Candidate Name: ______________________ Class: ____________

JC2 PRELIMINARY EXAMINATION

Higher 1

MATHEMATICS **8864/01**

Paper 1

14 September 2016

3 hours

Additional Materials: Cover page
Answer papers
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your full name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [35 marks]

1 **(i)** On a single diagram, sketch the graphs of $y=\dfrac{1-2x}{x+8}$ and $y=\mathrm{e}^x$. [2]

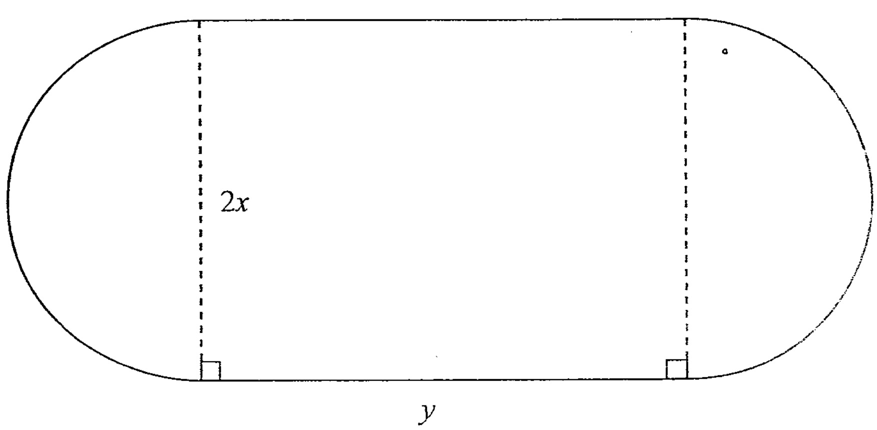
(ii) Solve the inequality $\dfrac{1-2x}{x+8}<\mathrm{e}^x$. Hence find the solutions of the inequality $\dfrac{\mathrm{e}^x(1+2x)}{8-x}<1$. [3]

2 The curve C has equation $y=x^3+4x^2+kx+k^2$, where k is a constant.

(i) Find, in terms of k, the equation of the tangent to C at the point P where $x=1$. [3]

(ii) Show that the tangent at P does not pass through the point $(2,1)$. [4]

3



A new flower-bed is being designed for a large garden. The flower-bed will occupy a rectangle $2x$ m by y m, flanked by two semicircles of diameter $2x$ m, as shown in the diagram. A low wall is to be built around the flower-bed. It costs \$100 per metre for the straight parts and \$125 per metre for the semicircular parts. The total cost of building the low wall is \$$C$. It is given that the area of the entire flower-bed is 2500π m^2.

(i) Find an expression for C in terms of x. [3]

(ii) Using differentiation, find the value of x for which C is a minimum. [4]

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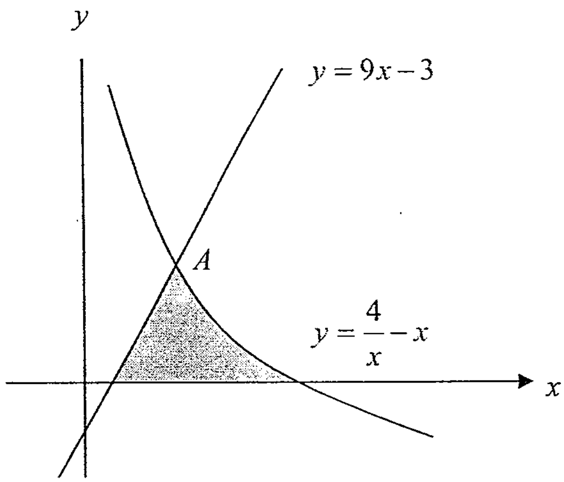


The diagram shows the line $y=9x-3$ intersecting the curve $y=\frac{4}{x}-x$ in the first quadrant at point A.

(i) Without using the calculator, find the x-coordinate of A. [3]

(ii) Find $\int \left(\frac{4}{x}-x\right) \mathrm{d}x$. [1]

(iii) Use your answers to parts **(i)** and **(ii)** to find the area of the shaded region. [4]

5 The curve C has equation $y=\ln(2x-1)-x^2$.

(i) Use a non-calculator method to find the coordinates of the stationary point of C. [3]

(ii) Sketch C, stating the coordinates of the stationary point and the equations of any asymptotes. [2]

(iii) By sketching a suitable graph on the same diagram, determine the number of real roots of the equation $x\left(\ln(2x-1)-x^2\right)+6=0$. [3]

Section B: Statistics [60 marks]

6 In a junior college, the number of male and female students in each level is summarised in the table below.

	Male	Female
Year 1	350	450
Year 2	300	400

The school administrative manager wishes to carry out a survey to gather students' feedback on the school facilities. She decides to select a random sample of 150 students.

(i) Describe how she may obtain a systematic sample. [2]

(ii) Describe how she may obtain a stratified sample. [2]

(iii) State, with a reason, whether a systematic sample or a stratified sample would be more appropriate in this context. [1]

7 A group of adults are asked whether they jog, swim or cycle once a week.

R is the event that the adult jog once a week.

S is the event that the adult swim once a week.

C is the event that the adult cycle once a week.

It is given that $\mathrm{P}(R)=0.4$, $\mathrm{P}(S)=0.3$, $\mathrm{P}(C)=0.45$ and $\mathrm{P}(R\cap S\cap C)=0.1$. It is given that the events R and S are independent and that events S and C are independent. Find

(i) $\mathrm{P}(R|S)$, [1]

(ii) $\mathrm{P}(S\cup C)$. [2]

Given also that events R and C are independent, find

(iii) $\mathrm{P}(R'\cap S'\cap C')$. [3]

Two adults are randomly chosen from the group. Find the probability that both of these adults each carry out exactly two of the three events (jog, swim or cycle). [2]

8 A salesman meets 20 different prospective customers each week to do a sales presentation. On average, 20% of these prospective customers will place an order with the salesman at the end of the presentation.

(i) Find the probability that there are more than 3 orders received in a week. [2]

(ii) Find the probability that the number of orders received in a week is within one standard deviation from the mean. [4]

(iii) Taking a year to consist of 52 weeks, estimate the probability that the mean number of orders received per week in a year is not more than 4.5. [3]

(iv) Use a suitable approximation to estimate the probability that there are more than 15 orders received in a 4-week period. You should state the mean and variance of the approximation. [4]

9 Cupcakes of a certain cafe are individually topped with frosting by the staff. The masses, in grams, of the unfrosted cupcakes and the frosting have normal distributions with means and standard deviations as shown in the table below.

	Mean	Standard Deviation
Unfrosted cupcakes	87	3
Frosting	27	2

(i) Find the probability that one randomly selected unfrosted cupcake has mass exceeding 86 grams. [1]

(ii) State the mean and variance of the mass of an individual **frosted** cupcake. Find the probability that a frosted cupcake has mass between 110 grams and 120 grams. [3]

(iii) State an assumption needed for your calculations in part **(ii)**. [1]

Each 100 grams of frosted cupcakes contains, on average 300 calories.

(iv) Find the probability that a frosted cupcake contains more than 340 calories, stating clearly the mean and variance of the distribution you use. [2]

A sample of n cupcakes was taken. The probability that the mean mass of each frosted cupcake is below 112 grams is at most 0.001. Find the least value of n. [4]

10 The owner of a fish farm claims that the mean length of a particular species of fish from a pond is greater than 25 cm. The length, x cm, of a fish is measured for a random sample of 80 fish. The results are summarised by

$$\sum(x-25)=24 \qquad \text{and} \qquad \sum(x-25)^2=230.$$

(i) Find the unbiased estimates of the population mean and variance. [3]

(ii) What do you understand by the term 'unbiased estimate'? [1]

(iii) Test at the 5% level of significance whether there is any evidence to doubt the owner's claim. [4]

The same species of fish has been genetically modified to improve its length and the new population variance is known to be 2 cm^2. A new random sample of 100 fishes is selected and the mean of this sample is k cm. A test at the 10% level of significance indicates that the owner's claim is valid for this genetically modified fish.

(iv) Find the set of values of k, giving your answer correct to 2 decimal places. [4]

11 Explain why it is advisable to plot a scatter diagram before interpreting a correlation coefficient calculated for a sample drawn from a bivariate distribution. [1]

In an investigation of the growth of ornamental stingray, the ages x, in months, and the width y (in cm), of a random sample of 9 stingrays are shown below:

x	0	1	2	3	4	5	6	7	8
y	5	5.2	8.3	9	11	15.6	16	19	23

(i) Give a sketch of the scatter diagram for the data as shown on your calculator. [2]

(ii) Calculate the equation of the regression line of y on x and draw this line on your scatter diagram. [2]

(iii) Find $\bar{x}$ and $\bar{y}$, and mark the point $(\bar{x}, \bar{y})$ on your scatter diagram. [2]

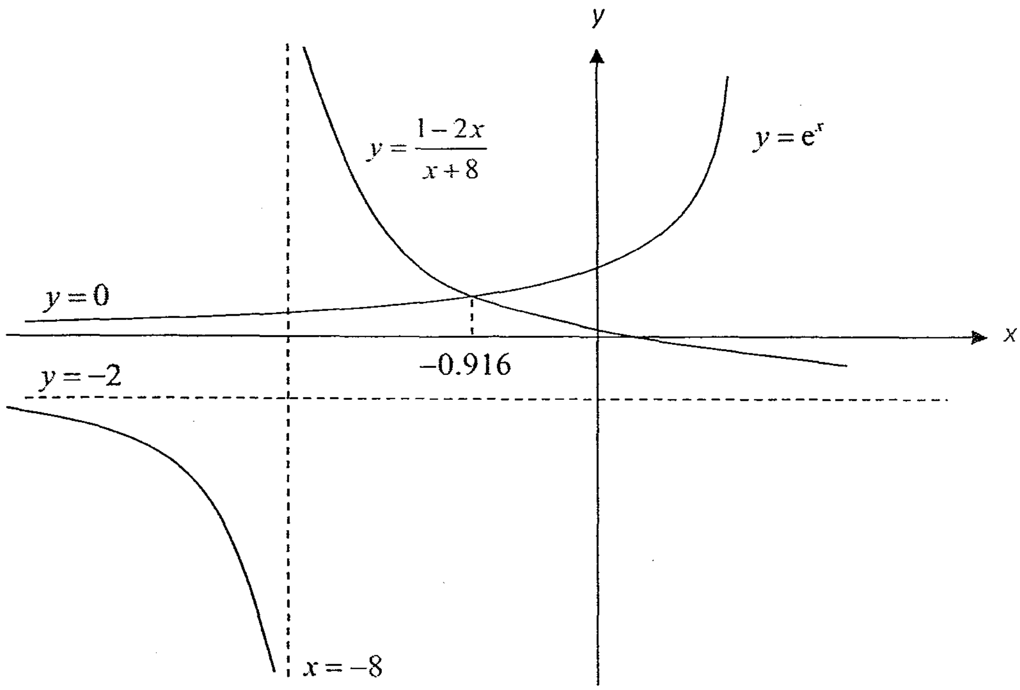
(iv) Calculate the product-moment correlation coefficient between width and age for these stingrays. Give a brief interpretation of your result. [2]

(v) Calculate an estimate of the width of a 4.5 month old stingray. Explain whether you would expect this to be a reliable estimate. [2]

- *End of paper* -

Solution

1(i)



(ii) $x<-8$ or $x>-0.916$

$\frac{e^x(1+2x)}{8-x}<1$

$\frac{1+2x}{8-x}<e^{-x}$ (since $e^x>0$)

Replace x by $-x$,

$-x<-8$ or $-x>-0.916$

$x>8$ or $x<0.916$

2(i) $\frac{dy}{dx} = 3x^2 + 8x + k$

At $x = 1$, $\frac{dy}{dx} = 11 + k$ and $y = 5 + k + k^2$

Equation of tangent at P:

$y - (5 + k + k^2) = (11 + k)(x - 1)$

$y = 11x - 11 + kx - k + 5 + k + k^2$

$y = (11 + k)x + k^2 - 6$

(ii) Sub $(2,1)$ into the equation of tangent,

$1 = (11 + k)(2) + k^2 - 6$

$k^2 + 2k + 15 = 0$

Using discriminant, $b^2 - 4ac = 2^2 - 4(15)$

$= -56 < 0$

Since discriminant < 0, there are no values of k such that the tangent passes through the point $(2,1)$.

3(i) $2xy + \pi x^2 = 2500\pi$

$2xy = 2500\pi - \pi x^2$

$y = \frac{1250\pi}{x} - \frac{\pi x}{2}$

$C = 100(2y) + 125(2\pi x)$

$C = 200\left(\frac{1250\pi}{x} - \frac{\pi x}{2}\right) + 250\pi x$

$C = \frac{250000\pi}{x} - 100\pi x + 250\pi x$

$C = \frac{250000\pi}{x} + 150\pi x$

(ii) $\frac{dC}{dx} = -\frac{250000\pi}{x^2} + 150\pi = 0$

$150\pi = \frac{250000\pi}{x^2}$

$x^2 = \frac{5000}{3}$

$x = 40.8$, -40.8 (NA since $x > 0$)

Using 1st Derivative Test,

x	40.8^-	40.8	40.8^+
$\frac{dC}{dx}$	-ve	0	+ve
Outline	\	—	/

$\therefore x = 40.8$ gives a minimum value of C.

Alternatively, using 2nd Derivative test,

$$\frac{d^2C}{dx^2} = \frac{500000\pi}{x^3} = 23.1 > 0$$

$\therefore x = 40.8$ gives a minimum value of C.

4(i) $9x - 3 = \frac{4}{x} - x$

$9x^2 - 3x = 4 - x^2$

$10x^2 - 3x - 4 = 0$

$$x = \frac{3 \pm \sqrt{3^2 - 4(10)(-4)}}{2(10)}$$

$$x = \frac{3 \pm 13}{20} = \frac{4}{5}, -\frac{1}{2} \quad \text{(NA since } x > 0\text{)}$$

(ii) $\int \left(\frac{4}{x} - x\right) dx = 4\ln x - \frac{1}{2}x^2 + C$

(iii) Line $y = 9x - 3$ cuts the x-axis at $\left(\frac{1}{3}, 0\right)$ and curve $y = \frac{4}{x} - x$ cuts the x-axis at $(2, 0)$

Area of shaded region

$$= \int_{\frac{1}{3}}^{\frac{4}{5}} (9x - 3)\, dx + \int_{\frac{4}{5}}^{2} \left(\frac{4}{x} - x\right) dx$$

$$= \left[\frac{9}{2}x^2 - 3x\right]_{\frac{1}{3}}^{\frac{4}{5}} + \left[4\ln x - \frac{1}{2}x^2\right]_{\frac{4}{5}}^{2}$$

$= 2.97$

5(i) $\frac{dy}{dx}=\frac{2}{2x-1}-2x=0$

$\frac{1}{2x-1}=x$

$2x^2-x-1=0$

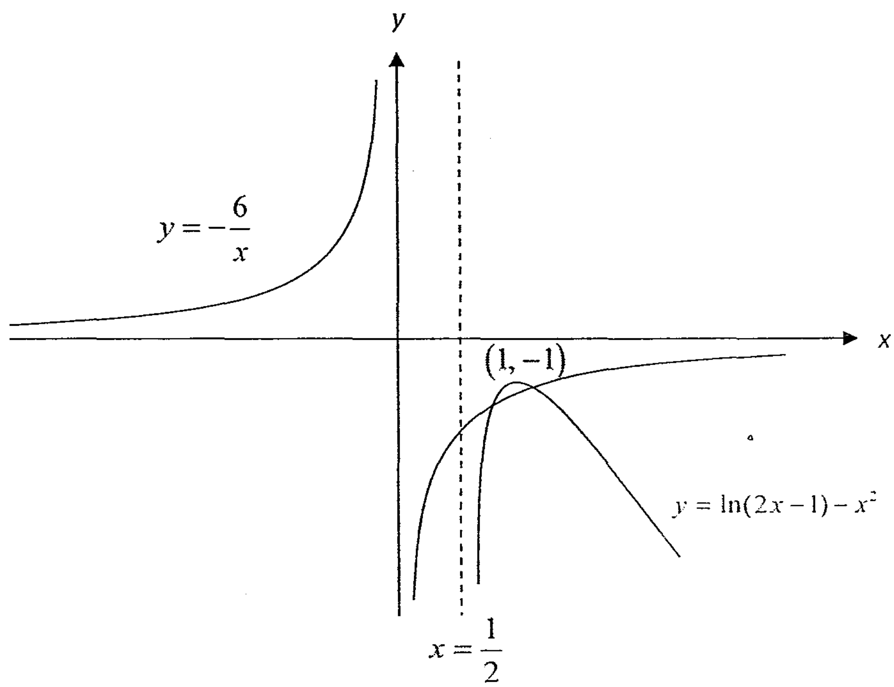
$(2x+1)(x-1)=0$

$x=1$ or $x=-\frac{1}{2}$ (NA since $x>\frac{1}{2}$)

When $x=1$, $y=\ln(2-1)-1=-1$

$\therefore$ The stationary point of C is $(1,-1)$.

(ii)



(iii) $x\left(\ln(2x-1)-x^2\right)+6=0$

$\ln(2x-1)-x^2=-\frac{6}{x}$

Sketch the graph of $y=-\frac{6}{x}$.

From the graph, there are 2 points of intersection. Hence, the equation has 2 real roots.

6(i) A list of all the pupils in the school is first obtained and each pupil is assigned a number from 1 to 1500.

A starting point is randomly chosen between 1 - 10, say Pupil 2.

$$k = \frac{1500}{150} = 10$$

For example, if Pupil 2 is chosen, then every subsequent 10^{th} pupil, i.e. Pupil 12, 22, 32, ... is chosen until the sample of 150 pupils is obtained.

(ii) Select the number of students from each strata based on the numbers in the table below

	Male	Female
Year 1	35	45
Year 2	30	40

Select the number of students in each strata by using <u>simple random sampling</u>.

(iii) Stratified sampling is more appropriate because the sample collected is proportionately representative of the population.

7 **(i)** Since R and S are independent,

$\mathrm{P}(R|S)=\mathrm{P}(R)=0.4$

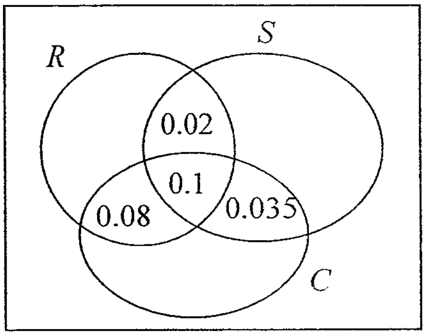
(ii) Since S and C are independent,

$\mathrm{P}(S\cap C)=\mathrm{P}(S)\mathrm{P}(C)=0.135$

$\mathrm{P}(S\cup C)=\mathrm{P}(S)+\mathrm{P}(C)-\mathrm{P}(S\cap C)=0.615$

(iii) Since R and C are independent,

$\mathrm{P}(R\cap C)=\mathrm{P}(R)\mathrm{P}(C)=0.18$



$\mathrm{P}(S\cup C\cup R)$

$=\mathrm{P}(S\cup C)+\mathrm{P}\left[(S\cup C)'\cap R\right]$
$=0.615+(0.4-0.08-0.1-0.02)$
$=0.815$

$\mathrm{P}(R'\cap S'\cap C')$

$=1-\mathrm{P}(S\cup C\cup R)=0.185$

Required probability $=(0.02+0.08+0.035)^2=0.018225\approx 0.0182$ (3sf)

Alternatively

$X\sim$ number of adults who exercise exactly two out the three events, out of 2

$X\sim \mathrm{B}(2,\ 0.135)$

$\mathrm{P}(X=2)=0.0182(3\mathrm{sf})$

8. Let $X \sim$ number of deals clinched, out of 20

$X \sim \mathrm{B}(20, 0.2)$

(i) $\mathrm{P}(X > 3) = 1 - \mathrm{P}(X \leq 3) = 0.58855 \approx 0.589$ (3sf)

(ii) $\mathrm{P}(\mu - \sigma < X < \mu + \sigma)$

$= \mathrm{P}(2.2111 < X < 5.7888)$

$= \mathrm{P}(3 \leq X \leq 5)$

$= \mathrm{P}(X \leq 5) - \mathrm{P}(X \leq 2)$

$= 0.59812 \approx 0.598$ (3sf)

(iii) $\mathrm{E}(X) = 20 \times 0.2 = 4$

$\mathrm{Var}(X) = 20 \times 0.2 \times (1 - 0.2) = 3.2$

As $n = 52$ is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(4, \frac{4}{65}\right)$ approx.

$\mathrm{P}(\overline{X} \leq 4.5) = 0.97808 \approx 0.978$

Let $Y \sim$ number of deals clinched, out of 80

$Y \sim \mathrm{B}(80, 0.2)$

Since $n = 80$ is large, $np = 16 > 5$, $n(1 - p) = 64 > 5$

$Y \sim \mathrm{N}(16, 12.8)$ approx.

$\mathrm{P}(Y > 15) = \mathrm{P}(Y > 15.5)$ (by continuity correction)

$= 0.55557 \approx 0.556$ (3sf)

9. $X \sim$ mass of an unfrosted cupcake. $X \sim \mathrm{N}(87, 9)$

$Y \sim$ mass of frosting. $Y \sim \mathrm{N}(27, 4)$

(i) $\mathrm{P}(X > 86) = 0.63055 \approx 0.631$ (3sf)

(ii) $W = X + Y$

$W \sim \mathrm{N}(114, 13)$

$\mathrm{P}(110 < W < 120) = 0.81833 \approx 0.818$

(iii) The mass of an unfrosted cupcake and the mass of frosting are assumed to be independent.

(iv) $P(3W > 340) = P(W > \frac{340}{3}) = 0.57335 \approx 0.573$

Alternatively,

$3W \sim N(342, 117)$

$P(3W > 340) = 0.57335 \approx 0.573$

$\overline{W} \sim N\left(114, \frac{13}{n}\right)$

$P(\overline{W} < 112) \leq 0.001$

n	$P(\overline{W} < 112)$
31	0.00101
32	0.00085
33	0.00071

Least $n = 32$

Alternatively

$P(\overline{W} < 112) \leq 0.001$

$P(Z < \frac{112-114}{\sqrt{\frac{13}{n}}}) \leq 0.001$

$\frac{-2}{\sqrt{\frac{13}{n}}} \leq -3.0902$

$2\sqrt{n} \geq 11.142$

$n \geq 31.036$

Least $n = 32$

10 **(i)** $\bar{x} = \frac{\sum(x-25)}{80} + 25 = 25.3$

$$s^2 = \frac{1}{80-1}\left[230 - \frac{(24)^2}{80}\right] = 2.8203 \approx 2.82\,(3\text{sf})$$

(ii) An estimate of a population parameter is an unbiased estimate if its expected value is equal to the true value of the population parameter.

(iii) Test $H_0: \mu = 25$ vs $H_1: \mu > 25$

Under H_0, since $n = 80$ is large, by Central Limit Theorem,

$\bar{X} \sim N\left(25, \frac{2.8203}{80}\right)$ approx.

Level of significance: 5%.

Critical region: $z \geq 1.6449$

Standardized test statistic: $z = \frac{\bar{x} - \text{"claimed value"}}{s/\sqrt{n}} = \frac{25.3 - 25}{\sqrt{\frac{2.8203}{80}}} = 1.5978$

From GC, p-value = $0.055045 > 0.05$

Since the p-value is more than the level of significance, we do not reject H_0.

There is sufficient evidence at 5% level of significance to doubt the owner's claim.

(iv) Test $H_0: \mu = 25$ vs $H_1: \mu > 25$

Standardised Test statistic, $z = \frac{\bar{x} - \mu}{\sqrt{\frac{\sigma^2}{n}}} = \frac{k - 25}{\sqrt{\frac{2}{100}}}$

Critical region: $z \geq 1.2816$

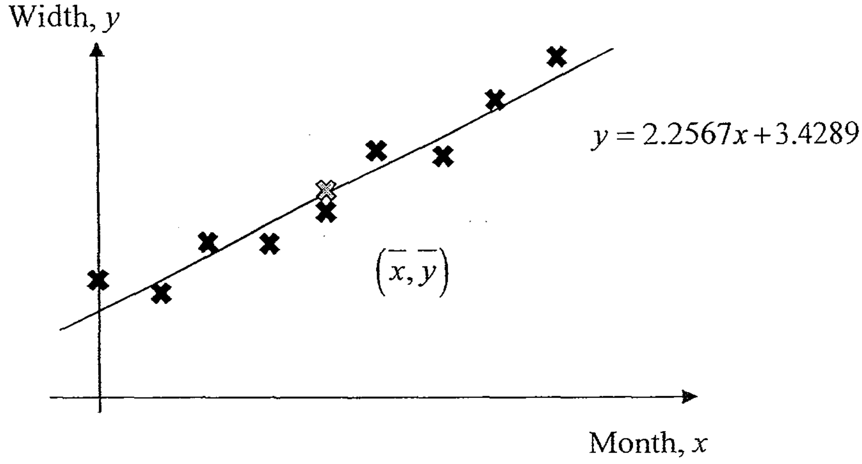
For fish farm's claim to be valid, H_0 rejected.

$$\frac{k-25}{\sqrt{\frac{2}{100}}} \geq 1.2816$$

$k \geq 25.18\,(2\text{dp})$

11 To assess whether a linear relationship between the two variables exist.

(i)



(ii) $y = 2.2567x + 3.4289 \approx 2.26x + 3.43$ (3sf)

(iii) Using GC $(\bar{x}, \bar{y}) = (4, 12.5)$ (use 2 variable stats)

(iv) $r = 0.983$ (3sf) is close to 1, there is a strong positive linear correlation between the width and the age (in months)

(v) $y = 2.2567(4.5) + 3.4289 = 13.584 \approx 13.6$

The width of a 4.5 month old stingray will be 13.6 cm. Since $x = 4.5$ is within the data set, it is an interpolation, and since $r = 0.983$ is close to 1, there is a strong positive linear correlation between the width and the age (in months), the estimate is reliable.