Name: ______________________ Class: ______________

Preliminary Examination 2016

MATHEMATICS
Higher 1

8864/01

30 August 2016

Paper 1

3 hours

Additional materials: Answer Paper
Cover Page
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work together securely, with the cover page in front.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 95.

This document consists of 6 printed pages.

Section A: Pure Mathematics [35 marks]

1 Find the exact value of $\int_{-5}^{1} \frac{1}{\sqrt{4-x}} \, dx$. [3]

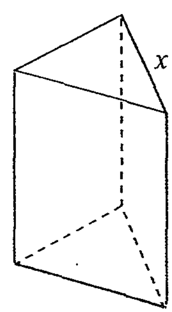
2 Show that the line $y = px + 5p$ intersect the curve $y = -x^2 + (2p-1)x + 4p + 3$ at two points for all real values of p. [4]

3 The curve C has equation $y = \ln\left(x e^{-x}\right) - (2-x)^3$.

(i) Find the exact value of the gradient of C at the point P where $x = 2$. [3]

(ii) The normal to C at P meets the y-axis at A and the x-axis at B. Find the area of triangle OAB. [4]

4 An advertising company designed a closed prism container made of aluminium to promote a new chocolate product. The container has a height of h cm, an equilateral triangular cross-sectional face of sides x cm, and a fixed volume of 10 cm^3.



(i) Show that the total surface area, S, of the closed prism container is given by

$$\left(\frac{\sqrt{3}}{2}x^2 + \frac{40\sqrt{3}}{x}\right)\text{cm}^2.$$ [3]

(ii) Find the value of x when the cost of aluminium needed to manufacture the container is minimised. [4]

(iii) Discuss the impact on the cost of aluminium used if x had been arbitrarily chosen other than the answer you obtained in **(ii)**. [1]

5 The curve C_1 has equation $y = \mathrm{f}(x) = ax^3 + 6x^2 + 18x + b$. The gradient of C_1 at the point $A(1, 13)$ is 24.

(i) Show that $a = -2$ and $b = -9$. [3]

The curve C_2 has equation $y = \mathrm{g}(x) = 10 - \mathrm{e}^{-\frac{1}{2}x}$.

(ii) Sketch the graphs of C_1 and C_2 on the same diagram, stating the equations of any asymptotes, the coordinates of any points of intersection with the axes, and the coordinates of the points where C_1 and C_2 intersect. [5]

(iii) Write down the values of x such that $\mathrm{f}(x) > \mathrm{g}(x)$. [1]

(iv) Find the exact area enclosed by C_1, C_2 and the lines $x = 2$ and $x = 3$. [4]

Section B: Statistics [60 marks]

6 A school has 700 Year 2 students. Of these students, 100 are in the Arts stream and 600 are in the Science stream. The Student Council is organising the Graduation Night, and intends to survey a sample of 100 Year 2 students to find out their music genre preferences.

(i) Describe how the sample could be chosen using systematic sampling. [2]

(ii) State one disadvantage of using a systematic sample in this context. [1]

(iii) What type of sampling method might be more appropriate? [1]

7 The weight of students in Omega College is known to have a normal distribution with mean μ_0 kg and standard deviation 7 kg. A random sample of 250 students undergoing a weight-reducing programme had their weights, x kg, measured to the nearest kg, and the following data were obtained:

$$\sum (x - 70) = 900 \text{ and } \sum (x - 70)^2 = 17000.$$

A test at the 5% significance level shows that there is insufficient evidence to conclude that the population mean weight of students has changed. Find the range of values of μ_0.

[5]

8 In a book, the authors claim that people who follow a new diet will lose an average of at least 3 kg in a month. The weight losses of the 180 people in a random sample who had followed the new diet for a month were noted. The mean and standard deviation of the sample were 2.7 kg and 2.8 kg respectively.

(i) Test at 5% level of significance whether the claim is valid, stating clearly any assumptions or approximations involved. [6]

(ii) Find the least value of α for which the authors' claim is not valid at the α% significance level. [1]

(iii) Explain what is meant by the phrase 'at the α% significance level' in this context. [1]

9 The monthly household expenditure, y thousand dollars, of 8 households of different income, x thousand dollars, are given in the table below.

Household	A	B	C	D	E	F	G	H
x	1.5	2.5	3.4	4.2	5.5	6.7	9.0	11.1
y	1.52	2.08	2.49	3.04	3.58	3.96	4.65	5.49

(i) Give a sketch of the scatter diagram for the data, as shown on your calculator. [2]

(ii) Find the product moment correlation coefficient, $\bar{x}$ and $\bar{y}$. [2]

The equation of the regression line of y on x is $y = 1.15 + 0.402x$, where the coefficients are given correct to 3 significant figures.

(iii) Find the equation of the regression line of x on y, and sketch the two regression lines on your scatter diagram. [3]

(iv) Use the appropriate regression line to estimate

(a) the expenditure of a household with an income of \$15 000. [1]

(b) the income of a household with an expenditure of \$1 800. [1]

(v) Comment briefly on the reliability of the estimates in part **(iv)**. [2]

10 Rose-shaped chocolates are packed in boxes of 24. Given that the probability of getting at most one mis-shaped chocolate in a box is 0.16215, find the probability that a randomly chosen chocolate is mis-shaped. [2]

The production process has improved and now only 5% of the chocolates produced are mis-shaped.

(i) Find the probability that more than one chocolate is mis-shaped in a randomly chosen box. [2]

A customer bought 10 boxes of chocolates, find the probability that

(ii) at least 3 but fewer than 8 boxes have at most one mis-shaped chocolate, [3]

(iii) more than 6.25% of the chocolates are mis-shaped using a suitable approximation. State the mean and variance of the distribution used in the approximation. [4]

11 **(a)** It is known that 30% of the adult residents of a city is aged 55 and above and 78% of the residents aged 55 and above access the internet daily. An adult resident is selected at random.

- E is the event that this person is aged 55 and older
- I is the event that this person accesses the internet daily

(i) Show that $\mathrm{P}(E \cap I) = 0.234$. [1]

(ii) Given further that $\mathrm{P}(E \cup I) = 0.986$, find $\mathrm{P}(I)$. [2]

(iii) State, with a reason, whether the events E and I are independent. [1]

(b) In a game that Royston played, he tossed two fair coins and a fair six-sided die. If the two fair coins both show heads, the score will be twice the number shown on the top of the die, otherwise the score will be the number shown on the top of the die. Find the probability that

(i) the score is at least 6, [2]

(ii) the score is not more than 8 given that the score is at least 6. [3]

12 Soap powder is packed in packets of two sizes. The mass in each small packet may be regarded as a normal variable with mean μ g and standard deviation 10 g while the mass in each large packet is another independent normal variable with mean 1005 g and standard deviation 20 g.

(i) Given the probability that a randomly chosen small packet has a mass more than 500 g is 0.7, show that $\mu = 505$ g. [2]

(ii) Three large packets are chosen at random. Find the probability that one of them has mass less than 1 kg while the other two have masses more than 1 kg. [3]

(iii) Find the probability that the total mass of two randomly chosen large packets is within 5 g of four times the mass of one randomly chosen small packet. [4]

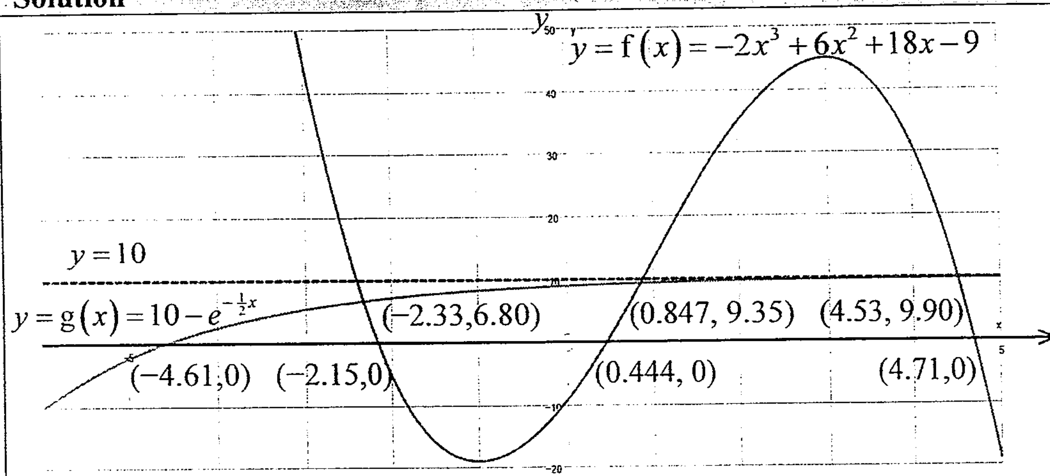
(iv) Five large packets and ten small packets are chosen at random and packed into a box for delivery to Baby Island Store. Find the probability that the average mass of all the fifteen packets chosen lies between 665 g and 675 g. [3]

2016 H1 Mathematics Preliminary Examination Solutions

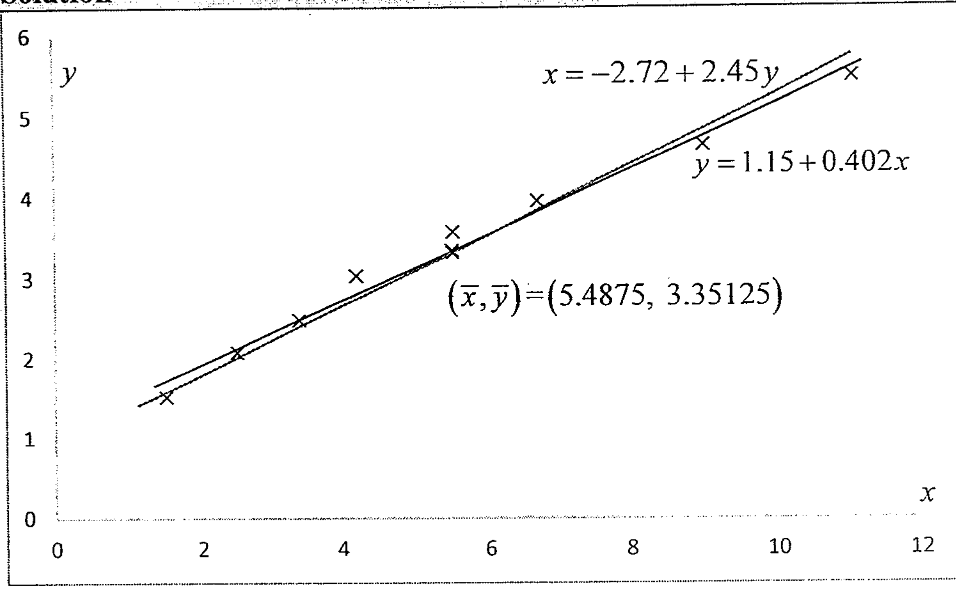
No	Solution
1	$\int_{-5}^{1} \frac{1}{\sqrt{4-x}}\,dx = \int_{-5}^{1} (4-x)^{-\frac{1}{2}}\,dx$ $= \left[\frac{(4-x)^{\frac{1}{2}}}{(-1)(\frac{1}{2})}\right]_{-5}^{1}$ $= \left[-2\sqrt{4-x}\right]_{-5}^{1}$ $= -2\sqrt{3} + 6$
2	$y = px + 5p$ ----------------------------(1) $y = -x^2 + (2p-1)x + 4p + 3$ -----------(2) Subst (1) into (2): $px + 5p = -x^2 + (2p-1)x + 4p + 3$ $-x^2 + (2p-1-p)x + 4p + 3 - 5p = 0$ $-x^2 + (p-1)x - p + 3 = 0$ --------------(3) Discriminant $= (p-1)^2 - 4(-1)(-p+3)$ $= p^2 - 2p + 1 - 4p + 12$ $= p^2 - 6p + 13$ $= (p-3)^2 + 4$ Since for all real values of p, $(p-3)^2 \geq 0 \Rightarrow$ Discriminant $(p-3)^2 + 4 > 0$ Equation (3) has two distinct solutions, the line (1) intersects the curve (2) at two points.
3(i)	$y = \ln(x\mathrm{e}^{-x}) - (2-x)^3 = \ln x - x - (2-x)^3$ $\frac{dy}{dx} = \frac{1}{x} - 1 + 3(2-x)^2$ At P, $x = 2$, $\frac{dy}{dx} = -\frac{1}{2}$
3(ii)	At P, $y = \ln 2 - 2$, eqn of normal: $y - (\ln 2 - 2) = 2(x-2)$ At A, $x = 0, y = \ln 2 - 6 \approx -5.30685$ At B, $y = 0, x = 3 - \frac{1}{2}\ln 2 \approx 2.65343$ Area of $\Delta OAB = \frac{1}{2}\left[-(\ln 2 - 6)\right]\left(3 - \frac{1}{2}\ln 2\right) = \frac{1}{4}(6 - \ln 2)^2 \approx 7.04$

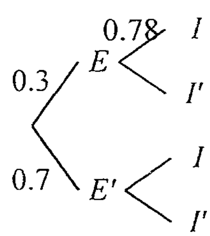
No	Solution
4(i)	Area of a triangular face $=\frac{1}{2}x^2\sin\frac{\pi}{3}=\frac{1}{2}x^2\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{4}x^2$ Volume of the container $=\frac{\sqrt{3}}{4}x^2h=10\Rightarrow h=\frac{40}{x^2\sqrt{3}}$ Area of a rectangular side $=xh=x\left(\frac{40}{x^2\sqrt{3}}\right)=\frac{40}{x\sqrt{3}}$ Total surface area of the container $S=2\left(\frac{\sqrt{3}}{4}x^2\right)+3\left(\frac{40}{x\sqrt{3}}\right)=\left(\frac{\sqrt{3}}{2}x^2+\frac{40\sqrt{3}}{x}\right)\text{cm}^2$ (Shown)
4(ii)	$\frac{dS}{dx}=\sqrt{3}x-\frac{40\sqrt{3}}{x^2}$ $\frac{dS}{dx}=0\Rightarrow\sqrt{3}x=\frac{40\sqrt{3}}{x^2}\Rightarrow x=\sqrt[3]{40}$ or 3.42 Table: x: $\left(\sqrt[3]{40}\right)^-$, $\sqrt[3]{40}$, $\left(\sqrt[3]{40}\right)^+$; $\frac{dS}{dx}$: −ve, 0, +ve; slope: \, _, / Alternative When $x=\sqrt[3]{40}$, $\frac{d^2S}{dx^2}=\sqrt{3}+\frac{80\sqrt{3}}{x^3}>0$ Cost is minimised when $x=\sqrt[3]{40}$ or 3.42cm
4(iii)	Any other value of x will result in higher production cost for the same volume of the container.
5(i)	$y=ax^3+6x^2+18x+b$ $\frac{dy}{dx}=3ax^2+12x+18$ When $x=1$, $y=13$, gradient = 24 $24=3a+12+18$ $3a=-6$ $a=-2$ (Shown) When $x=1$, $y=13$, $a=-2$ $13=-2+6+18+b$ $b=-9$ (Shown)

x	$\left(\sqrt[3]{40}\right)^-$	$\sqrt[3]{40}$	$\left(\sqrt[3]{40}\right)^+$
$\frac{dS}{dx}$	−ve	0	+ve
	\	_	/

No	Solution
5(ii)	 $y = -2x^3 + 6x^2 + 18x - 9$ Axial intercepts (−2.15,0), (0.444, 0), (4.71, 0), (0,−9) $y = 10 - e^{-\frac{1}{2}x}$ Asymptote: $y = 10$, Axial intercepts (−4.61,0), (0, 9) Points of Intersection: (−2.33,6.80), (0.847, 9.35), (4.53, 9.90)
5(iii)	For $f(x) > g(x)$, $x < -2.33$ or $0.847 < x < 4.53$
5(iv)	$\int_2^3 -2x^3 + 6x^2 + 18x - 9 - \left(10 - e^{-\frac{1}{2}x}\right) dx$ $= \int_2^3 -2x^3 + 6x^2 + 18x - 19 + e^{-\frac{1}{2}x}\, dx$ $= \left[\frac{-2x^4}{4} + \frac{6x^3}{3} + \frac{18x^2}{2} - 19x + \frac{e^{-\frac{1}{2}x}}{\left(-\frac{1}{2}\right)}\right]_2^3$ $= \left[-\frac{1}{2}x^4 + 2x^3 + 9x^2 - 19x - 2e^{-\frac{1}{2}x}\right]_2^3$ $= \left[-\frac{1}{2}(3)^4 + 2(3)^3 + 9(3)^2 - 19(3) - 2e^{-\frac{3}{2}}\right]$ $- \left[-\frac{1}{2}(2)^4 + 2(2)^3 + 9(2)^2 - 19(2) - 2e^{-1}\right]$ $= 31\frac{1}{2} - 2e^{-\frac{3}{2}} + 2e^{-1}$
6(i)	From the school's registration list for Year 2, **randomly select** a student from the first 7 on the list as a starting point. Then select **every 7th** student down the list from that starting point, until 100 names are obtained.
6(ii)	The sample obtained **may not be representative of the population**.
6(iii)	Stratified sampling.

No	Solution
7	Let X = weight of students and μ = mean weight $H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$ Under H_0, $\overline{X} \sim N\left(\mu_0, \frac{7^2}{250}\right)$ Test statistics $Z = \frac{\overline{X} - \mu_0}{\sqrt{\frac{49}{250}}} \sim N(0,1)$ $\overline{x} = 70 + \frac{900}{250} = 73.6$ (exact) At 5%, we reject H_0 if $z \leq -1.9600$ or $z \geq 1.9600$. Since we do not reject H_0 $-1.9600 < \frac{73.6 - \mu_0}{\sqrt{\frac{49}{250}}} < 1.9600$ $72.7 < \mu_0 < 74.5$ (3 s.f.)
8(i)	Let X = weight lost and μ = average weight lost in a month $s^2 = \frac{180}{179}(2.8)^2 = \frac{7056}{895}$ $H_0 : \mu = 3$ $H_1 : \mu < 3$ Under H_0, since n is large, by Central Limit Theorem, $\overline{X} \sim N\left(3, \frac{\frac{7065}{895}}{180}\right) = N\left(3, \frac{196}{4475}\right)$ approximately Test statistics $Z = \frac{\overline{X} - 3}{\sqrt{\frac{196}{4475}}} \sim N(0,1)$ approximately p-value = 0.075861 ≈ 0.0759 ($z = -1.4335$) Since p-value = 0.0759 > α = 0.05, we do not reject H_0 at 5% level of significance and conclude that there is insufficient evidence that the average weight loss is less than 3 kg. (insufficient evidence that the authors' claim is not valid so the claim could be true).
(ii)	We reject H_0 for the authors' claim to be not valid p-value $= 0.075861 \leq \frac{\alpha}{100} \Rightarrow \alpha \geq 7.5861$ Least value of α = 7.59 (3 s.f.)
(iii)	There is a probability of $\frac{\alpha}{100}$ of concluding that the average weight lost in a month is less than 3 kg when the average weight lost is 3 kg.

No	Solution
9(i)	
9(ii)	$r=0.992$ (3 s.f.) $\bar{x}=5.4875$ (exact) $\bar{y}=3.35125$ (exact)
9(iii)	$x=-2.7229+2.44996y$ $x=-2.72+2.45y$ [Draw: $y=\dfrac{1}{2.44996}x+\dfrac{2.7229}{2.44996}=0.408x+1.11$]
9(iv)	(a) When $x=15$, $y=1.15+0.402(15)=7.18$ The expenditure is \$7180 (b) When $y=4$, $x=-2.7229+2.44996(1.8)=1.6870\approx1.7$ (1 d.p.) The income is \$1 700
9(v)	Since $x=15$ is outside the data range, we are doing extrapolation, the estimate is not reliable Since $y=1.8$ is within the data range and r is close to 1 indicate that there is strong positive linear relationship between monthly household income and household expenditure, the estimate is reliable.
10	Let X be the number of mis-shaped chocolates in a box. Then $X\sim\mathrm{B}(24,p)$. $\mathrm{P}(X\le1)=0.16215$ Using GC, $p=0.130$. <u>Alternative</u> $\mathrm{P}(X=0)+\mathrm{P}(X=1)=0.16215$ $\binom{24}{0}p^0(1-p)^{24}+\binom{24}{1}p(1-p)^{23}=0.16215$ $(1-p)^{24}+24p(1-p)^{23}=0.16215$ Using GC, $p=0.130$.

No	Solution
10(i)	$X \sim \mathrm{B}(24, 0.05)$ $\mathrm{P}(X>1)=1-\mathrm{P}(X\le 1)=1-0.66082=0.33918\approx 0.339$
10(ii)	Let Y be the number of boxes with at most one mis-shaped chocolate $Y\sim \mathrm{B}(10, 0.66082)$ $\mathrm{P}(3\le Y<8)=\mathrm{P}(Y\le 7)-\mathrm{P}(Y\le 2)=0.71437-0.0038548=0.71051\approx 0.711$ <u>Alternative</u> $\mathrm{P}(3\le Y<8)=\mathrm{P}(Y=3)+\mathrm{P}(Y=4)+\mathrm{P}(Y=5)+\mathrm{P}(Y=6)+\mathrm{P}(Y=7)$ $=0.71051\approx 0.711$
10(iii)	$24\times 10=240$ and $0.0625\times 240=15$ Let W be the number of mis-shaped chocolates in 10 boxes $W\sim \mathrm{B}(240, 0.05)$ Since $n=240$ is large, $np=12>5$, $nq=228>5$ $W\sim N(12, 11.4)$ approximately $\mathrm{P}(W>15)\xrightarrow{cc}\mathrm{P}(W>15.5)=0.14996\approx 0.150$
11(a)(i)	$\mathrm{P}(E\cap I)=\mathrm{P}(E)\times \mathrm{P}(I\mid E)=0.3\times 0.78=0.234$ (Shown) 
11(a)(ii)	$\mathrm{P}(E\cup I)=\mathrm{P}(E)+\mathrm{P}(I)-\mathrm{P}(E\cap I)$ $0.986=0.3+\mathrm{P}(I)-0.234$ $\mathrm{P}(I)=0.92$
11(a)(iii)	E and I are not independent since $\mathrm{P}(I)\ne \mathrm{P}(I\mid E)=0.78$ or $\mathrm{P}(E)\times \mathrm{P}(I)=0.3\times 0.92=0.276\ne \mathrm{P}(E\cap I)=0.234$
11(b)(i)	P (Score is at least 6) = P(HH, die shows 3, 4, 5, 6) + P(HTor TH or TT, die shows 6)) $=\left(\frac{1}{4}\times\frac{4}{6}\right)+\left(\frac{3}{4}\times\frac{1}{6}\right)=\frac{7}{24}$ (exact) or 0.292 (3sf) <u>Alternative</u> P (Score is at least 6) = 1 − P(Score is at most 5) = 1 − P(HH, die shows 1, 2) − P(HTor TH or TT, die shows 1, 2, 3, 4, 5) $=1-\left(\frac{1}{4}\times\frac{2}{6}\right)-\left(\frac{3}{4}\times\frac{5}{6}\right)=\frac{7}{24}$ (exact) or 0.292 (3sf)

No	Solution

11(b)(ii)

P (score is not more than 8 | score is at least 6)

$$= \frac{P(6 \le \text{score} \le 8)}{P(\text{score} \ge 6)}$$

$$= \frac{P(\text{HH, die shows } 3, 4) + P(\text{HT or TH or TT, die shows } 6)}{P(\text{score} \ge 6)}$$

$$= \frac{\left(\frac{1}{4}\times\frac{2}{6}\right)+\left(\frac{3}{4}\times\frac{1}{6}\right)}{\frac{7}{24}} = \frac{5}{7} \text{ (exact)} \quad \text{or } 0.714 \text{ (3sf)}$$

Alternative for Q11(b)

Scores

Die / Coins	**1**	**2**	**3**	**4**	**5**	**6**
HH	2	4	6	8	10	12
HT	1	2	3	4	5	6
TH	1	2	3	4	5	6
TT	1	2	3	4	5	6

(i) P (Score is at least 6) $= \frac{1}{4}\times\frac{1}{6}\times 7 = \frac{7}{24}$

(ii) P (score is not more than 8 | score is at least 6) $= \frac{5}{7}$

12

Let S = Mass of soap powder in a small packet,

and L = Mass of soap powder in a large packet.

Then $S \sim N(\mu, 10^2)$ and $L \sim N(1005, 20^2)$.

12(i)

$$P(S > 500) = 0.7$$

$$P(S \le 500) = 0.3$$

$$P\left(Z \le \frac{500-\mu}{10}\right) = 0.3$$

$$\frac{500-\mu}{10} = -0.5244$$

$\mu = 505$ (Shown)

12(ii)

$$P(L < 1000) = 0.40129$$

Required probability

$$= 3P(L<1000)\left[P(L>1000)\right]^2$$

$$= 3(0.40129)(1-0.40129)^2$$

$= 0.43153 \approx 0.432$ (3 s.f.)

No	Solution
12(iii)	$L_1 + L_2 - 4S \sim N\left(2(1005) - 4(505), 2(20)^2 + 4^2(10)^2\right) = N(-10, 2400)$ $\mathrm{P}(-5 < L_1 + L_2 - 4S < 5) = 0.079622 \approx 0.0796$ (3 s.f.)
12(iv)	Average mass of all the fifteen packets chosen is $M = \frac{1}{15}(L_1 + \ldots + L_5 + S_1 + \ldots + S_{10})$ $\mathrm{E}(M) = \frac{1}{15}\left[5(1005) + 10(505)\right] = \frac{2015}{3}$ $\mathrm{Var}(M) = \left(\frac{1}{15}\right)^2\left[5(20^2) + 10(10^2)\right] = \frac{40}{3}$ $M \sim N\left(\frac{2015}{3}, \frac{40}{3}\right)$ $\mathrm{P}(665 < M < 675) = 0.785$