_1=3 \text{ and } X_2=4\right)$$
$$=\frac{\mathrm{P}\left(\frac{X_1+\cdots+X_{35}}{35}\geq 4 \cap (X_1=3 \text{ and } X_2=4)\right)}{\mathrm{P}(X_1=3 \text{ and } X_2=4)}$$
$$=\frac{\mathrm{P}\left(\frac{3+4+X_3+\cdots+X_{35}}{35}\geq 4\right)\mathrm{P}(X_1=3)\mathrm{P}(X_2=4)}{\mathrm{P}(X_1=3)\mathrm{P}(X_2=4)}$$
$$=\mathrm{P}(X_3+\cdots+X_{35}\geq 4\times 35-3-4)$$
$$=\mathrm{P}(X_3+\cdots+X_{35}\geq 133)$$

The sample size 33 is large. By Central Limit Theorem,

	$X_3+\cdots+X_{35}\sim \mathrm{N}\left(33\times\frac{15}{4}=\frac{495}{4},\ 33\times\frac{55}{16}=\frac{1815}{16}\right)$ approximately. $\mathrm{P}\left(X_3+\cdots+X_{35}\geq 133\right)=0.1925638=0.193$ (3sf)

9(i)	Unbiased estimate of population mean is $\frac{8.71}{200} = 0.04355$ thousand hrs = 43.55 hours Unbiased estimate of population variance is $\frac{1}{199}\left(0.505 - \frac{8.71^2}{200}\right) = 6.31555\times10^{-4}$ (thousand hours)^2 = 632 hours^2 (3sf)
(ii)	Let μ be the population mean amount of time, in thousand hours, an employee spends at work per week. Null Hypothesis $H_0: \mu = 0.041$ Alternative Hypothesis $H_1: \mu > 0.041$ Perform a 1-tail test at 5% significance level. Under H_0, since sample size 200 is large, $\overline{X} \sim N\left(0.041, \frac{6.31555\times10^{-4}}{200}\right)$ approximately by Central Limit Theorem. From the sample, $\overline{x} = 0.04355$ Using a z-test, p-value $= P\left(\overline{X} \geq 0.04355\right) = 0.0756 > 0.05$ We do not reject H_0. There is insufficient evidence, at the 5% significance level, that the average working hours of an employee per week is more than 41.
(iii) (a)	Assume that the number of working hours per week of an employee follows a normal distribution. Let Y be the number working hours per week of an employee after the restructuring. $H_0: \mu = 40$ $H_1: \mu \neq 40$ Under H_0, $\overline{Y} \sim N\left(40, \frac{k}{12}\right)$ If H_0 is not rejected at the 5% significance level, $P\left(\overline{Y} \geq 40.1\right) > 0.025$ $P\left(Z \geq \frac{40.1-40}{\sqrt{\frac{k}{12}}}\right) > 0.025$ $\frac{0.1}{\sqrt{\frac{k}{12}}} < 1.9600$ $\frac{k}{12} > \left(\frac{0.1}{1.9600}\right)^2$ $k > 0.0312$ (3sf)
(b)	The Central Limit Theorem does not apply here as the sample size 12 is small.

10(a)	Let X denote the lug width (in mm) of their customers' watches. $X \sim \mathrm{N}(\mathrm{E}(X), \sigma^2)$ $\mathrm{E}(X) = 20$ $\mathrm{P}(X < 21) = 0.85$ $\mathrm{P}\left(Z < \frac{21-20}{\sigma}\right) = 0.85$ From G.C. $\frac{1}{\sigma} = 1.03643$ $\sigma = 0.96484 = 0.965$ (to 3s.f.)
(b)	Let W denote the strap width (in mm) made by the company. $W \sim \mathrm{N}(19.6, 1.1^2)$ $\mathrm{P}(W > 20.2) = 0.29272$ Let Y denote the number of straps of width more than 20.2mm out of 40. $Y \sim \mathrm{B}(40, 0.29272)$. $\mathrm{E}(Y) = 40 \times 0.29272 = 11.708 = 11.7$ (to 3s.f.)
(c)	$\mathrm{P}(19.8 < W \le 20) = 0.069798$ Required probability = $0.069798^2 = 0.0048717 = 0.00487$ (to 3s.f.)
(d)	Let α be the probability that a pair of straps is usable for the 18.5mm-watch, and β be the probability that a pair of straps is usable for the 20mm-watch. Clearly, $\alpha = [\mathrm{P}(18.3 < W \le 18.5)]^2 = 0.0016013$ (to 7 d.p.). From part **(c)**, $\beta = [\mathrm{P}(19.8 < W \le 20)]^2 = 0.0048717$ (to 7d.p.) **<u>Method 1</u>** : Probability that a pair of straps is not usable for both watches $= 1 - \alpha - \beta$ Hence, the required probability = 1 – P(both pairs of straps are not usable for both watches) $= 1 - (1 - \alpha - \beta)^2$ ... (*) = 0.0129041904 = 0.01290 (5 d.p.)

	Method 2 : Let Q be the number of pairs of straps (out of 2) that are usable (for either the 18.5mm-watch or the 20mm-watch. $Q \sim \mathrm{B}(2,\ \alpha+\beta)$ Hence, the required probability $= \mathrm{P}(Q \geq 1) = 1 - \mathrm{P}(Q = 0)$ $= 1 - (1-(\alpha+\beta))^2 = 1 - (1-\alpha-\beta)^2$ (same as (*)) $= 0.0129041904 = 0.01290$ (5 d.p.)