H1 Mathematics 2017 Preliminary Exam Paper Question

Section A : Pure Mathematics [40 Marks]

1. Whole Food Grocer was having sales and some food items were on offer. Organic feed eggs were having a 15% discount. There was also a $1 discount for every 2 packets of chia seeds purchased. There was no promotion for organic quinoa. The table below shows the total bills and the number of packets of organic quinoa, organic feed eggs and chia seeds Stephanie, Weiwei and Leo bought from Whole Food Grocer. Calculate the original selling price for one packet of each of the 3 food items, giving your answers correct to the nearest cent.

	Quinoa	Eggs	Chia seeds	Total Bill ($)
Stephanie	3	1	2	71.28
Weiwei	2	2	5	91.85
Leo	6	3	3	144.43

[4]

2. Differentiate the following with respect to x.

(a) $\left(x+\ln x\right)^2$, [2]

(b) $e^{\left(\frac{1}{\sqrt{2-x}}\right)}$. [2]

3. On the same diagram, sketch the graphs of $y=\frac{x+1}{2x+3}$ and $y=x^2+x-3$, indicating clearly the equation(s) of any asymptote(s) and any point(s) of intersection. Hence solve the inequality $\frac{x+1}{2x+3}\geq x^2+x-3$. [5]

4. **(i)** Express $\frac{2x^2+1}{x-4}$ in the form $Ax+B+\frac{C}{x-4}$, where A, B and C are constants to be determined. [2]

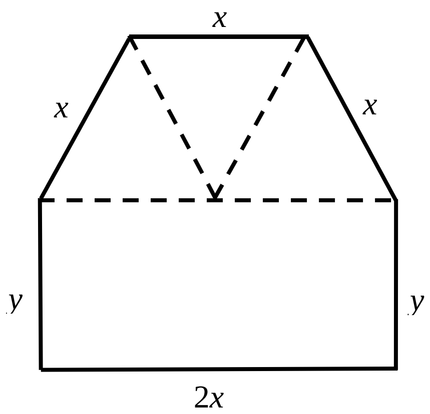
(ii) Hence, without the use of a calculator, find $\int_5^6 \frac{2x^2+1}{x-4}\,dx$ in exact form. [3]

5. **(i)** Sketch the curve C with equation $y=(x+3)(1-x)$, stating clearly the coordinates of the turning point and the axial intercepts. [2]

(ii) The line $y=x+k$ intersects C twice. Find the set of values of k. [3]

(iii) Without the use of a calculator, find the area of the region bounded by C and the line $y=x+5$. [5]

6. **(i)** Kim wants to fence up a vegetable plot in his backyard. The vegetable plot to be fenced up will occupy a rectangle of $2x$ m by y m together with half of a regular hexagon with sides of x m each, as shown in the diagram below. It is given that the area of the vegetable plot is 15 m^2.



Show that the perimeter $P=5x+\dfrac{15}{x}-\dfrac{3\sqrt{3}}{4}x$.

Find, using differentiation, the values of x and y such that P is minimum.

[7]

Two companies provide the cost for the fencing.

Company *A*	\$90 per metre or part thereof *
Company *B*	\$95 per metre for the first 10 metre \$84 for the subsequent metre or part thereof

* For example, it costs \$180 to build a fence of 1.2m using Company *A*

(ii) Find the range of the length of fencing to be built such that it is cheaper to engage Company *B*. [3]

(iii) Hence conclude which company Kim should engage to fence his backyard when P is minimum. [2]

Section B: Statistics [60 marks]

7. In an IT department, a staff is tasked to form 7-letter codes (need not be valid words) using the given word 'SPECIAL'. Find the number of codes that can be formed if

(a) there are no restrictions except the code 'SPECIAL' cannot be formed, [2]

(b) all the 3 vowels cannot be together, [2]

(c) the first and the last letters are consonants. [2]

8. A company uses 2 production lines, A and B, to produce lunch boxes. If the lunch box cannot be closed tightly, it will be considered as faulty. Of all the lunch boxes produced, 5% are faulty and 3% of the lunch boxes produced by B are faulty. Among the lunch boxes that are faulty, 60% of them are produced by line A.

One lunch box is selected at random.

(i) Show that the probability that it is produced by line B is $\frac{2}{3}$. [2]

(ii) Find the probability that it is faulty given that it is produced by A. [2]

Two lunch boxes are chosen at random.

(iii) Find the probability that both lunch boxes are produced by B given that exactly one is faulty. [3]

9. The probability of a diner choosing a burger during his visit to Cheeky Chick Café is 0.05. Among the diners who visited Cheeky Chick Café, 20 diners are randomly chosen.

(i) Find the probability that more than 3 diners choose a burger. [2]

(ii) Find the smallest value of n such that there is more than 90% chance of less than n diners choosing a burger. [3]

(iii) The probability of a diner buying a drink in the cafe is p, where $p > 0.5$. Given that the variance of a diner buying a drink is 4.55, find the value of p. [2]

10. The accountant of a company monitors the number of items produced per month by the company, x (in thousands), together with the total cost of production, \$$y$ (in thousands). The following table shows 12 sets of data collected for a random sample of 12 months.

Number of items produced (x)	21	39	48	24	72	75	15	35	62	81	12	56
Production cost (\$$y$)	40	58	67	45	89	96	37	53	83	102	35	75

(i) Draw a scatter diagram to illustrate the data. [2]

(ii) Find the product moment correlation coefficient and comment on its value in the context of the data. [2]

(iii) Find $\bar{x}$ and $\bar{y}$, and mark the point $\left(\bar{x},\bar{y}\right)$ on your scatter diagram drawn in part **(i)**. [2]

(iv) Find an equation for the regression line of y on x in the form $y = mx + c$, giving the values of m and c, correct to 2 decimal places. Sketch this line on your scatter diagram. Interpret the meanings of m and c in this context. [3]

(v) Use the equation of your regression line to calculate an estimate for the production cost of 70 thousand items. Comment on the reliability of your estimate. [2]

(vi) The selling price of each item produced is \$2.20. Find the minimum number of items to be produced per month at which the company does not suffer a loss. [2]

11. The weight of a packet of Calhwa potato chips is known to have a mean of 84 grams and standard deviation 5 grams. The manufacturer claims that the average weight of a packet of potato chips is at least 84 grams. To test this claim, a random sample of 100 such packets of potato chips are selected and tested. The average weight of the 100 packets of potato chips in the sample is 82.9 grams.

(i) State appropriate hypotheses for the test, defining any symbols you use. [2]

(ii) Test, at the 1% significance level, whether the manufacturer's claim is valid. [3]

(iii) State what you understand by the expression 'at the 1% significance level' in part **(ii)**. [1]

(iv) State, giving a reason, whether it is necessary for the weight of the packets of potato chips produced by a manufacturer to follow a normal distribution for the test in part **(ii)** to be valid. [1]

(v) Another random sample of 100 packets of potato chips from another batch gives an average weight of t grams. Find the range of values of t such that there is enough evidence to conclude that the average weight of the packets of potato chips has changed at the 5% level of significance. [5]

12. **(a)** The continuous random variable X has the distribution $\mathrm{N}\left(\mu, \sigma^2\right)$. It is given that $\mathrm{P}\left(X < 15\right) = 0.841$ and $\mathrm{P}\left(9 < X < 15\right) = 0.682$, find the values of μ and σ. [3]

(b) In a city, the minimum temperature in June, denoted by S, is assumed to be normally distributed with mean μ °C and standard deviation 3 °C.

(i) Find the probability that the minimum temperature in June differs from the mean μ by more than 2.5 °C. [3]

(ii) Find the value of μ such that there is a 75% chance that the minimum temperature in June is higher than 11 °C. [2]

In this city, the maximum temperature in June, denoted by T, is also assumed to be normally distributed with mean 20 °C and standard deviation 2.2 °C.

(iii) Find the probability that on a randomly chosen day in June, the maximum temperature is between 17.5 °C and 23 °C. [1]

(iv) Let $\mu = 12$ °C. Find the probability that the maximum temperature on a randomly chosen day in June is more than the average minimum temperature on 2 randomly chosen days in June by less than 10 °C. [4]

(v) State one assumption needed for your calculation in part **(iv)**. Give a reason why the assumption may be unrealistic. [2]

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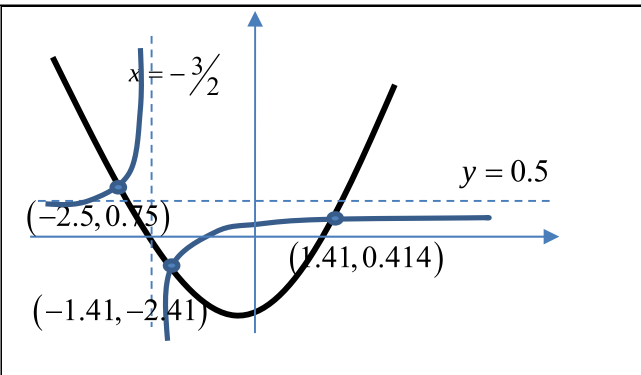
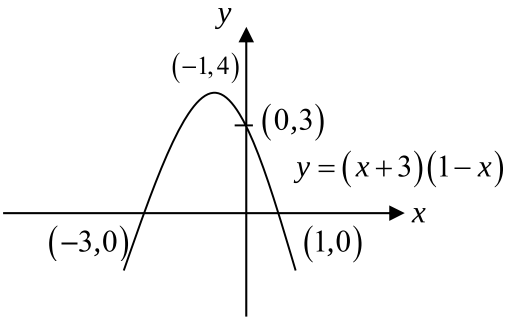
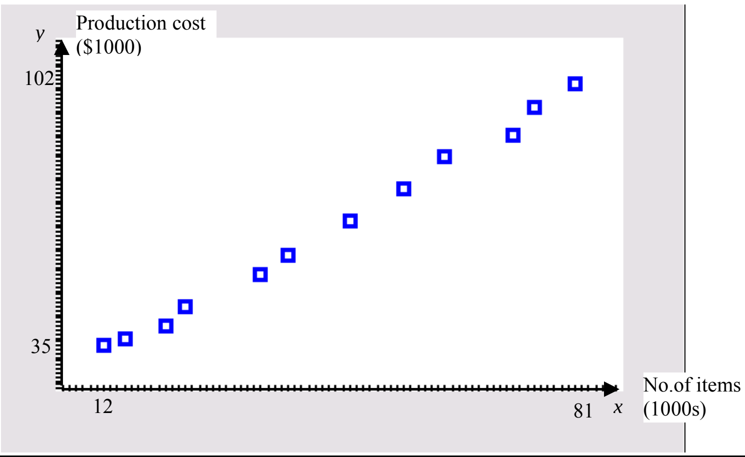
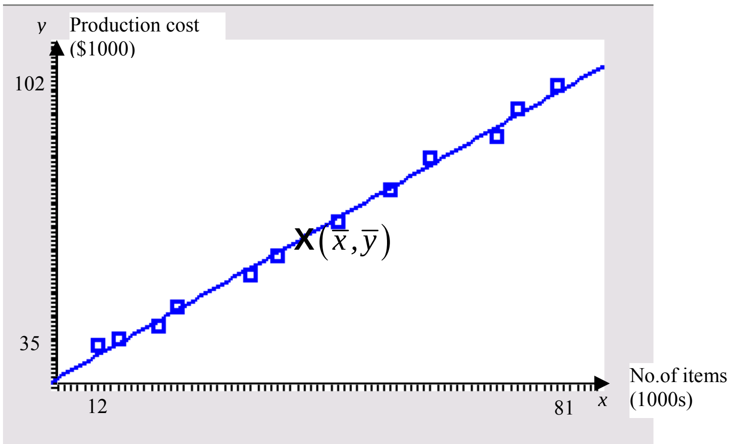
	Solutions
1	Let *x*, *y* and *z* be the original selling price per pack of organic quinoa, organic feed eggs and chia seeds in dollars. $3x+0.85y+2z=72.28$ --- (1) $2x+2(0.85)y+5z=93.85$ --- (2) $6x+3(0.85)y+3z=145.43$ --- (3) $x=\$14.90, y=\$11.49, z=\$8.90$
2(a)	$\frac{\mathrm{d}}{\mathrm{d}x}(x+\ln x)^2=2(x+\ln x)\left(1+\frac{1}{x}\right)$ $=\frac{2}{x}(x+\ln x)(x+1)$
2(b)	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\left(\frac{1}{\sqrt{2-x}}\right)}=\frac{1}{2(2-x)^{\frac{3}{2}}}\mathrm{e}^{\left(\frac{1}{\sqrt{2-x}}\right)}$
3	 $-2.5\le x<-1.41$ or $-1.41\le x\le 1.41$
4(i)	Using long division $\frac{2x^2+1}{x-4}=2x+8+\frac{33}{x-4}$ OR $\frac{2x^2+1}{x-4}=\frac{(Ax+B)(x-4)+C}{x-4}$ $2x^2+1=(Ax+B)(x-4)+C$ Compare coefficient: $2=A, B=8, C=33$
4(ii)	$\int_5^6\frac{2x^2+1}{x-4}\mathrm{d}x=\int_5^6 2x+8+\frac{33}{x-4}\mathrm{d}x$ $=\left[x^2+8x+33\ln\lvert x-4\rvert\right]_5^6$ $=84+33\ln 2-65$ $=19+33\ln 2$

5(i)	
5(ii)	$y=(x+3)(1-x)$ $\quad =x+3-x^2-3x$ $\quad = -x^2-2x+3$ $x+k=-x^2-2x+3$ $x^2+3x+k-3=0$ $b^2-4ac>0$ $3^2-4(k-3)>0$ $9-4k+12>0$ $4k<21$ $k<5.25$ $\{k \in \mathbb{R} : k<5.25\}$
5(iii)	$y=(x+3)(1-x)=x+5$ $-x^2-2x+3=x+5$ $x^2+3x+2=0$ $(x+2)(x+1)=0$ $x=-2, x=-1$ area $= \int_{-2}^{-1}\left(-x^2-2x+3-x-5\right)\mathrm{d}x$ $=\int_{-2}^{-1}\left(-x^2-3x-2\right)\mathrm{d}x$ $=\left[-\frac{x^3}{3}-\frac{3x^2}{2}-2x\right]_{-2}^{-1}$ $=\left(\frac{1}{3}-\frac{3}{2}+2\right)-\left(\frac{8}{3}-\frac{12}{2}+4\right)$ $=\frac{1}{6}\text{units}^2$ Or

	$\text{area} = \int_{-2}^{-1} (-x^2 - 2x + 3)\mathrm{d}x - \frac{1}{2} \times (3+4) \times 1$ $= \left[-\frac{x^3}{3} - \frac{2x^2}{2} + 3x \right]_{-2}^{-1} - \frac{7}{2}$ $= \frac{1}{6} \text{units}^2$
6(i)	$\text{Area} = 3\left(\frac{1}{2}\right)x^2 \sin\left(\frac{\pi}{3}\right) + 2xy = \frac{3\sqrt{3}}{4}x^2 + 2xy$ $\frac{3\sqrt{3}}{4}x^2 + 2xy = 15$ $2xy = 15 - \frac{3\sqrt{3}}{4}x^2$ $2y = \frac{15}{x} - \frac{3\sqrt{3}}{4}x$ Let P be the perimeter. $P = 5x + 2y$ $= 5x + \frac{15}{x} - \frac{3\sqrt{3}}{4}x$ $\frac{\mathrm{d}P}{\mathrm{d}x} = 5 - \frac{3\sqrt{3}}{4} - \frac{15}{x^2}$ For cost to be minimum, perimeter has to be minimum. $\frac{\mathrm{d}P}{\mathrm{d}x} = 0 \Rightarrow 5 - \frac{3\sqrt{3}}{4} - \frac{15}{x^2} = 0$ $\frac{15}{x^2} = 3.700962 \Rightarrow x^2 = 4.053$ $x = 2.0132$ $\therefore x = 2.01, \quad y = 2.42$ $\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = \frac{30}{x^3} > 0$ Therefore P is a minimum.

6(ii)	Let *N* be the perimeter of the fence in integral value Cost from Company $A = 90N$ Cost from Company $B = 10(95) + 84(N - 10)$ $= 110 + 84N$ $110 + 84N < 90N$ $6N > 110$ $N > 18.3 \Rightarrow N > 18$
6(iii)	When $x = 2.0132$, $y = 2.4177$ $P = 14.902$ Since $14.902 < 18$, therefore it is cheaper to choose Company *A*.
7(a)	No. of words that can be formed $= 7! - 1$ $= 5039$
(b)	No. of words if 3 vowels are altogether $= 3! \times 5!$ $= 720$ No. of words $= 5040 - 720$ $= 4320$
(c)	No. of words = ${}^4C_2 \times 2! \times 5!$ $= 1440$
8(i)	Let *B* be the event that the lunch box is produced by production line *B*. Let *F* be the event that the lunch box is faulty. $\mathrm{P}(F \cap B) = \mathrm{P}(B) \times \mathrm{P}(F \mid B)$ $= \mathrm{P}(F) \times \mathrm{P}(B \mid F)$ (*) $\mathrm{P}(B)(0.03) = 0.05(0.4)$ $\mathrm{P}(B) = \frac{2}{3}$
8(ii)	Let *A* be the event that the lunch box is produced by production line *A*. $\mathrm{P}(A \cap F) = 0.05 \times 0.6 = 0.03$ $\mathrm{P}(F \mid A) = \frac{0.03}{\frac{1}{3}} = 0.09$
8(iii)	$\mathrm{P}(B \cap F') = \frac{2}{3} - 0.02 = 0.64667$ $\mathrm{P}(B \mid \text{only 1 faulty}) = \frac{0.64667 \times 0.02 \times 2}{0.95 \times 0.05 \times 2}$ $= 0.272$
9(i)	Let *X* denote the number of diners, out of 20, who choose a burger. $X \sim \mathrm{B}(20, 0.05)$ $\mathrm{P}(X > 3) = 1 - \mathrm{P}(X \le 3) = 0.0159$

9(ii)	$\mathrm{P}(X < n) > 0.9$ --- (1) $\mathrm{P}(X \le n-1) > 0.9$ --- (2) Using GC, $\mathrm{P}(X \le 1) > 0.736$ $\mathrm{P}(X \le 2) > 0.925$ $\therefore$ smallest value of n is 3.
9(iii)	Let Y denote the number of diners, out of 20, buying a drink in the cafe. $Y \sim \mathrm{B}(20, p)$ $20p(1-p) = 4.55$ --- (1) $p^2 - p + 0.2275 = 0$ $p = 0.35$ or 0.65 Since $p > 0.5$, $p = 0.65$
10(i)	
10(ii)	Product moment correlation coefficient $r = 0.998$ which indicates a strong positive linear correlation between the number of items produced per month by the company together with the total cost of production
10(iii)	$\overline{x} = 45, \quad \overline{y} = 65$

10(iv)	$y = 0.98x + 20.99$ - 0.98 gives the rate at which the production costs are increasing i.e. for every additional item produced, the production cost increases by \$0.98 - OR For every increase in 1000 items produced, there is an increase in the total production cost by 980 dollars. - \$20,990 is the fixed cost of production. 
10(v)	$y = 0.9781x + 20.985$ When $x = 70$, $y = 0.9781 \times 70 + 20.985 = 89.452$, an estimate for the production cost of 70 thousand items is \$89452. Since $x = 70$ lies within the data range $12 \le x \le 81$ and $\lvert r \rvert$ is close to 1, therefore this estimate is reliable.
10(vi)	If x items(1000s) are produced, Total income = $\$2.20 \times x$ The total cost for producing x items is $y = 0.9781x + 20.985$ If there is no loss, $2.20x \ge 0.9781x + 20.985$ $1.2219x \ge 20.985$ $x \ge 17.17407$ Therefore the min number of items to be produced per month is 17175 items.
11(i)	Let X be the weight of a packet of Calhwa potato chips and μ denotes the population mean weight of a packet of potato chips in grams H_0: $\mu = 84$ H_1: $\mu < 84$

11(ii)	At 1% level of significance, under H_0, since $n = 100$ is large, by Central limit theorem, $\overline{X} \sim \text{N}\left(84, \frac{5^2}{100}\right)$ approximately Test statistic $Z = \frac{\overline{X} - \mu}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)$ $p = 0.0139 > 0.01$, we do not reject H_0 and conclude that at the 1% level of significance, there is insufficient evidence to say that the average weight of a packet of potato chips is less than 84 grams
11(iii)	When the level of significance is set at 1%, there is 1% chance that we wrongly conclude the mean weight of a packet of potato chips is less than 84 grams when in fact the mean weight of a packet of potato chips is at least 84 grams.
11(iv)	Since the sample size $n = 100$ is sufficiently large, the **sample mean** weight of the packets of potato chips will be normally distributed by the Central Limit Theorem. Therefore it is not necessary to assume the weight of packets of potato chips follow a normal distribution.
11(v)	$H_0: \mu = 84$ $H_1: \mu \neq 84$ Level of significance: 5% Under H_0, since n =100 is large, by Central limit theorem, $\overline{X} \sim \text{N}\left(84, \frac{5^2}{100}\right)$ approximately. Test statistic $\lvert Z \rvert = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim \text{N}(0,1)$ Rejection region: Reject H_0 if $z \le -1.95996$ or $z \ge 1.95996$ Since there is sufficient evidence, at 5% level significance to conclude that the average weight of the potato chip has changed, $\frac{t-84}{\frac{5}{\sqrt{100}}} \le -1.95996$ or $\frac{t-84}{\frac{5}{\sqrt{100}}} \ge 1.95996$ $\Rightarrow 2t - 84 \le -1.95996$ or $2t - 84 \ge 1.95996$ $t \le 83.020$ or $t \ge 84.979$ Range of t : $t \le 83.0$ or $t \ge 85.0$

12(a)	$\mathrm{P}(X<15)=0.841 \Rightarrow \mathrm{P}\left(Z<\frac{15-\mu}{\sigma}\right)=0.841$ --(1) $\Rightarrow \frac{15-\mu}{\sigma}=0.99858$ $\mathrm{P}(9<X<15)=0.682 \Rightarrow \mathrm{P}\left(\frac{9-\mu}{\sigma}<Z<\frac{15-\mu}{\sigma}\right)=0.682$ $\Rightarrow \mathrm{P}\left(Z<\frac{9-\mu}{\sigma}\right)=0.841-0.682=0.159$ ---(2) $\Rightarrow \frac{9-\mu}{\sigma}=-0.99858$ --- (2) Solving (1) & (2), $\mu=12$ and $\sigma=3.00$ **Alternatively** By observation , $\mu=12$. $\mathrm{P}(X<15)=0.841$ Using GC, $\sigma=3.00$
12(bi)	$S \sim \mathrm{N}(\mu,3^2)$ $\mathrm{P}(\lvert S-\mu\rvert>2.5)$ --- (1) $=\mathrm{P}\left(\left\lvert\frac{S-\mu}{3}\right\rvert>\frac{2.5}{3}\right)=\mathrm{P}\left(\lvert Z\rvert>\frac{2.5}{3}\right)=2\times\mathrm{P}\left(Z>\frac{2.5}{3}\right)$ --- (2) $=0.405$
b(ii)	$\mathrm{P}(S>11)=0.75$ --- (1) $\mathrm{P}\left(Z\le\frac{11-\mu}{3}\right)=0.25$ $\frac{11-\mu}{3}=-0.67449$, $\mu=13.0$ °C
b(iii)	$\mathrm{P}(17.5<T<23)=0.786$
b(iv)	Find $\mathrm{P}\left(0\le T-\frac{S_1+S_2}{2}<10\right)$ Let $W=T-\frac{S_1+S_2}{2}$ $\mathrm{E}(W)=8$ $\mathrm{Var}(W)=9.34$ $W \sim \mathrm{N}(8,9.34)$ $\mathrm{P}\left(0\le T-\frac{S_1+S_2}{2}<10\right)=0.73915=0.739(3\text{s.f})$
(v)	Assume that the minimum and maximum temperatures are independent of each other. It is unrealistic because the weather, e.g. wind direction, rainy weather, etc, will affect both the minimum and maximum temperature of the city.