General Certificate of Education Advanced Level

Higher 2

JC2 Preliminary Examination

MATHEMATICS

9740/01

Paper 1

24 Aug 2016

3 hours

Additional Materials: List of Formulae (MF15)

Name: ____________________________ **Class:** ________________

READ THESE INSTRUCTIONS FIRST

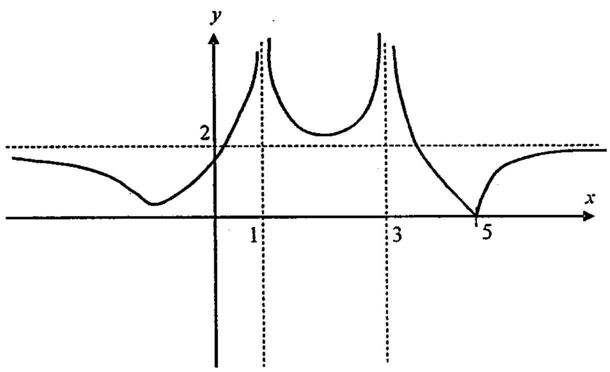
Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

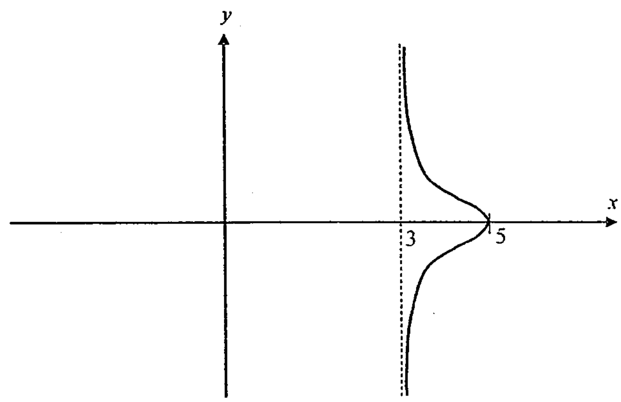
At the end of the examination, arrange your answers in NUMERICAL ORDER.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 **[In this question, sketches of the given graphs are not drawn to scale]**
The graphs of $y = |f(x)|$ and $y^2 = f(x)$ are given below.



$y = |f(x)|$



$y^2 = f(x)$

On separate diagrams, draw sketches of the graphs of

(a) $y^2 = f(-2x)$, [2]

(b) $y = f(x)$, [3]

stating the equations of any asymptotes and the coordinates of any points of intersection with the axes.

2 The vectors **a** and **b** are given by

$$\mathbf{a} = 4\mathbf{i} + 6p\mathbf{j} - 8\mathbf{k} \text{ and } \mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + 4p\mathbf{k}, \text{ where } p > 0.$$

It is given that $|\mathbf{a}| = 2|\mathbf{b}|$.

(i) Find p. [2]

(ii) Give a geometrical interpretation of $\frac{1}{|\mathbf{b}|}|\mathbf{b} \cdot \mathbf{a}|$. [1]

(iii) Using the value of p found in part **(i)**, find the exact value of $\frac{1}{|\mathbf{b}|}|\mathbf{b} \cdot \mathbf{a}|$. [2]

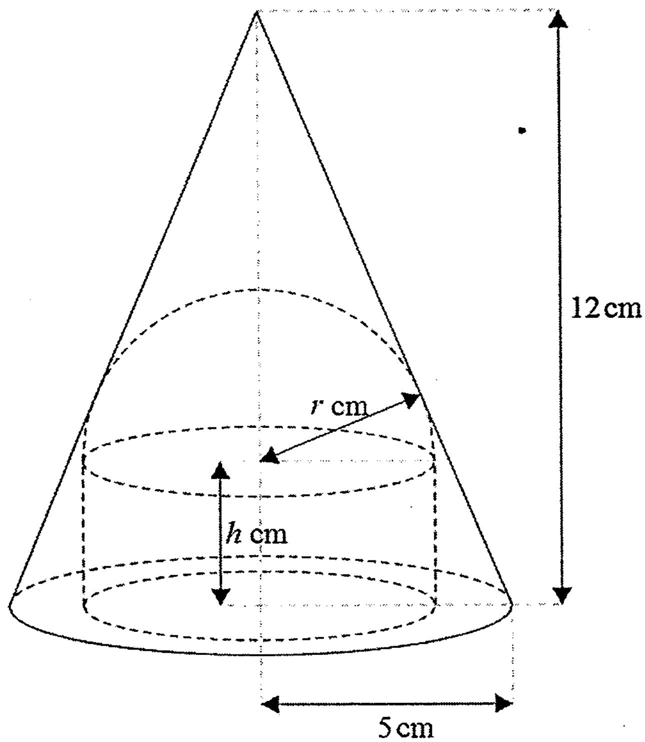
3 The cubic equation $x^3 + ax^2 + bx + c = 0$, where a, b and c are constants, has roots 3 + i and 2.

(i) One JC2 student remarked that the third root is 3 − i. State a necessary assumption the student made in order that the remark is true. [1]

(ii) Given that the assumption in part **(i)** holds, find the values of a, b and c. [4]

4 A closed container is made up of a cylinder of base radius r cm and height h cm, and a hemispherical top with the same radius r.

It is inscribed within a fixed right circular cone of base radius 5 cm and height 12 cm, as shown in the diagram below.



(i) By using similar triangles, show that $h = 12 - \frac{13}{5}r$.

Determine the exact range of possible values of length r. [3]

(ii) Find the total volume V of the closed container in terms of r.
By differentiation, find the exact value of r that produces the maximum container volume V, as r varies.

[Volume of a sphere with radius R is $\frac{4}{3}\pi R^3$.] [4]

5 A sequence $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation $u_n = \frac{n}{(n-1)^2} u_{n-1}$, for $n \geq 2$.

(a) Given that $u_1 = 2$, use the method of mathematical induction to prove that $u_n = \frac{2n}{(n-1)!}$ [4]

, for $n \geq 1$.

(b) Given that $u_1 = a$, where a is any constant. Write down u_2, u_3 and u_4 in terms of a.

Hence or otherwise, find u_n in terms of a. [3]

6 **(i)** Given that $1-2r = A(r+1)+Br$, find the constants A and B. [1]

(ii) Use the method of differences to find $\sum_{r=1}^{n} \frac{1-2r}{3^r}$. [3]

(iii) Hence find the value of $\sum_{r=1}^{\infty} \frac{2-2r}{3^r}$. [4]

7 **(i)** Given that $y=\sqrt{1+\ln(1+x)}$, find the exact range of values of x for y to be well defined. [1]

(ii) Show that $2y\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{1-y^2}$. [2]

(iii) Hence, find the Maclaurin's series for $\sqrt{1+\ln(1+x)}$, up to and including the term in x^2. [3]

(iv) Verify that the same result is obtained using the standard series expansions given in the List of Formulae (MF15). [3]

8 **Do not use a calculator in answering this question.**

(i) It is given that complex numbers z_1 and z_2 are the roots of the equation $z^2-6z+36=0$ such that $\arg(z_1) > \arg(z_2)$. Find exact expressions of z_1 and z_2 in the form $re^{i\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [4]

(ii) Find the complex number $\frac{z_1^{\,4}}{iz_2}$ in exact polar form. [3]

(iii) Find the smallest positive integer n such that $z_2^{\,n}$ is a positive real number. [2]

9 **(i)** By using the substitution $u=\sqrt{x+1}$, find $\int \frac{\sqrt{x+1}}{x-1}\,\mathrm{d}x$. [5]

(ii) The region R is bounded by the curve $y=\frac{\sqrt{x+1}}{x-1}$ and the lines $x=8$ and $y=1$.

Find

(a) the exact area of R, simplifying your answer in the form $A-\sqrt{2}\ln\left(\frac{B-\sqrt{2}}{B+\sqrt{2}}\right)$ where A and B are integers to be determined, [5]

(b) the volume of the solid generated when R is rotated 2π radians about the x-axis, giving your answer correct to 2 decimal places. [3]

10 The plane p passes through the points A, B and C with coordinates $(0, 1, 1)$, $(2, -1, 4)$ and $(-2, -1, 0)$ respectively.

(i) Show that a cartesian equation of the plane p is $2x - y - 2z = -3$. **[3]**

The line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R}$.

(ii) Find the acute angle between l and p. **[2]**

The point Q has position vector $5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

(iii) Show that Q lies on the line l. **[1]**

(iv) It is given that a variable point R lies on the plane p and is at a distance of $\sqrt{45}$ from the point Q. Find the foot of perpendicular from the point Q to the plane p and hence describe geometrically the locus of R. **[6]**

(v) Find a vector equation of the line which is a reflection of the line l in plane p. **[3]**

11 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{2x + k}{x - 2}, x \in \mathbb{R},\ x \neq 2,$$

where k is a positive constant.

(i) Sketch the graph of $y = \mathrm{f}(x)$, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the x and y axes. **[3]**

(ii) Describe fully a sequence of transformations which would transform the curve $y = \frac{1}{x}$ onto $y = \mathrm{f}(x)$. **[4]**

(iii) Find f^{-1} in a similar form and write down the range of f^{-1}. **[3]**

(iv) Hence or otherwise, find f^2.

Find the value of $\mathrm{f}^{2017}\left(\frac{1}{2}\right)$, leaving your answer in terms of k. **[4]**

The function g is defined by

$$\mathrm{g} : x \mapsto a + \sqrt{x - 3}, \quad x \in \mathbb{R},\ x > 3,$$

where a is a real constant.

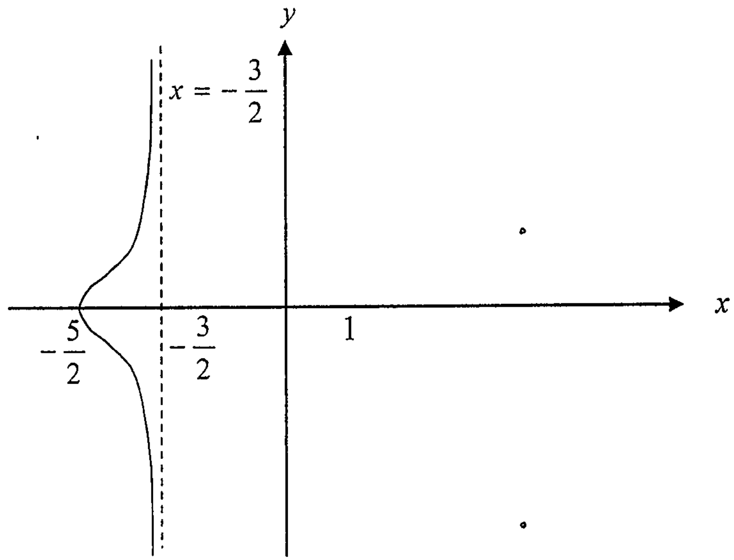
(v) Given that fg exists, write down an inequality for a and explain why gf does not exist. **[3]**

— THE END —

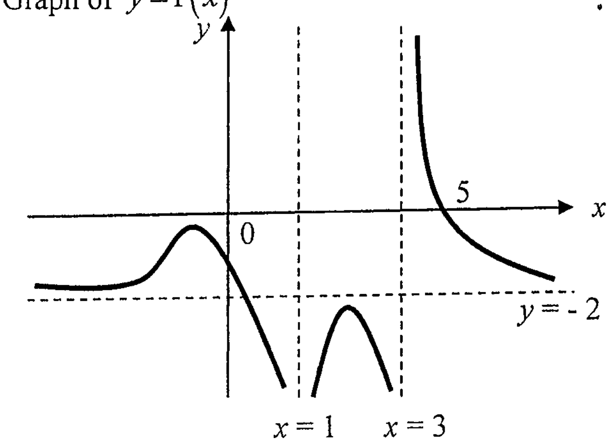
Topic: Transformation of Curves

Solution

(a) Graph of $y^2 = f(-2x)$



(b) Graph of $y = f(x)$



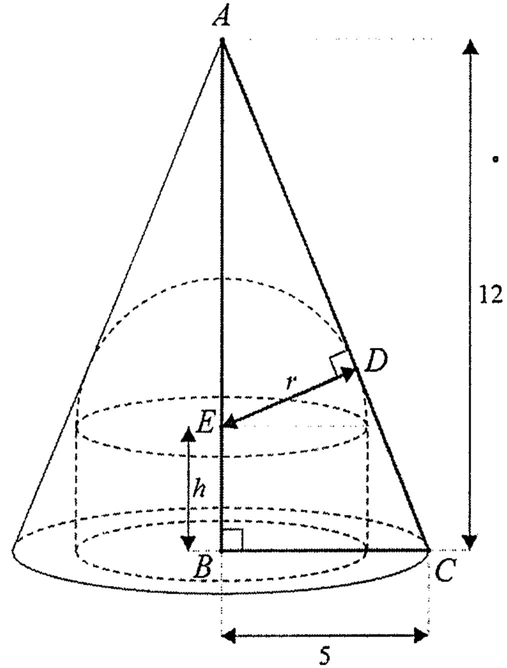
		Solution			
		(i) $\lvert\mathbf{a}\rvert = 2\lvert\mathbf{b}\rvert$ $\sqrt{16+36p^2+64} = 2\sqrt{4+9+16p^2}$ $80+36p^2 = 4\left(13+16p^2\right)$ $28p^2 = 28$ $p = 1$ or $p = -1$ (Reject $\because p > 0$) $\therefore p = 1$			
		(ii) Length of projection of **a** on **b**			
		(iii) $\frac{1}{\lvert\mathbf{b}\rvert}\lvert\mathbf{b}\cdot\mathbf{a}\rvert = \frac{\left\lvert\begin{pmatrix}2\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}4\\6\\-8\end{pmatrix}\right\rvert}{\sqrt{2^2+(-3)^2+4^2}} = \frac{42}{\sqrt{29}}$			

Topic: Complex Numbers

			Solution			
		(i)	The assumption is that a, b and c are all real.			
		(ii)	Let $x^3+ax^2+bx+c=(x-(3+\text{i}))(x-(3-\text{i}))(x-2)$ $=(x^2-6x+10)(x-2)$ $=x^3-8x^2+22x-20$ By comparing coefficients, we have $a=-8$, $b=22$ and $c=-20$.			

Solution

(i) Using the diagram provided,



Identify that ΔABC is similar to ΔADE.

$\therefore \ \frac{AE}{DE} = \frac{AC}{BC}$

Since $AC = \sqrt{5^2 + 12^2} = 13$, $BC = 5$, $DE = r$,

$\therefore \ \frac{AE}{r} = \frac{13}{5} \ \Rightarrow \ AE = \frac{13}{5}r$

$h = AB - AE = 12 - \frac{13}{5}r$

Since r and h are lengths,

$r \geq 0$ and $h \geq 0$ (i.e. $12 - \frac{13}{5}r \geq 0$, $r \leq \frac{60}{13}$).

$0 \leq r \leq \frac{60}{13}$

Solution

(ii) Volume of inscribed container,

$V = \pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= \pi r^2\left(12 - \frac{13}{5}r\right) + \frac{2}{3}\pi r^3$

$= 12\pi r^2 - \frac{29}{15}\pi r^3$

Differentiating this,

$\frac{dV}{dr} = 24\pi r - \frac{29}{5}\pi r^2$.

Consider $\frac{dV}{dr} = 0$.

i.e. $24\pi r - \frac{29}{5}\pi r^2 = 0$,

$\pi r(24 - \frac{29}{5}r) = 0$,

$r = 0$(rejected as $r \neq 0$) or $r = \frac{120}{29}$.

First derivative test

$\frac{dV}{dr} = 24\pi r - \frac{29}{5}\pi r^2$

r	$r = \frac{120}{29}^-$ e.g. $r = \frac{119}{29}$	$r = \frac{120}{29}$	$r = \frac{120}{29}^+$ e.g. $r = \frac{121}{29}$
$\frac{dV}{dr}$	$\frac{119}{145}\pi > 0$	0	$-\frac{121}{145}\pi < 0$

Maximum volume at $r = \frac{120}{29}$

		Solution			
		Second derivative test $\frac{dV}{dr} = 24\pi r - \frac{29}{5}\pi r^2$ $\frac{d^2V}{dr^2} = 24\pi - 2\left(\frac{29}{5}\right)\pi r$ $= 24\pi - 2\left(\frac{29}{5}\right)\pi r$ $= 24\pi - \frac{58}{5}\pi\left(\frac{120}{29}\right)$ $= -24\pi < 0$ (maximum volume)			

Solution

Let P_n be the statement that $u_n = \frac{2n}{(n-1)!}$ for $n \in \mathbf{Z}^+$, $n \geq 1$.

When $n=1$, LHS= $u_1 = 2$ (given)

$$\text{RHS} = \frac{2}{(0)!} = 2 = \text{LHS},$$

$\therefore P_1$ is true.

Assume that P_k is true for some k, where $k \in \mathbf{Z}^+, k \geq 1$ i.e.

$$u_k = \frac{2k}{(k-1)!}$$

To prove P_{k+1} is also true ,i.e., $u_{k+1} = \frac{2(k+1)}{k!}$

$$\text{LHS} = u_{k+1} = \frac{k+1}{(k)^2} u_k$$

$$= \frac{k+1}{(k)^2} \frac{2k}{(k-1)!}$$

$$= \frac{2(k+1)}{k(k-1)!}$$

$$= \frac{2(k+1)}{k!} = \text{RHS}$$

Since P_1 is true, and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n \in \mathbf{Z}^+$.

		Solution			
		(b) $u_2=\frac{2}{(1)^2}u_1=\frac{2}{1}a=2a=\frac{2}{1!}a$ $u_3=\frac{3}{(2)^2}u_2=\frac{3}{(2)^2}2a=\frac{3}{(2)}a=\frac{3}{2!}a$ $u_4=\frac{4}{(3)^2}u_3=\frac{4}{(3)^2}\frac{3}{(2)}a=\frac{4}{(3\times 2)}a=\frac{4}{3!}a$			
		"Hence" approach $u_n=\frac{n}{(n-1)!}a$ by observation from part (ii) "Otherwise" approach $u_n=\frac{n}{(n-1)^2}u_{n-1}$ $=\frac{n}{(n-1)^2}\frac{n-1}{(n-2)^2}u_{n-2}$ $=\frac{n}{(n-1)^2}\frac{n-1}{(n-2)^2}\frac{n-2}{(n-3)^2}u_{n-3}$ $=\frac{n}{(n-1)^2}\frac{n-1}{(n-2)^2}\frac{n-2}{(n-3)^2}\cdots\frac{3}{(2)^2}\frac{2}{(1)^2}u_1$ $=\frac{n}{(n-1)^1}\frac{1}{(n-2)^1}\frac{1}{(n-3)^1}\cdots\frac{1}{(2)^1}\frac{1}{(1)^1}a$ $=\frac{n}{(n-1)!}a$			

		Solution			
		(i) $1-2r = A(r+1)+Br$ $= (A+B)r + A$ $\therefore A=1, B=-3$			
		(ii) $\sum_{r=1}^{n} \frac{1-2r}{3^r} = \sum_{r=1}^{n} \frac{(r+1)-3r}{3^r}$ $= \sum_{r=1}^{n} \left(\frac{r+1}{3^r} - \frac{r}{3^{r-1}} \right)$ $= \left(\frac{2}{3^1} - \frac{1}{3^0} \right) +$ $\left(\frac{2}{3^2} - \frac{2}{3^1} \right) +$ $\vdots$ $\left(\frac{n}{3^{n-1}} - \frac{n-1}{3^{n-2}} \right) +$ $\left(\frac{n+1}{3^n} - \frac{n}{3^{n-1}} \right)$ $= \frac{n+1}{3^n} - 1$			

6	Sigma Notation					
			Solution			
			(iii) $\displaystyle\sum_{r=1}^{\infty}\frac{2-2r}{3^r}=\sum_{r=1}^{\infty}\frac{1-2r+1}{3^r}$ $\displaystyle=\sum_{r=1}^{\infty}\left(\frac{1-2r}{3^r}+\frac{1}{3^r}\right)$ $\displaystyle=\sum_{r=1}^{\infty}\left(\frac{1-2r}{3^r}\right)+\sum_{r=1}^{\infty}\left(\frac{1}{3^r}\right)$ $\displaystyle=\lim_{n\to\infty}\left(\frac{n+1}{3^n}-1\right)+\frac{\frac{1}{3}}{1-\frac{1}{3}}$ $\displaystyle=-1+\frac{1}{2}$ $\displaystyle=-\frac{1}{2}$			

			Solution			
		(i)	For $y=\sqrt{1+\ln(1+x)}$ to be well-defined, $1+x>0$ and $1+\ln(1+x)\geq 0$ $\ln(1+x)\geq -1$ $x>-1$ and $1+x\geq e^{-1}$ $x\geq e^{-1}-1$ $\therefore x\geq e^{-1}-1$			
		(ii)	**By Implicit Differentiation,** $y=\sqrt{1+\ln(1+x)}$ $y^2=1+\ln(1+x)$ $\Rightarrow \ln(1+x)=y^2-1$ $\Rightarrow 1+x=e^{y^2-1}$ Differentiate implicitly with respect to x, $2y\frac{dy}{dx}=\frac{1}{1+x}$ $=\frac{1}{e^{y^2-1}}$ $=e^{1-y^2}$ (shown)			
		(iii)	Differentiate the above results implicitly with respect to x, $2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=\frac{d}{dx}(1-y^2)\cdot e^{1-y^2}$ $2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=-2y\frac{dy}{dx}\cdot e^{1-y^2}$ When $x=0$, $y=\sqrt{1+\ln 1}=1$, $\frac{dy}{dx}=\frac{1}{2}$,			

		Solution			
		$\frac{d^2y}{dx^2} = -\frac{3}{4}$, $y = 1 + \frac{1}{2}x + \frac{-\frac{3}{4}}{2!}x^2 + \ldots$ $= 1 + \frac{1}{2}x - \frac{3}{8}x^2 + \ldots$			
		(iv) $y = \left(1 + \ln(1+x)\right)^{\frac{1}{2}}$ $= \left(1 + \left(x - \frac{x^2}{2} + \ldots\right)\right)^{\frac{1}{2}}$ $= 1 + \frac{1}{2}\left(x - \frac{x^2}{2} + \ldots\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(x - \frac{x^2}{2} + \ldots\right)^2 + \ldots$ $= 1 + \frac{1}{2}x - \frac{1}{4}x^2 - \frac{1}{8}\left(x - \frac{x^2}{2}\right)\left(x - \frac{x^2}{2}\right) + \ldots$ $= 1 + \frac{1}{2}x - \frac{1}{4}x^2 - \frac{1}{8}x^2 + \ldots$ $= 1 + \frac{1}{2}x - \frac{3}{8}x^2 + \ldots$			

		Solution			
		(i) $z^2 - 6z + 36 = 0 \Rightarrow z = \dfrac{6 \pm \sqrt{36 - 4(1)(36)}}{2} = 3 \pm 3\sqrt{3}\mathrm{i}$			
		Thus, $z_1 = 6e^{\mathrm{i}\frac{\pi}{3}}$ and $z_2 = 6e^{-\mathrm{i}\frac{\pi}{3}}$			
		(ii) $\dfrac{\left(6e^{i\frac{\pi}{3}}\right)^4}{\left(6e^{i\left(\frac{\pi}{2}-\frac{\pi}{3}\right)}\right)} = 6^3 e^{i(\frac{7\pi}{6})}$ $= 6^3\left[\cos\left(\dfrac{-5\pi}{6}\right) + i\sin\left(\dfrac{-5\pi}{6}\right)\right]$			
		(iii) $z_2 = 6e^{-\mathrm{i}\frac{\pi}{3}} \Rightarrow z_2{}^n = 6^n e^{\mathrm{i}\left(-\frac{n\pi}{3}\right)}$ Since $z_2{}^n \in \square^+$, $-\frac{n\pi}{3} = 2k\pi$ for some integer k. $n = -6k$. $n = \ldots, 12, 6, 0, -6, -12, \ldots$ Smallest positive integer $n = 6$.			

Topic: Techniques of Integration / Definite Integrals

		Solution			
		(i) $\frac{du}{dx}=\frac{1}{2\sqrt{x+1}}$ $=\frac{1}{2u}$ $\int\frac{\sqrt{x+1}}{x-1}dx=\int\frac{u}{u^2-2}.2u\,du$ $=2\int(1+\frac{2}{u^2-2})du$ $=2u+\frac{2}{\sqrt{2}}\ln\left\|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right\|+c$ $=2\sqrt{x+1}+\sqrt{2}\ln\left\|\frac{\sqrt{x+1}-\sqrt{2}}{\sqrt{x+1}+\sqrt{2}}\right\|+c$			
		(ii) (a) Area of R $=5(1)-\int_3^8\frac{\sqrt{x+1}}{x-1}dx$ $=5-\left[2\sqrt{x+1}+\sqrt{2}\ln\left\|\frac{\sqrt{x+1}-\sqrt{2}}{\sqrt{x+1}+\sqrt{2}}\right\|\right]_3^8$ $=5-\{2+\sqrt{2}\ln\left(\frac{3-\sqrt{2}}{3+\sqrt{2}}\right)-\sqrt{2}\ln\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\}$ $=3-\left(\sqrt{2}\ln\left(\frac{3-\sqrt{2}}{3+\sqrt{2}}.\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)\right)$			

			Solution			
			$= 3 - \left(\sqrt{2}\ln\left(\frac{4+\sqrt{2}}{4-\sqrt{2}}\right)\right)$ $= 3 - \sqrt{2}\ln\left(\frac{-4-\sqrt{2}}{-4+\sqrt{2}}\right)$			
		(ii) **(b)**	Vol. generated $= \pi(1^2).5 - \pi\int_3^8 \frac{x+1}{(x-1)^2}\,dx$ $= 9.52830$ $= 9.53 \text{ units}^3$			

Solution

(i)

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ -8 \end{pmatrix} = 4\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

Choose normal vector $\underset{\sim}{n}_p$ for plane $p = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

$$p: \ \mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = -3$$

A cartesian equation of the plane p is $2x - y - 2z = -3$

(ii) Let the acute angle between l and p be θ.

The angle between the normal vector $\underset{\sim}{n}_p$ (for plane p) and the direction vector $\underset{\sim}{m}_l$ (for line l),

$$\alpha = \cos^{-1} \frac{\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}{\left|\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\right|} = \cos^{-1} \frac{6}{3\sqrt{5}} = 26.565^\circ$$

			Solution			
			$\therefore \theta = 90° - 26.565° = 63.4°$ (to 1d.p.) or 1.11 rad			
			Alternative : Let the acute angle between l and p be θ. $\sin\theta = \dfrac{\left\lvert \begin{pmatrix}2\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix}\right\rvert}{\left\lvert \begin{pmatrix}2\\-1\\-2\end{pmatrix}\right\rvert \left\lvert \begin{pmatrix}1\\0\\-2\end{pmatrix}\right\rvert} = \dfrac{6}{3\sqrt{5}}$ $\theta = \sin^{-1}\dfrac{6}{3\sqrt{5}} = 63.4°$ (to 1d.p.) or 1.11 rad (3 s.f.)			
	2	(iii)	Assume Q lies on the line l. $\begin{pmatrix}5\\-1\\-2\end{pmatrix} = \begin{pmatrix}2\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-2\end{pmatrix} \Rightarrow \begin{cases}5 = 2+\lambda\\-1=-1\\-2=4-2\lambda\end{cases} \Rightarrow \begin{cases}\lambda=3\\ \\ \lambda=3\end{cases}$ Since $\lambda = 3$ is consistent throughout, Q lies on the line l. Alternative 1:			

Solution

Since $\overrightarrow{OQ} = \begin{pmatrix}5\\-1\\-2\end{pmatrix} = \begin{pmatrix}2\\-1\\4\end{pmatrix} + \begin{pmatrix}3\\0\\-6\end{pmatrix}$

$= \begin{pmatrix}2\\-1\\4\end{pmatrix} + 3\begin{pmatrix}1\\0\\-2\end{pmatrix}$

Q lies on the line l.

Alternative 2:

$l: \dfrac{x-2}{1} = \dfrac{z-4}{-2}, \; y = -1$

$Q = (5, -1, -2)$, i.e. $x = 5, \; y = -1, \; z = 4$.

$\dfrac{x-2}{1} = \dfrac{5-2}{1} = 3, \quad \dfrac{z-4}{-2} = \dfrac{-2-4}{-2} = 3$.

Hence, Q lies on the line l.

(iv) Let F be the foot of perpendicular from the point Q to the plane p.

$l_{QF} : \mathbf{r} = \begin{pmatrix}5\\-1\\-2\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}, \mu \in \square$

Since F lies on l_{QF}, $\overrightarrow{OF} = \begin{pmatrix}5\\-1\\-2\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}$, for some $\mu \in \square$.

Solution

Since F also lies on plane p, $\overrightarrow{OF}\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-3$.

$$\left[\begin{pmatrix}5\\-1\\-2\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-3$$

$$2(5+2\mu)-(-1-\mu)-2(-2-2\mu)=-3$$

$$15+9\mu=-3\Rightarrow\mu=-2$$

$$\therefore\overrightarrow{OF}=\begin{pmatrix}5\\-1\\-2\end{pmatrix}-2\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}$$

The foot of perpendicular from the point Q to the plane p is $(1,1,2)$.

<u>Alternative :</u>

$\overrightarrow{QF}=(\overrightarrow{QA}\cdot\hat{n})\,\hat{n}$, where $\underset{\sim}{n}$ is a normal vector of p

$$=\left(\begin{pmatrix}-5\\2\\3\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right)\frac{1}{3}\begin{pmatrix}2\\-1\\-2\end{pmatrix}$$

$$=-2\begin{pmatrix}2\\-1\\-2\end{pmatrix}$$

Solution

$$\therefore \overrightarrow{OF} = \overrightarrow{OQ} + \overrightarrow{QF} = \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

The foot of perpendicular from the point Q to the plane p is $(1,1,2)$.

$$\overrightarrow{QF} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$$

$$\left|\overrightarrow{QF}\right| = \sqrt{(-4)^2 + 2^2 + 4^2} = 6$$

$$RF = \sqrt{45-36} = 3$$

The locus of R is a circle that lies in plane p with centre $(1,1,2)$ and radius 3.

(v) Let Q' be the image of Q in plane p.

$$\overrightarrow{OF} = \tfrac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OQ'})$$

$$\overrightarrow{OQ'} = 2\overrightarrow{OF} - \overrightarrow{OQ}$$

$$= 2\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$$

Equivalent Methods

$$\overrightarrow{OQ'} = \overrightarrow{OF} + \overrightarrow{QF}$$

$$\overrightarrow{OQ'} = \overrightarrow{OQ} + \overrightarrow{QQ'}$$

$$= \overrightarrow{OQ} + 2\overrightarrow{QF}$$

Solution

$$\overrightarrow{BQ'} = \overrightarrow{OQ'} - \overrightarrow{OB}$$

$$= \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix}$$

<u>Alternative:</u>

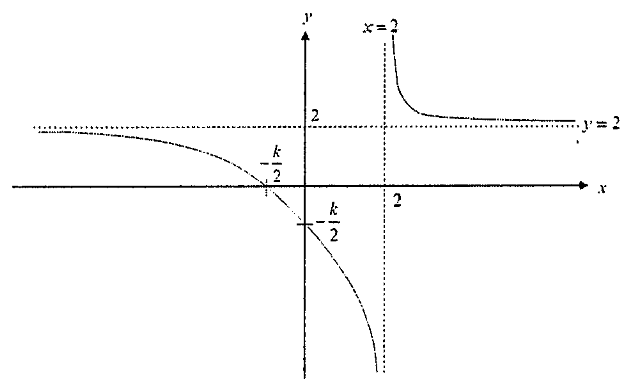
$$\overrightarrow{BQ'} = \overrightarrow{BQ} + \overrightarrow{QQ'}$$

$$= \left[\begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}\right] + 2\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix}$$

A vector equation of the line which is a reflection of the line l in plane p is

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \gamma \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix}, \gamma \in \mathbb{R}$$

or $$\mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix} + \gamma \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix}, \gamma \in \mathbb{R}$$

1	Topic: Graphing Techniques & Functions					
			Solution			
			(i) $y = \frac{2x+k}{x-2} = 2 + \frac{(k+4)}{x-2}$			
			Vertical Asymptote: $x = 2$ Horizontal Asymptote: $y = 2$			
			x-intercept: $x = -\frac{k}{2}$ y-intercept: $y = -\frac{k}{2}$			
						

Solution

$$f(x) = \frac{1}{x}$$

$$\downarrow \quad A$$

$$f(x-2) = \frac{1}{x-2}$$

$$\downarrow \quad B$$

$$(k+4)f(x-2) = \frac{(k+4)}{x-2}$$

$$\downarrow \quad C$$

$$2+(k+4)f(x-2) = 2+\frac{(k+4)}{x-2}$$

A: Translation in the positive x direction by 2 units

B: Scaling parallel to the x direction (// to y-axis) by a factor of $(k+4)$

C: Translate in the positive y direction by 2 units

(iii) Let $y = \frac{2x+k}{x-2}$

$$y(x-2) = 2x+k$$

$$yx - 2y = 2x + k$$

$$yx - 2x = 2y + k$$

$$x(y-2) = 2y+k$$

$$x = \frac{2y+k}{y-2}$$

$$f^{-1}(y) = \frac{2y+k}{y-2}$$

	Solution
	$f^{-1}(x)=\dfrac{2x+k}{x-2}$
	$D_{f^{-1}}=R_f=D_f=(-\infty,2)\cup(2,\infty)$
	$f^{-1}:x\mapsto\dfrac{2x+k}{x-2},\qquad x\in\square,\ x\neq 2,$
	$R_{f^{-1}}=D_f=R_f=(-\infty,2)\cup(2,\infty)$
(iv)	$\because f(x)=f^{-1}(x)$
	$\therefore f^2=ff(x)=f\left[f^{-1}(x)\right]=x$
	$D_{f^2}=D_f=(-\infty,2)\cup(2,\infty)$
	$\therefore f^{2017}\left(\dfrac{1}{2}\right)=f\left(\dfrac{1}{2}\right)$
	$=\dfrac{2\left(\frac{1}{2}\right)+k}{\left(\frac{1}{2}\right)-2}$ $=\dfrac{1+k}{-\frac{3}{2}}$ $=\dfrac{-2-2k}{3}$
(v)	$f(x)=\dfrac{2x+k}{x-2}$ $D_f=(-\infty,2)\cup(2,\infty)$ $R_f=(-\infty,2)\cup(2,\infty)$

		Solution			
		$g(x)=a+\sqrt{x-3}$ $D_g=(3,\infty)$ $R_g=(a,\infty)$ Since $fg(x)$ exists, $R_g \subseteq D_f$ $R_g=(a,\infty)$ $D_f=(-\infty,2)\cup(2,\infty)$ $\therefore a\geq 2$ or a subset of it.			
		For $gf(x)$ to exist, $R_f \subseteq D_g$ $R_f=(-\infty,2)\cup(2,\infty)$ $D_g=(3,\infty)$ Since $R_f \not\subset D_g$, $gf(x)$ does not exist.			

General Certificate of Education Advanced Level
Higher 2
JC2 Preliminary Examination

MATHEMATICS 9740/02

Paper 2

30 August 2016

3 hours

Additional Materials: List of Formulae (MF15)

Name: ____________________________ **Class:** ________________

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, arrange your answers in NUMERICAL ORDER.

The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 **(i)** Prove that the substitution $u = x^2 + y^2$ reduces the differential equation $y\frac{\mathrm{d}y}{\mathrm{d}x} + x = \sqrt{x^2 + y^2}$ to

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2\sqrt{u}.$$

Hence, show that the general solution of the differential equation $y\frac{\mathrm{d}y}{\mathrm{d}x} + x = \sqrt{x^2 + y^2}$ is given by $\sqrt{x^2 + y^2} = x + D$, where D is an arbitrary constant. **[4]**

(ii) The result in part **(i)** represents a family of curves. On a single diagram, sketch a non-linear member of the family which passes through the point $(-2, 0)$.
You should state the equation of the graph and axial intercepts clearly on the diagram. **[3]**

(iii) State an equation of the line of symmetry for the curve in part **(ii)**. **[1]**

2 The three distinct roots of the equation $x^3 - 1 = 0$ are denoted by $1, \omega$ and ω^2.

(a) Without first finding ω explicitly, show that $1 + \omega + \omega^2 = 0$. **[2]**

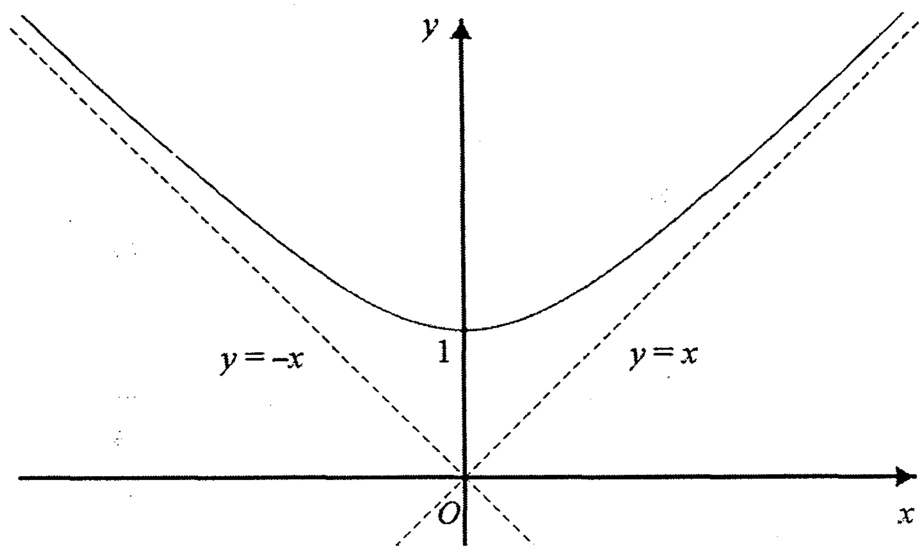
(b) Given now that $0 < \arg(\omega) < \pi$, sketch, on a single Argand diagram, the loci given by

(i) $|z - \omega| = |\omega|$ and **[3]**

(ii) $\arg(z + 1) = \pi + \arg(\omega^2)$. **[2]**

Hence, find the complex number that satisfies both loci, expressing your answer exactly in the form $a + \mathrm{i}b$, where a and b are real numbers. **[2]**

3 The diagram shows the graph of curve C represented by $y=\mathrm{f}(x)$, with oblique asymptotes $y=x$ and $y=-x$.



(a) On a separate diagram, sketch a graph of $y=\mathrm{f}'(x)$, clearly indicating the equation(s) of the asymptote(s) and axial-intercept(s). **[2]**

(b) The above curve C is represented by the parametric equations

$$x=\tan\theta,\ y=\sec\theta,\text{ where } -\frac{\pi}{2}<\theta<\frac{\pi}{2}.$$

(i) Show that the normal to the curve at point P, with coordinates $(\tan\theta,\ \sec\theta)$, for $0<\theta<\frac{\pi}{2}$, is given by $y=-x\operatorname{cosec}\theta+2\sec\theta$. **[2]**

(ii) The normal to the curve at point P intersects the x-axis at point N. Find the coordinates of the mid-point M of PN, in terms of θ. Hence find a cartesian equation of the locus of M, as θ varies. **[4]**

(iii) Taking O as the origin, show that the area of triangle OPN is $\tan\theta\sec\theta$.

Point P moves along the curve such that the rate of change of its parameter θ with respect to time t is given by $\frac{\mathrm{d}\theta}{\mathrm{d}t}=\cos\theta$. Find the exact rate of change of the area of triangle OPN when $\theta=\frac{\pi}{6}$. **[4]**

4 Adam and Gregory signed up for a marathon. In preparation for this marathon, Adam and Gregory each planned a 15-week personalised training programme. Adam runs 2.4 km on the first day of Week 1, and on the first day of each subsequent week, the distance covered is increased by 20% of the previous week. Gregory also runs 2.4 km on the first day of Week 1, but on the first day of each subsequent week, the distance covered is increased by d km, where d is a constant. Assume Adam and Gregory only run on the first day of each week.

(i) Find, in terms of d, the total distance covered by Gregory in these 15 weeks. [2]

(ii) Adam targets to cover a total distance of 170 km in these 15 weeks. Can Adam achieve this target? You must show sufficient working to justify your answer. [2]

(iii) It is given that Adam covers a longer distance than Gregory on the first day of the 15^{th} week. Find the maximum value of d, correct to 2 decimal places.
Using this value of d, show that the difference in the distance covered by Adam and Gregory for their 15^{th} week training is 0.134 km correct to 3 significant figures. [4]

(iv) Due to unforeseen circumstances, Adam has to end his training programme early. In order for Adam to cover a total distance of 170 km by the end of the 13 weeks, the distance covered has to be increased by x % of the previous week on each subsequent week from Week 1. Find x. [3]

Section B: Statistics [60 marks]

5 A bag contains four red and eight blue balls of which two of the red balls and six of the blue balls have the number "0" printed on them. The remaining balls have the number "1" printed on them. Three balls are randomly drawn from the bag without replacement.

(i) Show that the probability that at least one blue ball is drawn is $\frac{54}{55}$. [1]

Find the probability that

(ii) at least one ball of each colour is drawn, [2]

(iii) the sum of the numbers on the balls drawn is at least two. [3]

6 Packets of a particular brand of potato chips are delivered to a supermarket in boxes of 60. On average, 1.8 packets in a box are underweight. The number of underweight packets from a randomly chosen box is the random variable X.

Assume that X has a binomial distribution.

(i) Use a suitable approximation to find the probability that two randomly chosen boxes of potato chips contain more than 6 packets of underweight potato chips. State the parameter(s) of the distribution that you use. [4]

A batch of 50 boxes of potato chips is delivered to the supermarket.

(ii) Use a suitable approximation to find the probability that the mean number of underweight packets per box is more than 2. [3]

7 A group of 9 friends, including Albert and Ben, are having dinner at Albert's house. They sit in two groups: a row of 4 on a couch and a group of 5 at a round dining table with 5 identical seats.

Find the number of ways they can sit if

(i) there are no restrictions, [2]

(ii) Albert and Ben sit beside each other, [3]

(iii) Albert and Ben both sit on the couch or both sit at the round table, but they do not sit beside each other. [3]

8 A factory manufactures a certain product for sale. The following table gives the quantity of product manufactured, x units in thousands, and its corresponding cost of production, y dollars in thousands. The data is recorded during different months of a certain year.

Quantity of product, x	2.0	2.4	3.0	3.8	4.8	6.0	7.2	8.2	9.4
Cost of production, y	10	19	35	47	58	35	78	80	81

(i) Draw a scatter diagram for the data. [1]

One of the values of y appears to be incorrect.

(ii) Indicate the corresponding point in your diagram by labelling it P. [1]

Remove P from the set of data.

(iii) By using the scatter diagram for the remaining points, explain whether $y = a + bx$ or $y = a + b \ln x$ is the better model for the relationship between x and y. [1]

(iv) Using the better model chosen in part **(iii)**, find the product moment correlation coefficient and the equation of a suitable regression line.
Explain what happens to the product moment correlation coefficient and the equation of the regression line if the factory decides to include a fixed cost of M thousand dollars for purchasing a packing machine to the cost of production, y. [4]

(v) Use the regression line found in part **(iv)** to estimate the cost of production when the quantity produced is 6000 units and comment on its reliability. [2]

9 The finishing times in a 10km race with a large number of runners follow a normal distribution. After 40 minutes, 10% of the runners have completed the race. After one hour, 35% of the runners have yet to complete the race. The first 20% of runners who finish the race receive a medal.

(i) Show that, correct to 1 decimal place, the runners have running times with mean 55.4 minutes and standard deviation 12.0 minutes. **[2]**

(ii) Find the maximum time a runner can take to finish the race in order to receive a medal. **[2]**

A random sample of 12 runners is selected.

(iii) Find the probability that more than four runners receive a medal. **[2]**

(iv) Given that none of the runners receives a medal, find the probability that the slowest runner completes the race in under one hour. **[3]**

10 (a) A Physical Education teacher wants to plan a volleyball training programme for all students in a secondary school, where each student has exactly one CCA. In order to check on the current fitness level of students in the school, he selects a sample of students by choosing the Captains of every sports team and the Presidents of every Club and Society in the school.

(i) Explain briefly why this may not provide a representative random sample of the student population. **[1]**

(ii) Name a more appropriate sampling method which would provide a representative random sample and explain how it can be carried out in this context. **[3]**

(b) The vertical jump heights of players from a volleyball team are normally distributed with mean 40 cm. The coach claims that a particular training regime is effective in improving the players' jump heights. After the regime is implemented for a period of time, a random sample of 7 players is taken and their jump heights are recorded.

The sample mean is 42.1 cm and the sample standard deviation is k cm.
A test is to be carried out at the 10% level of significance to determine whether the training regime has been effective.

(i) State appropriate hypotheses for the test. **[1]**

(ii) Find the set of values of k for which the result of the test would be to reject the null hypothesis. **[3]**

(iii) State the conclusion of the test in the case where the sample variance is 15. **[2]**

11 The average number of calls per hour received by telephone operators at the Call Centre of bank ECBC is being reviewed.

(i) State, in context, two assumptions that need to be made for the number of calls received by a telephone operator to be well modelled by a Poisson distribution. **[2]**

The Call Centre has only three telephone operators at any point in time. One handles calls pertaining to credit card queries, another handles calls pertaining to business banking queries while the last operator handles calls pertaining to personal banking queries, with the numbers of calls received in one hour assumed to have the independent distributions $\text{Po}(\mu)$, $\text{Po}(6)$ and $\text{Po}(7)$ respectively.

(ii) It is given that the probability of receiving two calls pertaining to credit card queries within an hour is eight times that of receiving two calls pertaining to credit card queries within four hours.

Find the exact value of μ, expressing your answer in the form $\frac{a}{b}\ln 2$ where a and b are two positive integers to be found. **[3]**

(iii) On a certain day, the Call Centre receives more than 50 calls from 1200 to 1400 hours. Find the probability that there are no calls pertaining to credit card queries during this period. **[3]**

(iv) Using suitable approximations, find the probability that there are more calls pertaining to business banking queries than personal banking queries within a two-hour period. **[3]**

— **THE END** —

Section A: Pure Mathematics [40 marks]

Topic: Differential Equations

Solution

(i) Given $u = x^2 + y^2$, differentiate with respect to x:

$$\frac{du}{dx} = 2x + 2y\frac{dy}{dx}$$

$$x + y\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx} \quad \text{(I)}$$

Substitute (I) & $u = x^2 + y^2$ and into D.E:

$$\frac{1}{2}\frac{du}{dx} = \sqrt{u} :$$

$$\frac{du}{dx} = 2\sqrt{u} \text{ (shown)}$$

Hence,

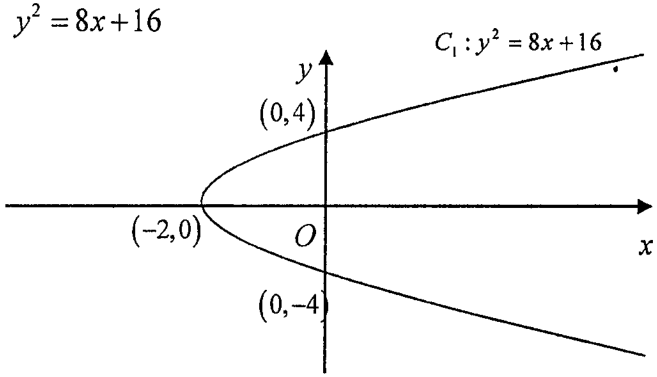
$$\frac{1}{\sqrt{u}}\frac{du}{dx} = 2$$

$$u^{-\frac{1}{2}}\frac{du}{dx} = 2$$

Integrate both sides with respect to x:

$$\int u^{-\frac{1}{2}}\frac{du}{dx}\,dx = \int 2\,dx$$

$$\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2x + C, \text{ where } C \text{ is an arbitrary constant}$$

		Solution			
		$2u^{\frac{1}{2}} = 2x + C$ $\sqrt{u} = x + \frac{C}{2}$ $\sqrt{x^2 + y^2} = x + D$, where $D = \frac{C}{2}$			
		(ii) $\sqrt{x^2 + y^2} = x + D$ $x^2 + y^2 = (x + D)^2$ $y^2 + x^2 = x^2 + 2Dx + D^2$ $y^2 = 2Dx + D^2$ $y = \pm\sqrt{2Dx + D^2}$ When $x = -2, y = 0$ $0 = -4D + D^2$ $D(D-4) = 0$ $D = 0$ (rej.) or $D = 4$ $y^2 = 8x + 16$  			
		(iii) The equation of line of symmetry is $y = 0$.			

Solution

Using Geometric series,

$$1+\omega+\omega^2=\frac{\omega^3-1}{\omega-1}=0 \qquad (\text{as } \omega^3=1)$$

Or

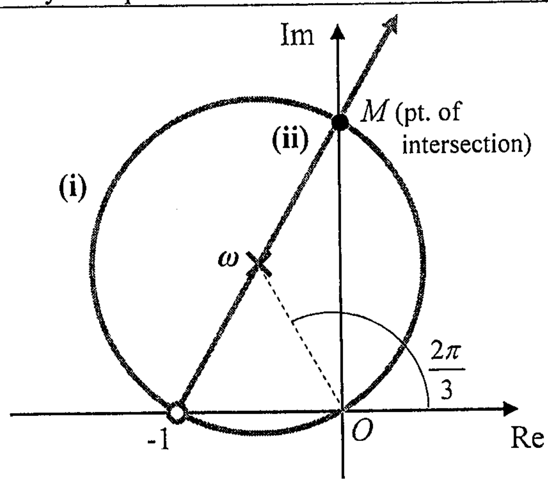
$$\omega^3-1=0$$

Since ω is a root of $x^3-1=0$, $(\omega-1)(\omega^2+\omega+1)=0$

Since $\omega\neq 1, \omega^2+\omega+1=0$

Or

Any acceptable method

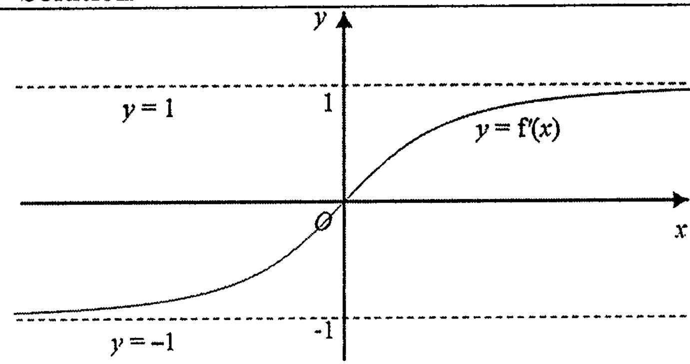


Using Pythagoras Theorem, $OM=\sqrt{2^2-1^2}=\sqrt{3}$.

$\therefore$ The complex number is $\sqrt{3}\mathrm{i}$.

Solution

(a)



(b) (i) Given $x=\tan\theta$, $y=\sec\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

$\frac{dx}{d\theta}=\sec^2\theta$, $\frac{dy}{d\theta}=\sec\theta\tan\theta$,

$\frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}$

$=\frac{\sec\theta\tan\theta}{\sec^2\theta}=\frac{\tan\theta}{\sec\theta}=\sin\theta$

At point P, gradient of normal $=-\frac{1}{\sin\theta}$.

Equation of the normal to the curve at P:

$$y-\sec\theta=-\frac{1}{\sin\theta}(x-\tan\theta),$$

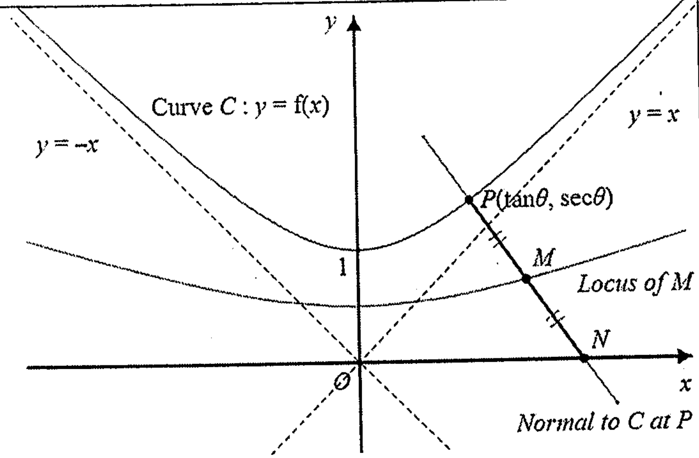
$$y-\frac{1}{\cos\theta}=-x\frac{1}{\sin\theta}+\frac{1}{\cos\theta},$$

$$y=-x\ \text{cosec}\,\theta+2\sec\theta. \qquad \text{(shown)}$$

Topic: Application of Differentiation

Solution

(b) (ii)



x-intercept of the normal at P :

$$0 = -x \operatorname{cosec}\theta + 2\sec\theta,$$

$$x = 2\frac{\sec\theta}{\operatorname{cosec}\theta} = 2\tan\theta .$$

$\therefore$ Point N is $(2\tan\theta,\ 0)$.

Mid-point of PN is M

$$\left(\frac{x_P+x_N}{2},\ \frac{y_P+y_N}{2}\right) = \left(\tfrac{3}{2}\tan\theta,\ \tfrac{1}{2}\sec\theta\right).$$

Locus of point M is given by the parametric equations

$$x = \tfrac{3}{2}\tan\theta,\ y = \tfrac{1}{2}\sec\theta, \text{ where } -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}.$$

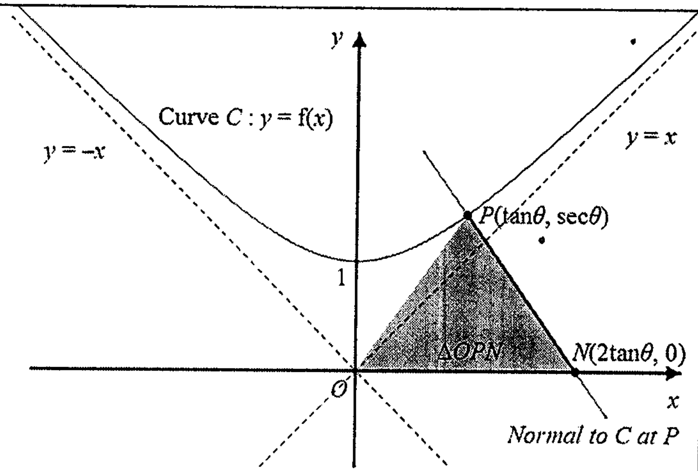
Since $\sec^2\theta - \tan^2\theta = 1$,

and that $\sec\theta = 2y$, $\tan\theta = \frac{2}{3}x$,

$\therefore\ (2y)^2 - (\frac{2}{3}x)^2 = 1$, a cartesian equation for the locus of M.

Solution

(b) (iii)



A, area of $\Delta OPN = \frac{1}{2}(ON)\left(\begin{smallmatrix}\text{Height of } P\\ \text{w.r.t. } x\text{-axis}\end{smallmatrix}\right)$

$= \frac{1}{2}(2\tan\theta)\ (\sec\theta) = \tan\theta\sec\theta$

($ON = 2\tan\theta$, from (b)(ii)

assuming $\theta > 0$.)

$\frac{dA}{d\theta} = \left(\sec^2\theta\right)\sec\theta + \tan\theta\left(\sec\theta\tan\theta\right)$

$= \sec^3\theta + \sec\theta\tan^2\theta$

Rate of change of area of ΔOPN,

$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

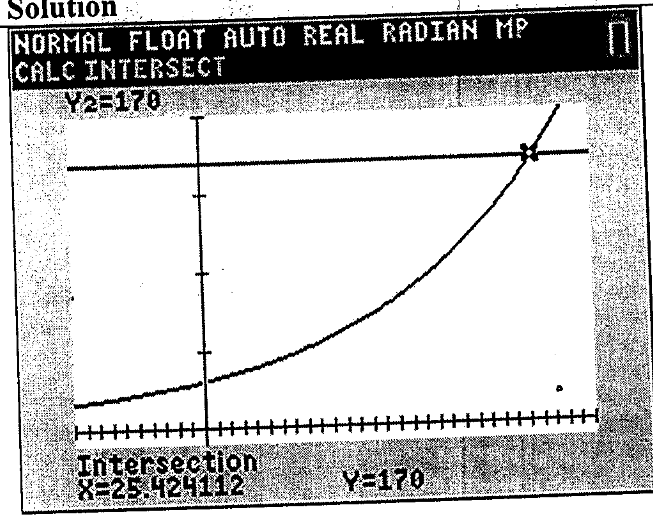
$= (\sec^3\theta + \sec\theta\tan^2\theta) \times \cos\theta$

$= \sec^2\theta + \tan^2\theta$

<u>Alternatively,</u>
Differentiating $A = \tan\theta\sec\theta$ implicitly with respect to time t,

		Solution			
		$\frac{dA}{dt}=\left[\left(\sec^2\theta\right)\sec\theta+\tan\theta\left(\sec\theta\tan\theta\right)\right]\times\frac{d\theta}{dt}$ $=(\sec^3\theta+\sec\theta\tan^2\theta)\times\cos\theta$ $=\sec^2\theta+\tan^2\theta$ When $\theta=\frac{\pi}{6}$, $\sec\theta=\frac{1}{\cos\theta}=\frac{2}{\sqrt{3}}$, $\tan\theta=\frac{1}{\sqrt{3}}$ $\therefore\ \frac{dA}{dt}=\sec^2\theta+\tan^2\theta=\frac{4}{3}+\frac{1}{3}=\frac{5}{3}$, rate of change of the area of ΔOPN when $\theta=\frac{\pi}{6}$.			

			Solution			
		(i)	S_{15} of B $=\frac{15}{2}(2(2.4)+14d)$ $=36+105d$			
		(ii)	S_{15} of A $=\frac{2.4\left((1.2)^{15}-1\right)}{1.2-1}$ $=172.88$ >170 Yes, Adam can achieve his target.			
		(iii)	U_{15} of A $> U_{15}$ of B U_{15} of A $-$ U_{15} of B >0 $2.4(1.2)^{14}-(2.4+14d)>0$ $d<2.02957$ $\max d=2.02(2\text{dp})$			
			$2.4(1.2)^{14}-(2.4+14(2.02))=0.134$ (shown)			
		(iv)	New S_{13} of A $=\frac{2.4\left(\left(1+\frac{r}{100}\right)^{13}-1\right)}{\left(1+\frac{r}{100}\right)-1}=170$ From GC,			

APGP						
		Solution  $x = 25.4\%$				

Section B: Statistics [60 marks]

Topic: Probability

		Solution			
		(i) P(at least one blue ball) = 1 – P(no blue balls) = 1 – P(3 red balls) $= 1 - \frac{^{4}C_3}{^{12}C_3}$ $= 1 - \frac{4}{220}$ $= \frac{54}{55}$ (Shown)			
		<u>Alternative Method</u> P(at least one blue ball) = 1 – P(3 red balls) $= 1 - \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$ $= \frac{54}{55}$ (Shown)			
		<u>Alternative Method (Direct Method)</u> P(at least one blue ball) = P(1 blue and 2 red) + P(2 blue and 1 red) + P(3 blue) $= \frac{^{8}C_1 \times {^{4}C_2}}{^{12}C_3} + \frac{^{8}C_2 \times {^{4}C_1}}{^{12}C_3} + \frac{^{8}C_3}{^{12}C_3}$ $= \frac{48}{220} + \frac{112}{220} + \frac{56}{220}$ $= \frac{54}{55}$ (Shown)			

		Solution			
		Alternative Method P(at least one blue ball) = P(1 blue and 2 red) + P(2 blue and 1 red) + P(3 blue) $= \left(\frac{8}{12}\times\frac{4}{11}\times\frac{3}{10}\times\frac{3!}{2!}\right)+\left(\frac{8}{12}\times\frac{7}{11}\times\frac{4}{10}\times\frac{3!}{2!}\right)+\left(\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\right)$ $= \frac{48}{220}+\frac{112}{220}+\frac{56}{220}$ $= \frac{54}{55}$ (Shown)			
		(ii) P(at least one of each colour drawn) = 1 – P(3 red) – P(3 blue) $= 1 - \frac{{}^{4}C_3}{{}^{12}C_3} - \frac{{}^{8}C_3}{{}^{12}C_3}$			
		$= 1 - \frac{4}{220} - \frac{56}{220}$ $= \frac{8}{11}$ or ≈ 0.727			
		Alternative Method P(at least one of each colour drawn) = P(1 blue and 2 red) + P(2 blue and 1 red) $= \frac{{}^{8}C_1 \times {}^{4}C_2}{{}^{12}C_3} + \frac{{}^{8}C_2 \times {}^{4}C_1}{{}^{12}C_3}$ $= \frac{48}{220}+\frac{112}{220}$ $= \frac{8}{11}$ or ≈ 0.727			
		(iii) There are 8 balls with "0" and 4 balls with "1" P(sum is at least two) = P(1, 1, 0) + P(1, 1, 1)			

		Solution			
		$= \frac{{}^4C_2 \times {}^8C_1}{{}^{12}C_3} + \frac{{}^4C_3}{{}^{12}C_3}$			
		$= \frac{48}{220} + \frac{4}{220}$ $= \frac{13}{55}$ or ≈ 0.236			
		<u>Alternative Method</u> P(sum is at least two) $= P(1, 1, 0) + P(1, 1, 1)$ $= \left(\frac{4}{12} \times \frac{3}{11} \times \frac{8}{10} \times \frac{3!}{2!}\right) + \left(\frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}\right)$ $= \frac{13}{55}$ or ≈ 0.236			

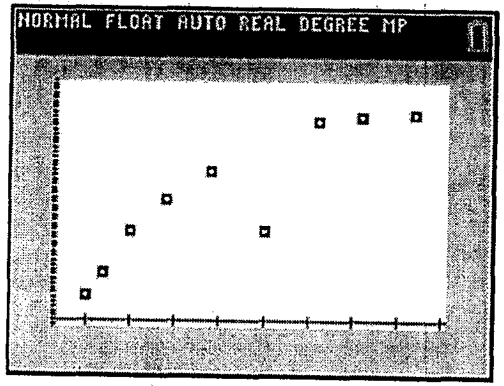
		Solution			
	(i)	$X \sim B(60, 0.03)$ Since $n = 60 > 50$ is large, $p = 0.03 < 0.1$ is small and $np = 1.8 < 5$, $X \sim \text{Po}(1.8)$ approximately $X_1 + X_2 \sim \text{Po}(3.6)$ approximately $P(X_1 + X_2 > 6) = 1 - P(X_1 + X_2 \leq 6)$ ≈ 0.073273 $= 0.0733\ (3\text{s.f.})$			
	(ii)	Since sample size = 50 is large, $\bar{X} \sim \text{N}\left(1.8, \frac{1.746}{50}\right)$ approximately by Central Limit Theorem $P(\bar{X} > 2) \approx 0.14225$ $= 0.142\ (3\text{s.f.})$			

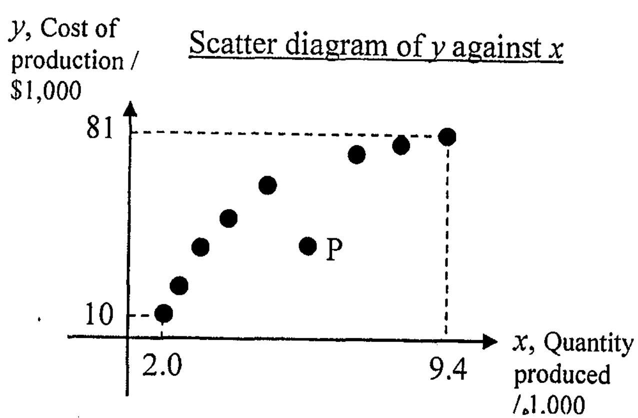
		(i) No. of ways with no restriction = (no. of ways to separate 9 people into a group of 4 and a group of 5) × (no. of ways to arrange the row of 4 people) × (no. of ways to arrange the circle of 5 people) $= {}^9C_4 \times 4! \times {}^5C_5 \times (5-1)!$ = 72 576			
		<u>Alternative Method</u> No. of ways with no restriction = (no. of ways to separate 9 people into a group of 5 and a group of 4) × (no. of ways to arrange the circle of 5 people) × (no. of ways to arrange the row of 4 people) $= {}^9C_5 \times (5-1)! \times {}^4C_4 \times 4!$ [Note: ${}^9C_4 = {}^9C_5$] = 72 576			
		(ii) No. of ways if Albert and Ben sit together = (Albert and Ben in the row) + (Albert and Ben in the circle) = (no. of ways to pick 2 remaining people for the row × arrange row people × A and B swap × arrange circle people) + (no. of ways to pick 3 remaining people for the circle × arrange circle people × A and B swap × arrange row people) $= ({}^7C_2 \times 3! \times 2 \times (5-1)!) + ({}^7C_3 \times (4-1)! \times 2! \times 4!)$			
		= 6048 + 10080 = 16128			
		(iii) No. of ways if Albert and Ben both sit on the couch or both sit at the table			

		= (Albert and Ben in the row) + (Albert and Ben in the circle) = (no. of ways to pick 2 remaining people for the row × arrange row people × arrange circle people) + (no. of ways to pick 3 remaining people for the circle × arrange row people × arrange circle people) $({}^7C_2 \times 4! \times (5-1)!) + ({}^7C_3 \times 4! \times (5-1)!)$ = 32256 No. of ways if Albert and Ben sit separately = 32256 – 16128 = 16128			
		Alternative Method No. of ways if Albert and Ben sit separately = (Albert and Ben in the row) + (Albert and Ben in the circle) = (no. of ways to pick 2 remaining people for the row × arrangement in row × arrangement in circle) + (no. of ways to pick 3 remaining people for the circle × arrangement in circle × arrangement in row) = (no. of ways to pick 2 remaining people for the row × [arrange remaining 2 people in row × slot in Albert and Ben] × arrangement in circle) + (no. of ways to pick 3 remaining people for the circle × [arrange remaining 3 people in circle × slot in Albert and Bert] × arrangement in row) $= ({}^7C_2 \times [\,2! \times {}^3C_2 \times 2!\,] \times (5-1)!) + ({}^7C_3 \times [\,(3-1)! \times {}^3C_2 \times 2!\,) \times 4!)$			
		= 10080 + 6048 = 16128			

Solution

(i) GC screenshot :



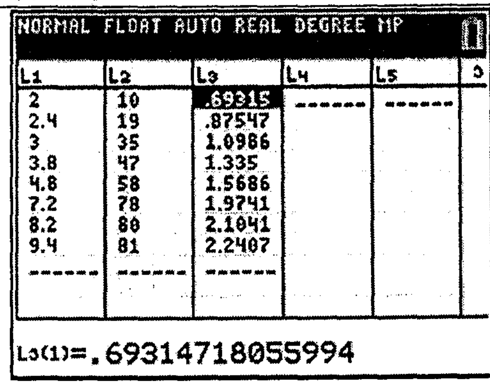


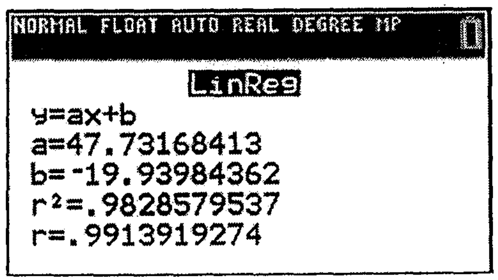
(ii) P = (6.0, 35).

(iii) The better model is $y = a + b \ln x$ since the set of data points in the scatter diagram exhibits a non-linear trend in which y increases at a decreasing rate as x increases, rather than a linear trend in which y increases at a constant rate with x.

Solution

(iv)





$r_{(\ln x),y} = 0.991$.

Line of regression y on $\ln x$.

$y = (47.73168413\ldots) \ln x + (-19.93984362\ldots)$

$\Rightarrow y = 47.7 \ln x - 19.9$

If a fixed cost is included in y, the value of $r_{(\ln x),y} = 0.991$ <u>is unchanged</u>, as there is no change in the strength of the linear relationship between ($\ln x$) and y, under translation.

However, as a constant value M is added to the y value of each data point, the regression line would be also <u>translated in the direction of the positive y-axis by M units</u>,
i.e. with resultant regression line equation being

Solution

$y = 47.7 \ln x - 19.9 + M$

or

$y - M = 47.7 \ln x - 19.9$

(v) When $x = 6$,

$$y = (47.73168413\ldots)\ln(6) - 19.93984362\ldots$$
$$= 65.5838534\ldots$$
$$\approx 65.6$$

Hence, the
estimated cost of production is \$ 65,600 (to 3 s.f).

This value is reliable, since $r = 0.991$ is close to 1 indicating a strong positive linear relationship between y and $\ln x$, and estimating y at $x = 6$ is an interpolation as $x = 6$ is within the range of values of x in the data used to construct the regression line (i.e. $2.0 \le x \le 9.4$).

		Solution			
		(i) Let X minutes be the random variable denoting the finishing time of a randomly selected runner in the race. $X \sim N(\mu, \sigma^2)$ $P(X \le 40) = 0.1 \Rightarrow P\left(Z \le \frac{40-\mu}{\sigma}\right) = 0.1$ $\Rightarrow \frac{40-\mu}{\sigma} = -1.2816$ $\Rightarrow \mu - 1.2816\sigma = 40 \quad (1)$ $P(X > 60) = 0.35 \Rightarrow P\left(Z \le \frac{60-\mu}{\sigma}\right) = 0.65$ $\Rightarrow \frac{60-\mu}{\sigma} = 0.38532$ $\Rightarrow \mu + 0.38532\sigma = 60 \quad (2)$ Solving (1) and (2), $\mu \approx 55.377 = 55.4(3\text{s.f.})$ and $\sigma \approx 11.998 = 12.0(3\text{s.f.})$			
		(ii) For $P(X \le a) = 0.2 \Rightarrow a \approx 45.300 = 45.3(3\text{s.f.})$ Maximal timing is 45.3 minutes. [Accept $a \approx 45.279$ for 5s.f. intermediate]			
		(iii) Let Y be the number of runners, out of 12, who receive a medal. $Y \sim B(12, 0.2)$ $P(Y > 4) = 1 - P(Y \le 4) \approx 0.072555 = 0.0726$			
		(iv) $P(\text{slowest runner finishes within 1 hour} \mid \text{all do not receive medal})$ $= \frac{P(\text{slowest runner finishes within 1 hour and all do not receive medal})}{P(\text{all do not receive medal})}$			

Solution

$$= \frac{P(\text{all runners finish within 1 hour and do not receive medal})}{(1-0.2)^{12}}$$

$$= \frac{P(45.279 \le X_1 \le 60 \text{ and } 45.279 \le X_2 \le 60 \text{ and } \ldots \text{ and } 45.279 \le X_{12} \le 60)}{(1-0.2)^{12}}$$

$$= \frac{[0.65-0.2]^{12}}{(0.8)^{12}}$$

$$= \left(\frac{0.45}{0.8}\right)^{12}$$

$$\approx 0.0010034 = 0.00100(3\text{s.f.})$$

Method 2:

Let A be the r.v. denoting the number of runners who finish in an hour and do not receive medal, out of 12.

$A \sim B(12, 0.65-0.2)$

$A \sim B(12, 0.45)$

Required probability $= \dfrac{P(A=12)}{P(Y=0)}$

	Solution
a(i)	This method is quota sampling and is not representative of the student population as there is no consideration of the size of each sports team and each club and society relative to the student population, and the student from each group is selected non-randomly i.e. only Captains and Presidents selected.
a(ii)	Stratified sampling. Divide school population by year/level (strata), calculate the proportion of each strata relative to the population and select the respective number randomly.
b(i)	Let X cm be the vertical jump height of a randomly selected volleyball player. $H_0: \mu = 40$ vs $H_1: \mu > 40$
b(ii)	Under H_0, test statistics: $T = \frac{\overline{X} - \mu}{s / \sqrt{n}} \sim t(n-1)$ Where $\mu = 40, \bar{x} = 42.1, n = 7, s^2 = \frac{7}{6}k^2$ To reject H_0, p-value < 0.1 $\Rightarrow P(T > t) < 0.1$ $\Rightarrow P\left(T > \frac{42.1 - 40}{\sqrt{7/6}\ \alpha / \sqrt{7}}\right) < 0.1$ $\Rightarrow P\left(T \leq \frac{2.1}{k / \sqrt{6}}\right) > 0.9$ $\Rightarrow \frac{2.1}{k / \sqrt{6}} > 1.4398$

			Solution			
			$\Rightarrow k < \dfrac{2.1\sqrt{6}}{1.4398} \approx 3.5728$ Required set is $\{k \in \square : 0 < k < 3.57\}$ or $(0, 3.57)$			
		b(iii)	Since $k^2 = 15 \Rightarrow k = \sqrt{15} > 3.57$, we do not reject H_0 at 10% level of significance and conclude that there is insufficient evidence to support the claim that the training regime is effective.			

Topic: Poisson Distribution

	Solution
(i)	Any two of the following 4 assumptions: (1) The calls are made independently of each other. (2) The probability of receiving two or more calls within a very short interval of time is negligible. OR Calls are received one at a time. (3) The average number of calls received over any time interval of the same duration within the day is constant. (4) The calls received occur randomly.
(ii)	Let W_n, X_n and Y_n be the random variables denoting the number of calls received in n hours pertaining to credit card queries, business banking queries and personal banking queries respectively. Then $W_n \sim \text{Po}(n\mu), X_n \sim \text{Po}(6n)$ and $Y_n \sim \text{Po}(7n)$ $8P(W_4 = 2) = P(W_1 = 2)$ where $W_1 \sim Po(\mu)$ and $\Rightarrow \frac{8e^{-4\mu}(4\mu)^2}{2!} = \frac{e^{-\mu}\mu^2}{2!} \Rightarrow e^{3\mu} = 128$ $\Rightarrow \mu = \frac{1}{3}\ln 128 = \frac{7}{3}\ln 2$
(iii)	Since W_2, X_2, and Y_2 are independent, $P(W_2 = 0 \mid W_2 + X_2 + Y_2 > 50) = \frac{P(W_2 = 0 \text{ and } W_2 + X_2 + Y_2 > 50)}{P(W_2 + X_2 + Y_2 > 50)}$ $= \frac{P(W_2 = 0 \text{ and } X_2 + Y_2 > 50)}{1 - P(W_2 + X_2 + Y_2 \leq 50)}$

			Solution			
			where $W_2 \sim \text{Po}\left(\frac{14}{3}\ln 2\right)$ and $W_2 + X_2 + Y_2 \sim \text{Po}\left(\frac{14}{3}\ln 2 + 26\right)$, $X_2 + Y_2 \sim \text{Po}(26)$ we have $= \frac{P(W_2 = 0) \cdot [1 - P(X_2 + Y_2 \le 50)]}{1 - P(W_2 + X_2 + Y_2 \le 50)}$ $\approx 0.0022359 = 0.00224$ (3s.f.)			
		(iv)	$X_2 \sim \text{Po}(12)$ and $Y_2 \sim \text{Po}(14)$. Since $\lambda_1 = 12 > 10$ and $\lambda_2 = 14 > 10$, $X_2 \sim \text{N}(12,12)$ approximately and $Y_2 \sim \text{N}(14,14)$ approximately Since X_2 and Y_2 are independent, $X_2 - Y_2 \sim \text{N}(-2, 26)$ approximately $P(X_2 > Y_2) = P(X_2 - Y_2 > 0)$ $\approx P(X_2 - Y_2 > 0.5)$ by continuity correction ≈ 0.31196 $= 0.312$ (3s.f.)			