### JC2 PRELIMINARY EXAMINATION

Higher 1

#### MATHEMATICS 8864/01

Paper 1 15 September 2016

3 hours

Additional Materials:

Cover Page

Answer Paper

List of Formulae (MF15)

#### READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use a soft pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

#### Answer all the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [] at the end of each question or part question.

### Section A: Pure Mathematics [35 marks]

1 Find, algebraically, the range of values of x for which 
$$\frac{3}{4^x} + \frac{7}{2^x} - 6 < 0$$
. [4]

Sketch, on a single diagram, the graphs of y = ln(1-4x²) and y = x² + 2x - 5, stating clearly the equations of any asymptotes and the coordinates of the point(s) of intersection between the two graphs.
[4]
Find the numerical value of the area of the region between the two graphs.
[2]

3 (a) Differentiate 
$$\frac{1}{\sqrt{1-2x}}$$
. [2]

- **(b)** Use a non-calculator method to find the exact value of  $\int_0^1 (e^{1-x} + e^x)^2 dx$ . [5]
- 4 [It is given that a closed cylinder of base radius r and height h has volume  $\pi r^2 h$  and total external surface area  $2\pi r^2 + 2\pi r h$ ]
  - (a) A company requires a closed, cylindrical tin can to hold 330 cm<sup>3</sup> of liquid when full. The base radius of the can is r cm and the height is h cm. Assuming that the tin can has negligible thickness, by expressing h in terms of r, find the value of r which gives a minimum total external surface area of the can. State the corresponding minimum total external surface area and explain briefly how you can tell that it is a minimum rather than a maximum.
  - (b) In the manufacturing of another closed cylindrical can, the can is heated such that the volume increases at a constant rate of  $\pi$  cm<sup>3</sup>s<sup>-1</sup> while the height of the can is such that it is always twice the radius of the can. Assuming that the can has negligible thickness. find the exact rate at which the radius is increasing at the instant when the radius is 3 cm.

- 5 The curve C has equation  $y = e^{3-x} 3x + 1$ .
  - (i) Sketch C, stating the coordinates of the point(s) of intersection with the axes. [3]
  - (ii) Find the equation of the normal to the curve at the point P where x = 3, giving your answer in the form ax + by = c, where a, b and c are integers to be determined. [3]
  - (iii) The normal at P meets the y-axis at the point N. Find the exact area of the triangle ONP.

### Section B: Statistics [60 marks]

The random variable X has a normal distribution with mean 79 and standard deviation  $\sigma$ . It is known that  $P(79 - a \le X \le 79 + b) = 0.6463$ , where a and b are positive constants.

Given that  $P(X \ge 79 + b) = 2P(X \le 79 - a)$ .

(i) Find the value of 
$$P(X < 79 - a)$$
. [1]

- (ii) Find the value of b when  $\sigma = 5$ . [3]
- 7 A and B are two events such that P(A) = p,  $P(B) = \frac{2}{3}$  and  $P(B|A') = \frac{1}{2}p$ .
  - (i) Determine whether the events A and B are independent, justifying your answer. [2]
  - (ii) Given that  $P(A \cap B) = \frac{5}{9}$ , show that  $9p^2 9p + 2 = 0$  and hence find the value of p.[3]
- A group of students from NYJC Green Club intend to conduct a survey to find out the opinion of residents in Braddell Heights Estate on whether they would like eco-friendly features to be introduced to their homes.
  - (i) Describe how the club might obtain a quota sample of size 130. [2]
  - (ii) State one disadvantage of obtaining the sample using this sampling method. [1]
  - (iii) Another group of students from the club suggested to use simple random sampling to conduct the survey instead. State one advantage of this sampling method and explain why it would not be appropriate to use this method in this context.[2]

- During the first day of the holidays, Roy plans to revise Mathematics each morning. However, sometimes he does not revise in the morning. The probability that he will revise Mathematics the first morning is 0.8. Thereafter, if he had revised that morning, the probability that he will revise the next morning will be 0.7. If he did not revise that morning, the probability that he will revise the next morning will be 0.9.
  - (i) Draw a tree diagram to represent the possible outcomes for the first two mornings. [2]
  - (ii) Find the probability that Roy revised on one out of the two mornings. [2].
  - (iii) Given that Roy revised on at least one out of the two mornings, find the probability that he revised on the second morning. [3]

It is known that the rank points y, obtained by students is dependent on the number of hours, x, spent on revision per day. The results of a random sample of 9 students are given in the following table.

| Student | A   | В   | C   | D   | E   | F   | G   | Н   | I   |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| x       | 4.3 | 5.1 | 3.1 | 6.3 | 4.7 | 3.2 | 7.2 | 6.4 | 8.0 |
| у       | 68  | 74  | 64  | 76  | 75  | 66  | 85  | 81  | 86  |

- (i) Give a sketch of the scatter diagram for the data, as shown on your calculator. [2]
- (ii) Find the product moment correlation coefficient and comment on its value in the context of the data. [2]
- (iii) Find the equation of the regression line of y on x, in the form y = mx + c, giving the values of m and c correct to 4 significant figures and explain the meaning of m in this context.
- iv) Sketch this regression line on your scatter diagram. [1]
- (v) Estimate the number of hours a student spent on revision per day if he/she obtained 90 rank points. Comment on the reliability of your estimate. [2]

- In clinical trials, a certain drug has a 28% success rate of curing a known disease.
  - (i) If this drug is administered to 15 patients in a hospital who have the disease, find the probability that
    - (a) no patients will be cured, [1]
    - (b) at least 5 patients will be cured. [2]
  - (ii) The drug is now administered to 120 patients. Using a suitable approximation, estimate the probability that the number of patients that will be cured is between 30 and 40 inclusive. You should state the mean and variance of any distribution that you use.
  - (iii) This drug is later used by 24 hospitals. In each hospital, it is administered to 15 patients who have the disease. Find the probability that more than one third of the hospitals have at least 5 patients who will be cured.
    [3]

A shop that sells cupcakes claims that the mean mass of a cupcake is 100 grams. A random sample of 80 cupcakes is weighed and the mass, x grams, of each cupcake is recorded. The results are summarised by

$$\sum (x-100) = -19.2, \quad \sum (x-100)^2 = 129.3$$

- (i) Find unbiased estimates of the population mean and variance, giving your answers correct to 2 decimal places. [2]
- (ii) A customer complained that the shop overestimated the mass of their cupcakes. Test. at the 5% significance level, whether the complaint is valid. [4]
- (iii) Another large sample of n cupcakes is taken. Assume that the sample mean and standard deviation remain unchanged, find the largest possible value of n for the customer's complaint to be not valid at the 5% level of significance.
  [4]

- The mass of eggs produced by an organic farm follows a normal distribution with mean 55 grams and standard deviation 3.9 grams.
  - (i) Find the probability that an egg chosen at random has mass within 5 grams of the mean. [1]
  - (ii) Find the probability that the masses of two randomly chosen eggs differ by more than 4 grams. [3]

Six randomly chosen eggs are packed together in a cardboard box. The mass of an empty box is normally distributed with mean 28 grams and standard deviation 1.2 grams. The masses of the eggs and box are independent.

(iii) Find the probability that the total mass of a box containing six eggs is less than 350 grams. [3]

The eggs are sold at \$0.004 per gram and the box is charged at a fixed price of \$0.10.

(iv) Find the probability that the total selling price of a randomly chosen box containing six eggs is more than \$1.40. [3]

# 2016 JC 2 H1 Mathematics Preliminary Exam solutions

| 2016   | JC 2 H1 Mathematics Preliminary Exam solutions  |
|--------|---|
| Q<br>1 | Answers   |
| 1      | $\frac{3}{4^x} + \frac{7}{2^x} - 6 < 0$   |
|        | Let $y = 2^x$   |
|        | $\frac{3}{v^2} + \frac{7}{v} - 6 < 0$   |
|        |   |
|        | $3+7y-6y^2 < 0$   |
|        | $y < -\frac{1}{3} \text{ or } y > \frac{3}{2}$  |
|        | $2^{x} < -\frac{1}{3}$ (rejected) or $2^{x} > \frac{3}{2}$  |
|        | $x > \log_2 \frac{3}{2}$  |
|        | Alternative:  |
|        | $\frac{3}{4^x} + \frac{7}{2^x} - 6 < 0$   |
|        | <b>'</b>  |
|        | Let $y = \frac{1}{2^x}$   |
|        | $3y^2 + 7y - 6 < 0$   |
|        | $-3 < y < \frac{2}{3}$  |
|        | $-3 < \frac{1}{2^x} < \frac{2}{3}$ .  |
|        | $\Rightarrow 0 < \frac{1}{2^x} < \frac{2}{3}  \text{as}  2^x > 0$   |
|        |   |
|        | $2^{x} > \frac{3}{2} \implies x > \log_2 \frac{3}{2}$   |
|        | $\ln\left(\frac{3}{2}\right)$   |
|        | Alternative answers accepted: $x > \frac{\ln\left(\frac{3}{2}\right)}{\ln 2}$ or $x > \frac{\ln 3}{\ln 2} - 1$ or $x > 0.585$ |
| 2      | i y i   |
|        |   |
|        | $\sqrt{y} = x^2 + 2x - 5$   |
|        |   |
|        | -3.4495 $0$ $1.4495$  |
|        |   |
|        | $y = \ln\left(1 - 4x^{\frac{1}{2}}\right)$  |
|        | (0.4943.77)   |

(-0.499, -5.75)

Area required = 
$$\int_{-0.499}^{0.494} \ln(1-4x^2) - (x^2 + 2x - 5) dx$$
  
  $\approx 4.31 \text{ sq units}$ 

3 (a) 
$$\frac{d}{dx} \left( \frac{1}{\sqrt{1-2x}} \right) = -\frac{1}{2} (1-2x)^{\frac{3}{2}} (-2)$$
  
=  $(1-2x)^{\frac{3}{2}}$ 

(b) 
$$\int_0^1 (e^{1-x} + e^x)^2 dx$$

$$= \int_0^1 (e^{2-2x} + 2e + e^{2x}) dx$$

$$= \left[ \frac{e^{2-2x}}{-2} + 2ex + \frac{e^{2x}}{2} \right]_0^1 \cdot \left[ -\frac{1}{2} + 2e + \frac{e^2}{2} \right] - \left( -\frac{e^2}{2} + \frac{1}{2} \right)$$

$$= -1 + 2e + e^2$$

4 (a) 
$$V = \pi r^2 h$$
  

$$330 = \pi r^2 h$$

$$h = \frac{330}{\pi r^2}$$

$$S = 2\pi r^{2} + 2\pi rh$$
$$S = 2\pi r^{2} + 2\pi r \frac{330}{\pi r^{2}}$$

$$S = 2\pi r^2 + \frac{660}{r}$$

$$\frac{\mathrm{d}S}{\mathrm{d}r} = 4\pi r - \frac{660}{r^2}$$

For minimum S,  $\frac{dS}{dr} = 0$ 

$$4\pi r - \frac{660}{r^2} = 0$$

$$r^3 = \frac{660}{4\pi}$$

 $r \approx 3.74 \text{ cm}$ 

| r             | 3.74- | 3.74 | 3.74+ |
|---------------|-------|------|-------|
| <u>dS</u>     | -ve   | 0    | + ve  |
| $\mathrm{d}r$ |       |      | _     |
| Slope         | 1     | _    | /     |

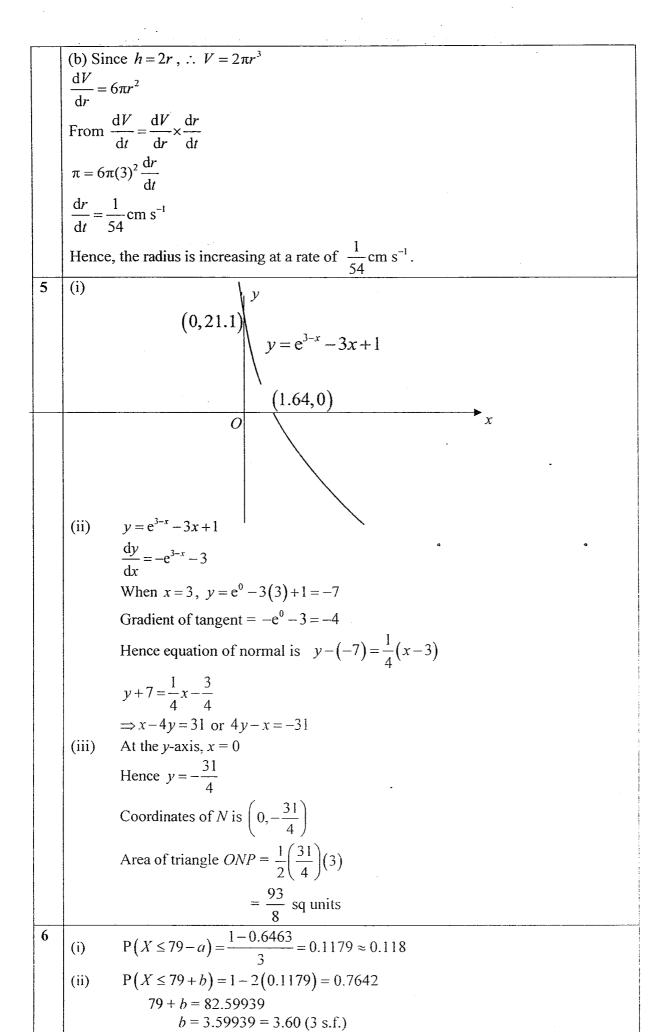
S is minimum when r = 3.74 cm

Alternative:

$$\frac{d^2S}{dr^2} = 4\pi + \frac{1320}{r^3} > 0 \text{ when } r = 3.74$$

 $\Rightarrow$  S is minimum when r = 3.74 cm

Hence, minimum  $S \approx 264 \text{ cm}^2$ .



$$P\left(Z \le \frac{(79+b)-79}{5}\right) = 0.7642$$

$$\frac{b}{5} = 0.71988$$

$$b = 5(0.71988) = 3.5994 = 3.60 \quad (3 \text{ s.f.})$$

7 (i) If A and B are independent,

$$P(B \mid A') = P(B)$$

$$\frac{1}{2}p = \frac{2}{3} \Rightarrow p = \frac{4}{3}$$
 which is impossible.

Hence, A and B are not independent.

### Alternative:

$$P(B \mid A') = \frac{P(B \cap A')}{P(A')}$$

$$\frac{1}{2}p = \frac{\frac{2}{3} - P(A \cap B)}{1 - p}$$

$$\frac{1}{2}p - \frac{1}{2}p^2 = \frac{2}{3} - P(A \cap B)$$

If A and B are independent,  $P(A \cap B) = P(A) \times P(B) = \frac{2}{3}p$ 

$$\frac{1}{2}p - \frac{1}{2}p^2 = \frac{2}{3} - \frac{2}{3}p$$

$$3p^2 - 7p + 4 = 0$$

$$p = \frac{4}{3}$$
 (NA) or  $p = 1$  (NA)

Hence, A and B are not independent.

(ii) 
$$P(B \mid A') = \frac{1}{2}p$$

$$\frac{P(B \cap A')}{P(A')} = \frac{1}{2}p$$

$$\frac{P(B) - P(A \cap B)}{1 - P(A)} = \frac{1}{2}p$$

$$\frac{\frac{2}{3} - \frac{5}{9}}{1 - p} = \frac{1}{2}p$$

$$\frac{1}{9} = \frac{1}{2}p(1-p)$$

$$2 = 9p - 9p^2$$

$$9p^2 - 9p + 2 = 0$$
 (shown)

Using GC, 
$$p = \frac{2}{3}$$
 or  $p = \frac{1}{3}$  (rejected)

8 (i) He could select 130 people according to the following mutually exclusive subgroups.

| And the second s | and the second s |           |           |       |
|--|--|-----------|-----------|-------|
| 15 years   | 16 to 30   | 31 to 45  | 46 to 60  | 60    |
| and  | years old  | years old | years old | years |
| below  |  |           |           | and   |
|  |  |           |           | above |
| 20   | 30   | 30        | 30        | 20    |

He could station himself at a busy street in the estate and is free to interview the residents until each quota has been obtained.

(ii) The interviewer could be biased in his selection of residents to be interviewed as he may only get to interview those residents who are cooperative.

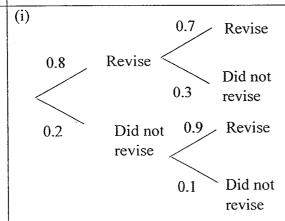
OR

The sample chosen might not be representative of the population in this particular estate.

(iii) It is free from bias.

It is not appropriate to use this method in this context as it is difficult to obtain the sampling frame.





- (ii) P(revised one out of the two mornings)
- $= 0.8 \times 0.3 + 0.2 \times 0.9$
- = 0.42
- (iii) P(revised on the 2nd morning | revised at least 1 out of the 2 mornings)

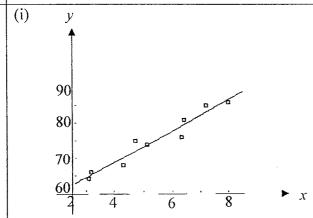
P(revised on 2nd morning & revised at least 1 out of 2 mornings)

P(revised at least 1 out of 2 mornings)

$$=\frac{0.8\times0.7+0.2\times0.9}{0.42+0.8\times0.7}$$

= 0.755

# 10



(ii) 
$$r = 0.969$$
 (to 3 sig.fig.)

As the number of hours spent on revision per day increases, the rank points obtained by the student increases in a strong linear manner.

(iii) 
$$y = 4.494x + 50.88$$
 (to 4 sig.fig.)

When the number of hours spent on revision per day increases by 1, the rank points obtained increases by 4.494.

- (iv) See scatter diagram
- (v) Use y = 4.494x + 50.88 as it is obvious that rank points is dependent on the no. of hours spent on revision. So for y = 90, x = 8.70 (to 3 sig.fig.)

The estimate is unreliable as the value of y = 90 lies outside the data range (it is an extrapolation).

(i) Let X be the random variable denoting number of patients out of 15 that will be cured.

$$X \sim B(15, 0.28)$$

(a) 
$$P(X = 0) \approx 0.00724$$

(b) 
$$P(X \ge 5) \approx 1 - P(X \le 4) \approx 0.41545 \approx 0.415$$

(ii) Let Y be the random variable denoting number of patients out of 120 that will be cured.

$$Y \sim B(120, 0.28)$$

Since 
$$n = 120$$
 is large,  $np = 33.6 > 5$  and  $nq = 86.4 > 5$ ,

$$Y \sim N\left(33.6, \left(\sqrt{24.192}\right)^2\right)$$
 approximately

$$P(30 \le Y \le 40) = P(29.5 < Y < 40.5)$$
 (continuity correction)  
= 0.717 (to 3 sig.fig.)

(iii) Let *H* be the random variable denoting number of hospitals out of 24 with at least two patients that will be cured.

$$H \sim B(24, 0.41545)$$
  
 $H > 8) = 1 - P(H < 8)$ 

$$P(H > 8) = 1 - P(H \le 8)$$
  
= 0.726 (to 3 sig.fig.)

12 (i) Unbiased estimates of the population mean,

$$\overline{x} = 100 + \frac{1}{80}\Sigma(x - 100)$$

$$=100 + \frac{1}{80} \left(-19.2\right)$$

$$=99.76$$

Unbiased estimates of the population variance,

$$s^{2} = \frac{1}{79} \left( \sum (x - 100)^{2} - \frac{\left(\sum (x - 100)\right)^{2}}{80} \right)$$
$$= \frac{1}{79} \left( 129.3 - \frac{\left(-19.2\right)^{2}}{80} \right)$$

 $=1.5784 \approx 1.58$ 

(ii) Let X be the random variable denoting the mass of a cupcake.

To test  $H_0$ :  $\mu = 100$ 

 $H_1: \mu < 100$ 

Significance level: 5%

Test Statistics: Under  $H_0$ , by Central Limit Theorem,

$$\overline{X} \sim N\left(100, \frac{1.5784}{80}\right)$$
 approximately.

Rejection criteria: Reject  $H_0$  if p-value < 0.05

$$\overline{x}$$
 = 99.76,  $n$  = 80,  $s$  =  $\sqrt{1.5784}$ 

Using GC, p-value = 0.043760

Since p-value < 0.05, there is sufficient evidence to reject  $H_0$  at 5% level of significance, i.e. there is sufficient evidence to conclude that the shop has overestimated the mean mass of a cupcake, ie. complaint is valid.

(iii) Let X be the random variable denoting the mass of a cupcake.

To test  $H_0$ :  $\mu = 100$ 

 $H_1: \mu < 100$ 

Significance level: 5%

Test Statistics: Under  $H_0$ , by Central Limit Theorem,

$$\overline{X} \sim N\left(100, \frac{1.5784}{n}\right)$$
 approximately.

Rejection criteria: Reject  $H_0$  if p-value < 0.05

For the customer's complaint to be not valid,  $H_0$  not to be rejected.

p-value  $\geq 0.05$ 

$$P(\overline{X} \le 99.76) \ge 0.05$$

$$P\left(\frac{\overline{X} - 100}{\left(\sqrt{\frac{1.5784}{n}}\right)} \le \frac{99.76 - 100}{\left(\sqrt{\frac{1.5784}{n}}\right)} \ge 0.05$$

$$P\left(Z \le \frac{-0.24}{\left(\sqrt{\frac{1.5784}{n}}\right)} \ge 0.05$$

$$\frac{-0.24}{\left(\sqrt{\frac{1.5784}{n}}\right)} \ge -1.64485$$

$$\frac{0.24}{\left(\sqrt{\frac{1.5784}{n}}\right)} \le 1.64485$$

$$\sqrt{n} \le \left(\frac{1.64485}{0.24}\right) \sqrt{1.5784}$$

$$n \le 74.139$$
Largest possible *n* is 74.

- 13 (i) Let X be the random variable denoting the mass of an egg i.e.  $X \sim N(55, 3.9^2)$ Required probability is P(50 < X < 60) = 0.800
  - (ii)Required probability is  $P(|X_1 X_2| > 4)$

$$E(X_1 - X_2) = E(X_1) - E(X_2) = E(X) - E(X) = 0$$

$$Var(X_1 - X_2) = Var(X_1) + Var(X_2)$$

$$= 2 \operatorname{Var}(X) = 2(3.9^2) = 30.42$$

i.e. 
$$X_1 - X_2 \sim N(0, (\sqrt{30.42})^2)$$

$$P(|X_1 - X_2| > 4) = P(X_1 - X_2 < -4) + P(X_1 - X_2 > 4)$$

$$= 0.46831 = 0.468$$
 (3 s.f.)

OR 
$$P(|X_1 - X_2| > 4) = 1 - P(|X_1 - X_2| \le 4)$$

$$=1-P(-4 \le X_1 - X_2 \le 4)$$

$$= 0.46831 = 0.468 (3 \text{ s.f.})$$

(iii) Let Y be the random variable denoting the mass of an empty box i.e.  $Y \sim N(28, 1.2^2)$ 

Required probability is

$$P(X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + Y < 350) = P(T < 350)$$

$$E(T) = E(Y) + 6E(X) = 28 + 6(55) = 358$$

$$Var(T) = Var(Y) + 6Var(X) = 1.2^2 + 6(3.9^2) = 92.7$$

i.e. 
$$T \sim N(358, (\sqrt{92.7})^2)$$

$$\therefore$$
 P(T < 350) = 0.20301 = 0.203 (3 s.f.)

(iv) Let selling price be  $C = 0.004(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) + 0.1$ 

Required probability is P(C > 1.40)

$$E(C) = E[0.004(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) + 0.1]$$
  
= 0.004[6E(X)]+0.1

$$=0.004(6)(55)+0.1$$

$$=1.42$$

$$Var(C) = Var[0.004(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) + 0.1]$$

$$= 0.004^2 [6E(X)] + 0$$

$$= 0.004^2 (6)(3.9^2)$$

$$= 0.00146016$$
i.e.  $C \sim N(1.42, (\sqrt{0.00146016})^2)$ 

$$\therefore P(C > 1.40) = 0.700 \quad (3 \text{ s.f.})$$

# Alternative solution:

$$C = 0.004(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) + 0.1$$

Let 
$$G = 0.004(X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$$

Required probability is P(C > 1.40) = P(G > 1.30)

$$E(G) = E[0.004(X_1 + X_2 + X_3 + X_4 + X_5 + X_6)]$$
  
= 0.004[6E(X)]

$$=0.004(6)(55)=1.32$$

$$Var(C) = Var[0.004(X_1 + X_2 + X_3 + X_4 + X_5 + X_6)]$$
  
= 0.004<sup>2</sup> \[ 6E(X) \]

$$= 0.004^{2} (6)(3.9^{2})$$

$$= 0.00146016$$

i.e. 
$$G \sim N\left(1.32, \left(\sqrt{0.00146016}\right)^2\right)$$

$$\therefore P(G > 1.30) = 0.700 \quad (3 \text{ s.f.})$$