Candidate Name: ______________________ Class: ____________

JC2 PRELIMINARY EXAM
Higher 2

MATHEMATICS
Paper 1

9740/01
14 Sep 2016
3 hours

Additional Materials: Cover page
Answer papers
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your full name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

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1 The police wish to crack a 3-digit passcode. The sum of the digits is 14. When the digits in the number are reversed, the new number becomes 495 more than the original number. The digit in the tens position is 3 more than the digit in the hundreds position. What is the passcode? [4]

2

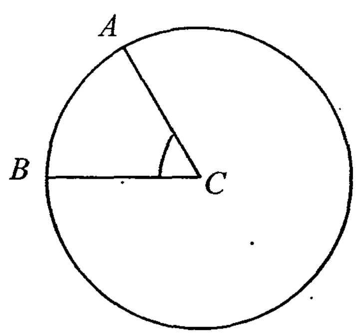


Fig. 1

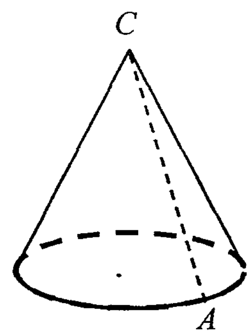


Fig. 2

Fig. 1 shows a circular card with centre C. A sector CAB is removed from the card, and the remaining card is folded such that AC and BC meet without overlapping to form a cone, as shown in Fig. 2 (A will meet B). Use differentiation to find the angle ACB exactly such that the volume of the cone is as large as possible. [6]

[It is given that a cone with radius r and height h has volume $\frac{1}{3}\pi r^2 h$ and curve surface area $\pi r l$ where l is the slant height.]

3 **(i)** Show that $\dfrac{4}{4r^2+12r+5}$ can be expressed as $\dfrac{A}{2r+1}+\dfrac{B}{2r+5}$, where A and B are constants to be determined. [2]

(ii) Hence, find an expression for $\displaystyle\sum_{r=1}^{n-1}\frac{2}{4r^2+12r+5}$ in terms of n. [3]

(iii) Hence, find the smallest value of n for which $\displaystyle\sum_{r=1}^{n-1}\frac{2}{4r^2+12r+5}$ is at least 99% of its sum to infinity. [3]

4 A curve C has parametric equations

$$x = 2a\cos^3\theta,\ \ y = a\sin^3\theta,$$

where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and a is a positive constant.

(i) A point P lies on C. Find, in terms of a, the exact coordinates of P, whose tangent is parallel to the line $2y = -x$. [4]

(ii) The tangent to C at the point $Q\left(2a\cos^3 t,\ \ a\sin^3 t\right)$, where $0 < t < \frac{\pi}{2}$, meets the x- and y-axes at R and S respectively. Find a cartesian equation of the locus of the mid-point of RS as t varies. [4]

5 The sum, S_n, of the first n terms of a sequence is given by

$$S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}.$$

(i) Find the values of S_1, S_2, S_3 and S_4. [2]

(ii) By expressing S_n in the form $[1 - \mathrm{f}(n)]$ for $n = 1,2,3,4$, find a conjecture for S_n in terms of n. [2]

(iii) Hence prove by mathematical induction the result of S_n for all positive integers n. [4]

6 Referred to the origin O, points A and B have position vectors **a** and **b** respectively. Point C lies on OA produced such that $OA : OC = 2 : 5$. Point D is on AB, between A and B such that $AD : DB = 4 : 1$.

(i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of **a** and **b**. [2]

(ii) Find a vector equation of line CD. [2]

(iii) Point E lies on CD produced, and it is also on OB, between O and B. Find $\overrightarrow{OE}$ and the ratio $OE : EB$. [5]

7 Newton's law of cooling states that the rate of cooling in t minutes is proportional to the difference between the body temperature $T\,^{\circ}C$ and its immediate surrounding temperature $T_o\,^{\circ}C$. Show that $T = T_o + Ae^{-kt}$, where A and k are positive constants.

[3]

Nurul is the chef of a dessert shop and she leaves her work place at 9pm daily. Before she leaves, she is required to cook a big pot of dessert and leave it to cool, before placing it in the refrigerator for the next business day. She takes 30 minutes to cook the pot of dessert to $100\,^{\circ}C$, and then leaves it to cool. After 15 minutes, the pot of dessert cools to $70\,^{\circ}C$.

The room temperature of the kitchen is $30\,^{\circ}C$, and the refrigerator can only accommodate items with temperature of at most $35\,^{\circ}C$. By what time, correct to the nearest minute, must Nurul start to cook the pot of dessert so that she will be able to leave her work place on time? [5]

8 A lion eyes its prey which is k m away and starts its chase with a leap of 2.5 m. Each subsequent leap of the lion is shorter than its preceding leap by 0.05 m. Its prey notices the lion's chase and runs away with a first leap of 1.5 m, with each subsequent leap 5% less than the previous leap. You may assume that the lion and the prey start running at the same moment and they complete the same number of leaps after the first leap.

(i) Find the total distance covered by the lion after n leaps. [2]

(ii) Find the total distance covered by the prey after n leaps. Deduce that the distance covered by the prey can never be greater than 30 m. [3]

(iii) Given $k = 25$, find the least number of leaps the lion needs to take to catch its prey. [3]

(iv) Assuming that the lion can cover a maximum of 30 leaps, find the least integer k, so that the prey will survive the hunt. [3]

9 **(a)** **(i)** If $t=\tan\frac{\theta}{2}$, show that $\sin\theta=\frac{2t}{1+t^2}$. [2]

(ii) Use the substitution $t=\tan\frac{\theta}{2}$ to find the exact value of

$$\int_0^{\frac{\pi}{2}}\left(\frac{\tan\frac{\theta}{2}+1}{\sin\theta+1}\right)\mathrm{d}\theta.$$ [5]

(b) Find $\int \mathrm{e}^{2v}\cos 3v\,\mathrm{d}v$. [4]

10 The point A has coordinates $(18,2,0)$. The plane p_1 has the equation $x+3y+z=a$, where a is a constant. It is given that p_1 contains the line l_1 with equation $\frac{x-1}{2}=y=\frac{z-1}{-5}$.

(i) Show that $a=2$. [2]

(ii) Find the coordinates of the foot of perpendicular from the point A to p_1. [3]

(iii) B is given to be a general point on l_1. Find an expression for the distance between the point A and B. Hence find the position vector of B that is nearest to A. [4]

The planes p_2 and p_3 have the equations $x+z=1$ and $2x+by+z=4$ respectively, where b is a constant.

(iv) Given that p_2 and p_3 intersect at l_2, show that l_2 is parallel to the vector $\begin{pmatrix}-b\\1\\b\end{pmatrix}$. By finding a point that lies on both planes, find a vector equation of l_2.

[3]

11 **(a)** The complex number w is such that $w = re^{i\theta}$, where $r > 0$ and $0 < \theta \le \frac{\pi}{2}$. The complex conjugate of w is denoted by w^*. Given that $\frac{w^2}{w^*} = -3$, find the exact values of r and θ. Hence find the three smallest positive integer n for which w^n is a real number. [5]

(b) The complex number z is such that $z^5 - 1 - i = 0$.

(i) Find the modulus and argument of each of the possible values of z. [5]

(ii) Two of these values are z_1 and z_2, where $\frac{\pi}{2} < \arg z_1 < \pi$ and $-\pi < \arg z_2 < -\frac{\pi}{2}$. Find the exact value of $\arg(z_1 - z_2)$ in terms of π and illustrate the locus $\arg(z - z_1) = \arg(z_1 - z_2)$ on an Argand diagram. [5]

H2 Mathematics

2016 JC 2 Preliminary Examination Paper 1 Solution

1

Let the passcode be xyz.

$x+y+z=14$........(1)

$100z+10y+x=100x+10y+z+495$

$99z-99x=495$......(2)

$y-x=3$..................(3)

Using the GC, $x=2, y=5, z=7$

$\therefore$ the passcode is 257

2

Let the radius and height of the cone be r and h respectively.

Let the radius of the circular card be x and angle ACB be θ.

By Pythagoras Theorem,

$$x^2=r^2+h^2 \Rightarrow r^2=x^2-h^2$$

$$V=\frac{1}{3}\pi r^2 h \quad =\frac{1}{3}\pi\left(x^2-h^2\right)h \quad =\frac{1}{3}\pi\left(x^2h-h^3\right)$$

$$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}\pi\left(x^2-3h^2\right)=0$$

$$h^2=\frac{x^2}{3} \Rightarrow r^2=\frac{2}{3}x^2$$

Consider the circumference of the circle without sector:

$$2\pi r=\frac{2\pi-\theta}{2\pi}(2\pi x)$$

$$2\pi\sqrt{\frac{2}{3}}x=(2\pi-\theta)(x)$$

$$\theta=2\left(1-\sqrt{\frac{2}{3}}\right)\pi$$

Alternatively, consider the curve surface area of the cone,

$$\pi x^2\left(\frac{2\pi-\theta}{2\pi}\right)=\pi r x$$

$$\pi x^2-\pi x^2\left(\frac{\theta}{2\pi}\right)=\pi\sqrt{\frac{2}{3}x^2}\,x$$

$$1-\left(\frac{\theta}{2\pi}\right)=\sqrt{\frac{2}{3}}$$

$$\theta=2\left(1-\sqrt{\frac{2}{3}}\right)\pi$$

$$\frac{d^2V}{dh^2}=\frac{1}{3}\pi(-6h)=-2\pi h<0 \text{ (Max)}$$

3

(i) $$\frac{4}{4r^2+12r+5}=\frac{4}{(2r+1)(2r+5)}=\frac{A}{2r+1}+\frac{B}{2r+5}$$

$$4=A(2r+5)+B(2r+1)$$

when $r=-\frac{5}{2}$: $4=B\left[2\left(-\frac{5}{2}\right)+1\right] \Rightarrow B=-1$

when $r=-\frac{1}{2}$: $4=A\left[2\left(-\frac{1}{2}\right)+5\right] \Rightarrow A=1$

$$\therefore \frac{4}{4r^2+12r+5}=\frac{1}{2r+1}-\frac{1}{2r+5}$$

(ii) $$\sum_{r=1}^{n-1}\frac{2}{4r^2+12r+5}=\frac{1}{2}\sum_{r=1}^{n-1}\frac{4}{4r^2+12r+5}$$

$$=\frac{1}{2}\left\{\frac{1}{3}-\frac{1}{7}\right.$$
$$+\frac{1}{5}-\frac{1}{9}$$
$$+\frac{1}{7}-\frac{1}{11}$$
$$+\frac{1}{9}-\frac{1}{13}$$
$$+\ldots\ldots\ldots$$
$$+\frac{1}{2n-5}-\frac{1}{2n-1}$$
$$+\frac{1}{2n-3}-\frac{1}{2n+1}$$
$$\left.+\frac{1}{2n-1}-\frac{1}{2n+3}\right\}$$

$$=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2n+1}-\frac{1}{2n+3}\right)=\frac{1}{2}\left(\frac{8}{15}-\frac{4n+4}{4n^2+8n+3}\right)=\frac{4}{15}-\frac{2(n+1)}{4n^2+8n+3}$$

(iii) $S_{n-1}\geq 0.99S_\infty$

$$\frac{2n+2}{4n^2+8n+3} \le \left(\frac{1}{100}\right)\left(\frac{4}{15}\right)$$

$$\frac{1500(2n+2)-4(4n^2+8n+3)}{(1500)(4n^2+8n+3)} \le 0$$

$$\frac{-4n^2+742n+747}{(1500)(4n^2+8n+3)} \le 0$$

$$\frac{(-n+186.501)(n+1.001)}{1500(2n+1)(2n+3)} \le 0$$

$-n+186.501 \le 0$ since $(2n+1)>0, (2n+3)>0, (n+1.001)>0$

$n \ge 186.501$

Alternatively

$$\frac{4}{15}-\frac{2n+2}{4n^2+8n+3} \ge 0.99\left(\frac{4}{15}\right)$$

$$\frac{2n+2}{4n^2+8n+3} \le \left(\frac{1}{100}\right)\left(\frac{4}{15}\right)$$

$4n^2+8n+3 \ge 750n+750$ (Since n is positive integer)

$4n^2-742n-747 \ge 0$

$n \ge 186.5$

minimum $n = 187$ (Alternative solution)

4 (i)

$x = 2a\cos^3\theta$ $\qquad$ $y = a\sin^3\theta$

$\frac{dx}{dt} = 2a(2\cos^2\theta)(-\sin\theta)$ $\qquad$ $\frac{dy}{dt} = a(3\sin^2\theta)(\cos\theta)$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$= \frac{3a\cos\theta\sin^2\theta}{-6a\cos^2\theta\sin\theta}$$

$$= -\frac{1}{2}\tan\theta$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

$$\theta = \frac{\pi}{4}$$

Point P: $\left(2a\cos^3\left(\frac{\pi}{4}\right), a\sin^3\left(\frac{\pi}{4}\right)\right) = \left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{4}\right)$

(ii) The equation of tangent at Q is

$$y-a\sin^3 t=-\frac{1}{2}\tan t(x-2a\cos^3 t)$$

$$y=-\left(\frac{1}{2}\tan t\right)x+a\sin t\cos^2 t+a\sin^3 t$$

$$y=-\left(\frac{1}{2}\tan t\right)x+a\sin t$$

$R(2a\cos t,0), S(0,a\sin t)$

Midpoint of $RS=\left(a\cos t,\frac{1}{2}a\sin t\right)$

$$x=a\cos t \Rightarrow \cos t=\frac{x}{a}$$

$$y=\frac{1}{2}a\sin t \Rightarrow \sin t=\frac{2y}{a}$$

$$\cos^2 t+\sin^2 t=1$$

$$\left(\frac{x}{a}\right)^2+\left(\frac{2y}{a}\right)^2=1$$

$$x^2+4y^2=a^2$$

Since $0<t<\frac{\pi}{2}, 0<x<a$ or $0<y<\frac{a}{2}$

5 (i) $S_1=\frac{1}{2!}=\frac{1}{2}$

$$S_2=\frac{1}{2!}+\frac{2}{3!}=\frac{5}{6}$$

$$S_3=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}=\frac{23}{24}$$

$$S_4=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}=\frac{119}{120}$$

(ii)

$$S_1=\frac{1}{2}=1-\frac{1}{2}$$

$$S_2=\frac{5}{6}=1-\frac{1}{6}$$

$$S_3=\frac{23}{24}=1-\frac{1}{24}$$

$$S_4=\frac{119}{120}=1-\frac{1}{120}$$

$$S_n=1-\frac{1}{(n+1)!}$$

(iii)

Let P_n be the statement $S_n = 1 - \frac{1}{(n+1)!}$ for $n = 1, 2, 3, \ldots$

when $n = 1$

$\text{LHS} = S_1 = \frac{1}{2}$

$\text{RHS} = 1 - \frac{1}{(1+1)!} = \frac{1}{2}$

$\therefore P_1$ is true

Assume P_k is true for some $k = 1, 2, 3, \ldots$

$S_k = 1 - \frac{1}{(k+1)!}$

We want to prove that P_{k+1} is also true

$S_{k+1} = 1 - \frac{1}{(k+2)!}$

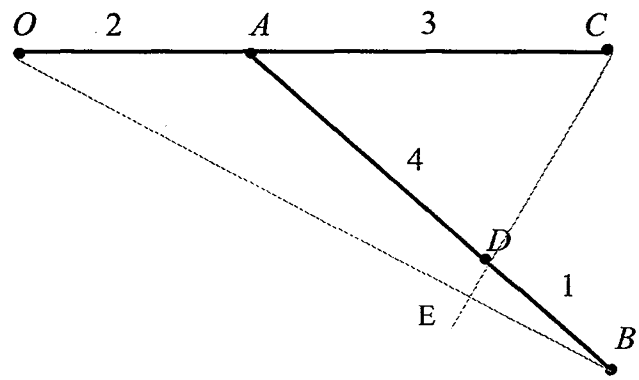
$$\begin{aligned}
\text{LHS} &= S_{k+1} \\
&= S_k + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \left[\frac{(k+2)-(k+1)}{(k+2)!}\right] \\
&= 1 - \frac{1}{(k+2)!} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P_{k+1}$ is true

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true ,
by mathematical induction P_n is true for all $n = 1, 2, 3, \ldots$

6

(i) $\dfrac{OA}{OC}=\dfrac{2}{5}$



$\dfrac{OC}{OA}=\dfrac{5}{2}$

$\overrightarrow{OC}=\dfrac{5}{2}\overrightarrow{OA}=\dfrac{5}{2}\mathbf{a}$

By ratio theorem,

$\overrightarrow{OD}=\dfrac{\mathbf{a}+4\mathbf{b}}{5}$

$\overrightarrow{OD}=\dfrac{1}{5}\mathbf{a}+\dfrac{4}{5}\mathbf{b}$

(ii) $\overrightarrow{CD}=\dfrac{1}{5}\mathbf{a}+\dfrac{4}{5}\mathbf{b}-\dfrac{5}{2}\mathbf{a}=-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}$

$l_{CD}:\mathbf{r}=\dfrac{5}{2}\mathbf{a}+\lambda\left(-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}\right)\quad \lambda\in\mathbf{R}$

(iii) Since E is a point on CD produced,

$\overrightarrow{OE}=\dfrac{5}{2}\mathbf{a}+\lambda\left(-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}\right)\quad \lambda\in\mathbf{R}$

Since E is a point on OB,

$\overrightarrow{OE}=\alpha\mathbf{b}\quad \alpha\in\mathbf{R}$

$\dfrac{5}{2}\mathbf{a}+\lambda\left(-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}\right)=\alpha\mathbf{b}$

$\left(\dfrac{5}{2}-\dfrac{23}{10}\lambda\right)\mathbf{a}+\dfrac{4}{5}\lambda\mathbf{b}=\alpha\mathbf{b}$

$\dfrac{5}{2}-\dfrac{23}{10}\lambda=0\Rightarrow\lambda=\dfrac{25}{23}$

$\dfrac{4}{5}\lambda=\alpha\Rightarrow\alpha=\dfrac{20}{23}$

$\therefore\overrightarrow{OE}=\dfrac{20}{23}\mathbf{b}$

$OE: EB=20:3$

7

$$\frac{dT}{dt} = -k(T - T_o), \quad k > 0$$

$$\int \frac{1}{T - T_o} dT = \int -k dt$$

Note that no modulus required since $T > T_0$

$\ln(T - T_o) = -kt + C$, where C is an arbitary constant

$T - T_o = e^{-kt+C}$

$T - T_o = e^{-kt}e^{C}$

$T - T_o = Ae^{-kt}$, where $A = e^{C}$

$\therefore T = T_o + Ae^{-kt}$ (shown)

$T_o = 30°C$

At $t = 0$: $100 = 30 + Ae^{-k(0)}$

$A = 70$

At $t = 15$: $70 = 30 + 70e^{-15k}$

$40 = 70e^{-15k}$

$e^{-15k} = \frac{4}{7}$

$k = -\frac{1}{15}\ln\frac{4}{7} \approx 0.0373077$

To find time taken for pot of dessert to cool to at most $35°C$:

$30 + 70e^{-kt} \le 35$

$70e^{-kt} \le 5$

$e^{-kt} \le \frac{5}{70}$

$-kt \le \ln\frac{5}{70}$

$$t \ge \frac{\ln(5/70)}{-\frac{1}{15}\ln(4/7)}$$

$t \ge 70.74$

$t = 71$ minutes

It takes at least 71 minutes for the pot of dessert to cool to $35°C$ and 30 minutes to cook. Hence Nurul must start preparing the pot of dessert at 7.19pm the latest.

8

(i) Let L be the distance covered by the lion.

$a = 2.5$ and $d = -0.05$

$$L = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$= \frac{n}{2}\left[2(2.5) + (n-1)(-0.05)\right]$$

$$= -\frac{1}{40}n^2 + \frac{101}{40}n$$

(ii) Let P be the distance covered by the prey.

$a = 1.5$ and $r = 0.95$

$$P = \frac{1.5\left(1 - 0.95^n\right)}{1 - 0.95}$$

$$= 30(1 - 0.95^n)$$

When $n \to \infty$, $P \to 30$

So the distance covered by the prey can never exceed 30m

(iii) In order for the lion to catch its prey,

$L \geq P + 25$

$$-\frac{1}{40}n^2 + \frac{101}{40}n \geq 30(1 - 0.95^n) + 25$$

$$-\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) \geq 55$$

$$n = 24, -\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) = 54.96 < 55$$

$$n = 25, -\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) = 55.822 > 55$$

$$n = 26, -\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) = 56.556 > 55$$

least $n = 25$

Hence, the lion will need at least 25 leaps to catch its prey.

(iv) Let the initial distance be k

In order for the prey to escape the hunt,

$P + k \geq L$

$$30(1 - 0.95^{30}) + k \geq -\frac{1}{40}(30^2) + \frac{101}{40}(30)$$

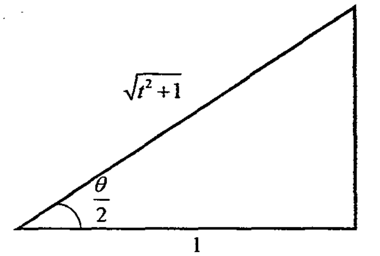
$$23.561 + k \geq 53.25$$

$$k \geq 29.689$$

$\therefore$ the shortest distance is 30 m.

9

(a)(i) $t=\tan\frac{\theta}{2}$



$$\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=\frac{2t}{1-t^2}$$

by triangle rule:

$$\sin\theta=\frac{2t}{1+t^2}\quad\text{(shown)}$$

Alternatively RHS $=\dfrac{2t}{1+t^2}=\dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\dfrac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}=\dfrac{2\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\times\cos^2\frac{\theta}{2}=\sin\theta$ =LHS

Alternatively

Use double angle formula: $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cos^2\frac{\theta}{2}=\dfrac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}=\dfrac{2t}{1+t^2}$

(ii)

$$\int_0^{\frac{\pi}{2}}\frac{\tan\frac{\theta}{2}+1}{\sin\theta+1}\,\mathrm{d}\theta$$
$$=\int_0^1\frac{t+1}{\frac{2t}{1+t^2}+1}\left(\frac{2}{1+t^2}\mathrm{dt}\right)$$
$$=\int_0^1\frac{t+1}{\frac{2t+1+t^2}{1+t^2}}\left(\frac{2}{1+t^2}\mathrm{dt}\right)$$
$$=\int_0^1\frac{2(t+1)}{2t+1+t^2}\mathrm{dt}$$
$$=\int_0^1\frac{2t+2}{t^2+2t+1}\mathrm{dt}=\int_0^1\frac{2}{t+1}\mathrm{dt}$$
$$=2\left[\ln(t+1)\right]_0^1$$
$$=2\ln 2$$

$t=\tan\frac{\theta}{2}$

when $\theta=\frac{\pi}{2}$: $t=\tan\frac{\pi/2}{2}=1$

when $\theta=0$: $t=\tan\frac{0}{2}=0$

$\tan^{-1}t=\frac{\theta}{2}$

$\frac{1}{1+t^2}\frac{dt}{d\theta}=\frac{1}{2}$

$\frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{1+t^2}{2}$

$\frac{\mathrm{d}\theta}{\mathrm{dt}}=\frac{2}{1+t^2}$

(b)

$$\int e^{2v}\cos 3v\,dv$$

$$= \frac{1}{3}e^{2v}\sin 3v - \int \frac{2}{3}e^{2v}\sin 3v\,dv$$

$u = e^{2v}$	$\frac{dy}{dv} = \cos(3v)$
$\frac{du}{dv} = 2e^{2v}$	$y = \frac{1}{3}\sin(3v)$

$$= \frac{1}{3}e^{2v}\sin 3v - \frac{2}{3}\left[-\frac{1}{3}e^{2v}\cos(3v) + \int \frac{2}{3}e^{2v}\cos(3v)dv\right]$$

$u = e^{2v}$	$\frac{dy}{dv} = \sin(3v)$
$\frac{du}{dv} = 2e^{2v}$	$y = -\frac{1}{3}\cos(3v)$

$$= \frac{1}{3}e^{2v}\sin 3v + \frac{2}{9}e^{2v}\cos(3v) - \int \frac{4}{9}e^{2v}\cos(3v)dv$$

$$\frac{13}{9}\int e^{2v}\cos 3v\,dv = \frac{1}{3}e^{2v}\sin 3v + \frac{2}{9}e^{2v}\cos(3v)$$

$$\int e^{2v}\cos 3v\,dv = \frac{3}{13}e^{2v}\sin 3v + \frac{2}{13}e^{2v}\cos(3v) + c$$

Alternatively

$$\int e^{2v}\cos 3v\,dv$$

$$= \frac{1}{2}e^{2v}\cos 3v + \int \frac{3}{2}e^{2v}\sin 3v\,dv$$

$u = \cos(3v)$	$\frac{dy}{dv} = e^{2v}$
$\frac{du}{dv} = -3\sin(3v)$	$y = \frac{1}{2}e^{2v}$

$$= \frac{1}{2}e^{2v}\cos 3v + \frac{3}{2}\left[\frac{1}{2}e^{2v}\sin(3v) - \int \frac{3}{2}e^{2v}\cos(3v)dv\right]$$

$u = \sin(3v)$	$\frac{dy}{dv} = e^{2v}$
$\frac{du}{dv} = 3\cos(3v)$	$y = \frac{1}{2}e^{2v}$

$$= \frac{1}{2}e^{2v}\cos 3v + \frac{3}{4}e^{2v}\sin(3v) - \frac{9}{4}\int e^{2v}\cos(3v)dv$$

$$\frac{13}{4}\int e^{2v}\cos 3v\,dv = \frac{1}{2}e^{2v}\cos 3v + \frac{3}{4}e^{2v}\sin(3v)$$

$$\int e^{2v}\cos 3v\,dv = \frac{3}{13}e^{2v}\sin 3v + \frac{2}{13}e^{2v}\cos(3v) + c$$

10

(i) $\ell_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}, \ \lambda \in \mathbf{R}$

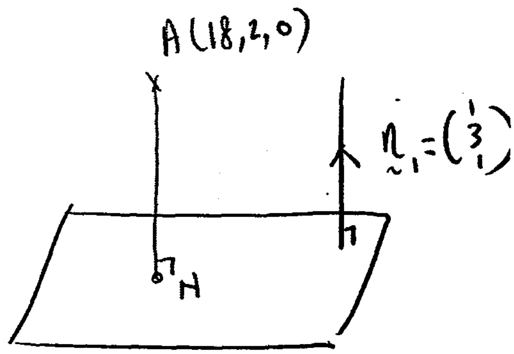
Since (1, 0, 1) is on ℓ_1 and p_1

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = a$$

$1+0+1=a$

$a=2$ (shown)

(ii) Let N be the foot of perpendicular from A to p_1



$$\ell_{AN} : \mathbf{r} = \begin{pmatrix} 18 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \ \alpha \in \mathbf{R}$$

let $\overrightarrow{ON} = \begin{pmatrix} 18+\alpha \\ 2+3\alpha \\ \alpha \end{pmatrix}$ for some value of α

Since N is a point on p_1

$$\begin{pmatrix} 18+\alpha \\ 2+3\alpha \\ \alpha \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 2$$

$18+\alpha+6+9\alpha+\alpha=2$

$24+11\alpha=2$

$11\alpha=-22$

$\alpha=-2$

$$\overrightarrow{ON} = \begin{pmatrix} 18-2 \\ 2-6 \\ -2 \end{pmatrix} = \begin{pmatrix} 16 \\ -4 \\ -2 \end{pmatrix} \therefore N(16,-4,-2)$$

(iii) Since B is on ℓ_1

$$\overrightarrow{OB} = \begin{pmatrix} 1+2\lambda \\ \lambda \\ 1-5\lambda \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 1+2\lambda \\ \lambda \\ 1-5\lambda \end{pmatrix} - \begin{pmatrix} 18 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -17+2\lambda \\ -2+\lambda \\ 1-5\lambda \end{pmatrix}$$

$$|\overrightarrow{AB}|=\sqrt{(-17+2\lambda)^2+(-2+\lambda)^2+(1-5\lambda)^2}$$

$$|\overrightarrow{AB}|=\sqrt{(289-68\lambda+4\lambda^2)+(4-4\lambda+\lambda^2)+(1-10\lambda+25\lambda^2)}$$

$$|\overrightarrow{AB}|=\sqrt{294-72\lambda+30\lambda^2}$$

$$|\overrightarrow{AB}|^2=294-72\lambda+30\lambda^2$$

For shortest distance from A to ℓ_1

$|\overrightarrow{AB}|^2$ must be minimum

$$\therefore |\overrightarrow{AB}|^2=30\lambda^2-72\lambda+294$$

$$2|\overrightarrow{AB}|\frac{\mathrm{d}|\overrightarrow{AB}|^2}{\mathrm{d}\lambda}=60\lambda-72$$

$$\frac{\mathrm{d}|\overrightarrow{AB}|^2}{\mathrm{d}\lambda}=0$$

$$60\lambda-72=0$$

$$\lambda=\frac{6}{5}$$

$$\overline{OB}=\begin{pmatrix}1+2\lambda\\ \lambda\\ 1+5\lambda\end{pmatrix}=\begin{pmatrix}17/5\\ 6/5\\ 7\end{pmatrix}\text{ or }\frac{1}{5}\begin{pmatrix}17\\ 6\\ 35\end{pmatrix}$$

(iv) direction vector of $\ell_2=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}2\\b\\1\end{pmatrix}=\begin{pmatrix}-b\\-(1-2)\\b\end{pmatrix}=\begin{pmatrix}-b\\1\\b\end{pmatrix}$

To find a common point between p_2 and p_3 by letting $y=0$:

$x+z=1$ --- (1)

$2x+z=4$ --- (2)

Solve (1) and (2):

$x=3,\ z=-2$

Hence ℓ_2: $\mathbf{r}=\begin{pmatrix}3\\0\\-2\end{pmatrix}+\mu\begin{pmatrix}-b\\1\\b\end{pmatrix},\ \mu\in\mathbf{R}$ (shown)

11

(i) $w = re^{i\theta}$

$w^* = re^{-i\theta}$

$$\frac{w^2}{w^*} = \frac{\left(re^{i\theta}\right)^2}{re^{-i\theta}}$$

$$= \frac{r^2e^{i2\theta}}{re^{-i\theta}}$$

$$= re^{i3\theta} = -3 = 3e^{i\pi}$$

$$3\theta = \pi \Rightarrow \theta = \frac{\pi}{3} (0 < \theta \le \frac{1}{2}\pi)$$

$r = 3$

$w = 3e^{i\frac{\pi}{3}}$, $w^n = 3^n e^{i\frac{n\pi}{3}}$

w^n is real $, \Rightarrow \frac{n\pi}{3} = 0, \pi, 2\pi, \ldots$, so $n = 3, 6, 9, \ldots$

(b) **(i)** $z^5 = 1 + i$

$$= \sqrt{2}e^{\left(2k\pi + \frac{\pi}{4}\right)i}$$

$$z = 2^{\frac{1}{10}} e^{\left(\frac{2k\pi}{5} + \frac{\pi}{20}\right)i}, \ k = 0, \pm 1, \pm 2$$

$$z = 2^{\frac{1}{10}}e^{\frac{\pi}{20}i},\ 2^{\frac{1}{10}}e^{\frac{9\pi}{20}i},\ 2^{\frac{1}{10}}e^{-\frac{7\pi}{20}i},\ 2^{\frac{1}{10}}e^{\frac{17\pi}{20}i},\ 2^{\frac{1}{10}}e^{-\frac{3\pi}{4}i}$$

So $|z| = 2^{\frac{1}{10}}$ for all z

$$\arg(z) = \frac{\pi}{20}, \frac{9\pi}{20}, -\frac{7\pi}{20}, \frac{17\pi}{20}, -\frac{3\pi}{4}$$

(ii) $z_1 = 2^{\frac{1}{10}}e^{\frac{17\pi}{20}i}$

$z_2 = 2^{\frac{1}{10}}e^{-\frac{15\pi}{20}i}$

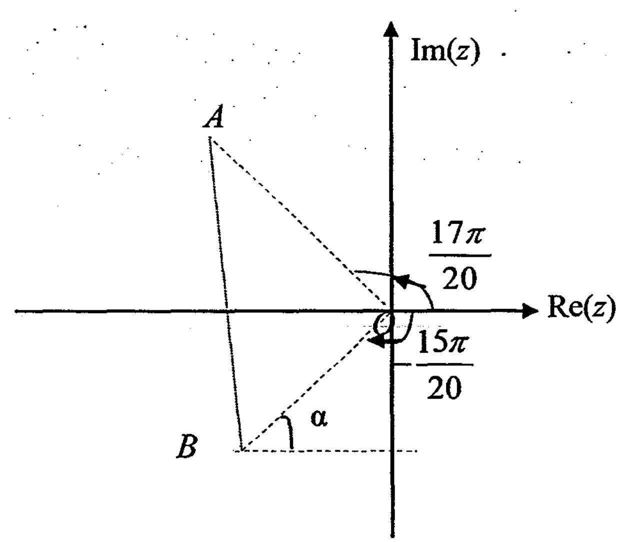
Let Point A and B represent z_1 and z_2 respectively.

$|z_1| = |z_2| \Rightarrow OAB$ is an isosceles triangle.

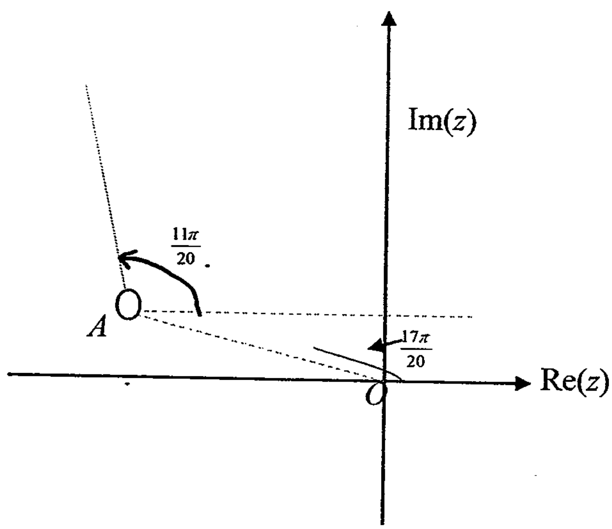
$$\square AOB = \frac{2\pi}{5}$$

$$\square OAB = \square OBA = \frac{1}{2}[\pi - \square AOB]$$

$$= \frac{1}{2}\left[\pi - \frac{2\pi}{5}\right] = \frac{3\pi}{10}$$



$$\arg(z_1 - z_2) = \alpha + \angle OBA$$
$$= \left(\pi - \frac{15\pi}{20}\right) + \frac{3\pi}{10}$$
$$= \frac{11\pi}{20}$$



Candidate Name: ______________________ Class: ____________

JC2 PRELIMINARY EXAM
Higher 2

MATHEMATICS **9740/02**
Paper 2 **16 Sep 2016**
3 hours

Additional Materials: Cover page
Answer papers
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your full name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 Two complex numbers a and b are given by $2+3\mathrm{i}$ and $-4-5\mathrm{i}$ respectively.

(i) On a single Argand diagram, sketch the loci

(a) $|2z-a-b|=|a-b|$,

(b) $0 \le \arg(z-b) \le \arg(a-b)$. [4]

(ii) Find range of arg(z) where z is the complex number satisfies the relations in part **(i)**. [4]

2 A curve C has equation $y=\dfrac{ax}{x-1}$ where $a>0$.

(i) By writing the equation of C as $y=A+\dfrac{B}{x-1}$, state a sequence of transformations which transform the graph of $y=\dfrac{1}{x}$ to C. [3]

(ii) Sketch C, giving the equations of any asymptotes and the coordinates of any points of intersection with the axes. [2]

(iii) The region R is bounded by C, the lines $x=2$, $x=4$ and $y=a$. Find the exact volume in terms of a when R is rotated through 2π radians about the x-axis. [3]

(iv) The region S is bounded by $y=\dfrac{a}{x}$, the lines $x=1$, $x=3$ and $y=0$. State the exact volume in terms of a when S is rotated through 2π radians about the line $y=-a$. [1]

3 The function f is defined by $\mathrm{f}(x)=\begin{cases}\dfrac{x}{2} & \text{if } x\le 0,\\ 2\sin x & \text{if } 0<x\le 4.\end{cases}$

(i) Sketch the graph of $y=\mathrm{f}(x)$. [2]

(ii) If the domain of f is restricted to $x\le k$, state the largest value of k, in exact form, for which the function f^{-1} exist. [1]

(iii) Using the domain from part **(ii)**, define f^{-1} in a similar form. [4]

(iv) Solve $\mathrm{f}^{-1}(x)=\mathrm{f}(x)$. [2]

In the rest of the question, the domain of f is as originally defined.

The function g is defined by $\mathrm{g}: x \mapsto -x^3, x\in □, x>0$.

(v) Find an expression for $\mathrm{fg}(x)$. [2]

4 **(i)** Differentiate $\tan^{-1}\left(\dfrac{\sqrt{3}}{2}x\right)$ with respect to x. [2]

(ii) Find the binomial expansion for $\dfrac{1}{3x^2+4}$ up to and including the term in x^6, giving the coefficients as exact fractions in their simplest form. Find the set of values of x for which the expansion is valid. [5]

(iii) Hence, find the first four non-zero terms of the Maclaurin series for $\tan^{-1}\left(\dfrac{\sqrt{3}}{2}x\right)$. Give the coefficients as exact fractions in their simplest form.

[5]

Section B: Statistics [60 marks]

5 A pharmaceutical company has invented a new drug for diabetic patients and wishes to carry out a trial of the new drug involving 5% of the patients from a local hospital.

(i) Explain how a systematic sample could be carried out. [2]

(ii) State one disadvantage of systematic sampling in this context and name a more appropriate sampling method. [2]

6 Given that $\mathrm{P}(A \mid B') = 3\mathrm{P}(A \mid B)$ and $\mathrm{P}(B') = 4\mathrm{P}(B)$.

(i) Show that $\mathrm{P}(B') = \frac{4}{5}$. [2]

(ii) Using $\mathrm{P}(A \cap B') = \mathrm{P}(A) - \mathrm{P}(A \cap B)$, find $\mathrm{P}(B' \mid A)$. [3]

7 The sales department of a company consists of 3 teams led by Mrs Wong, Miss Tan and Mr Lim. Each team is made up of 1 team leader and 5 sales executives. The number of male and female sales executives within each team is given in the table below:

	Team A	Team B	Team C
Team Leader	Mrs Wong	Miss Tan	Mr Lim
Number of Male Executive(s)	3	4	0
Number of Female Executive(s)	2	1	5

A taskforce is to be formed by selecting 7 representatives from the 18 members of the department. Find the number of different taskforces that can be formed if the taskforce must include

(i) Miss Tan and 1 other team leader, [2]

(ii) more females than males, [2]

(iii) at least 1 representative from each team. [3]

8 In this question you should state clearly the values of the parameters of any normal distribution you use.

The mass in kilograms of an Atlantic salmon is a normally distributed continuous random variable X with mean μ and standard deviation σ.

(i) It is known that $P(X<22)=0.159$ and $P(X>31)=0.106$. Show that $\mu=26.0$ and $\sigma=4.01$. [3]

(ii) In a random sample of 40 Atlantic salmon, estimate the probability that at least 35 of them have a mass of at most 31 kilograms. [3]

It is also known that the mass in kilograms of the Bluefin tuna has the distribution $N(380,10^2)$.

(iii) Find the probability that the average mass of 2 randomly chosen Bluefin tuna and 3 randomly chosen Atlantic salmon is at most 170 kg. [2]

9 The table gives the world record time, in seconds, for the 100 metres free style swimming event at the Olympic Games in the past years.

Year, x	1908	1920	1956	1968	1976	2000
Time, t	65.60	60.40	55.40	52.20	49.99	48.18

(i) Draw a scatter diagram to illustrate the data. [1]

(ii) Comment on whether a linear model would be appropriate, referring both to the scatter diagram and the context of the question. [2]

(iii) It is thought that the time can be modelled by one of the formulae $\ln t = a + bx$ or $\frac{1}{t} = a + bx$. Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) $\ln t$ and x,

(b) $\frac{1}{t}$ and x. [2]

(iv) Use your answers to part **(iii)** to explain which of $\ln t = a + bx$ or $\frac{1}{t} = a + bx$ is the better model. [1]

(v) The time corresponding to 1964 was added to obtain the equation with appropriate model chosen in part **(iv)** where $a = -0.09836$ and $b = 5.96846 \times 10^{-5}$. Find the time in 1964. [2]

10 A car manufacturer launches a new car model "Green Leaf" that is marketed to be environmentally friendly. It is claimed that the carbon emission of the "Green Leaf" is at most 80 g/km. The transport authority suspects that the figure is understated, and requests the manufacturer to submit test data from 20 units of the "Green Leaf". The test data submitted is as follows.

Carbon Emission (g/km)	78	79	80	81	82
No. of units	2	3	6	4	5

(i) Calculate unbiased estimates of the population mean and variance. [2]

(ii) Stating a necessary assumption, test at the 10% level of significance whether there is any evidence to doubt the manufacturer's claim. [4]

The transport authority subsequently decides to conduct their own test, and invites 10 owners of the "Green Leaf" to form a sample. The mean and variance of this sample is found to be 80.6 g/km and m^2 g^2/km^2 respectively.

(iii) Find the set of values of m for which the result of the test would be to reject the manufacturer's claim, at the 1% significance level. [3]

11 There are 2 main types of T-cells in the human body. T4-cells are "helper" cells that lead attacks against infections in the human body, while T8-cells are "suppressor" cells that kill cancer and virus infected cells in the human body. It is to be assumed that the number of T4-cells per 0.01 mm^3 of blood can be modelled by the distribution Po(5) and the number of T8-cells per 0.01 mm^3 of blood can be modelled by the independent distribution Po(1.5).

A patient is considered healthy if he or she has at least 4 T4-cells and at least 1 T8-cells in 0.01 mm^3 of blood.

(i) Find the probability that a randomly selected patient is healthy. [3]

(ii) Find the probability that only 1 out of 3 randomly selected patients is healthy. [2]

A patient is susceptible to infections if his or her T4-cells count falls below 3 per 0.01 mm^3 of blood.

(iii) Use a suitable approximation, which should be stated, to find the probability that, in 100 randomly selected patients, the number of patients susceptible to infections is between 20 and 50 inclusive. [4]

12 Mr Ouyang, a car manufacturer, finds that on average, 2% of his cars have faulty gearboxes. On a particular occasion, he selects n cars randomly for inspection, and the number of cars with faulty gearbox is denoted by the random variable C.

(i) State in context of this question, what must be assumed for C to be well modelled by a binomial distribution. [2]

(ii) Given that $n = 20$, find the probability that C is between 2 and 6. [2]

(iii) The probability that there are less than 2 cars with faulty gearbox in a sample of n cars is at most 0.95. Write down an inequality in terms of n, and find the least possible value of n. [3]

(iv) Mr Ouyang selects 100 batches of 20 cars. Estimate the probability that the average number of cars with faulty gearbox per batch is at least 0.3. [3]

H2 Mathematics

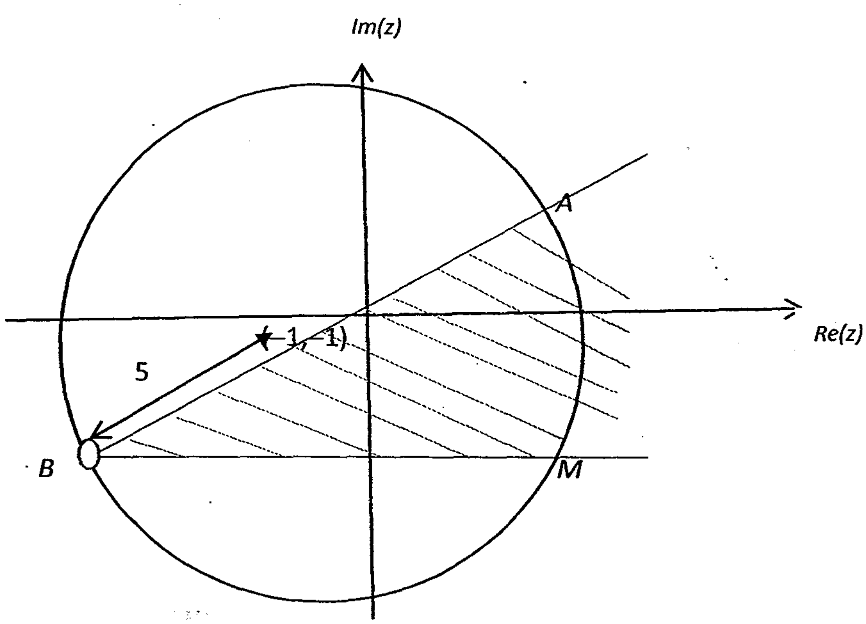
JC 2 Preliminary Examination Paper 2 Solution

1

(i) $|2z-a-b|=|a-b|$

Centre of circle $C = \dfrac{(2+3i)+(-4-5i)}{2} = -1-i$

Radius of circle $C = \dfrac{\sqrt{(2+4)^2+(3+5)^2}}{2} = 5$



(ii) AM is the common region satisfies both (i) and (ii).

$\angle AMB = 90°$ since AB is diameter and angle in semicircle is a right angle

Max arg(z) = arg(a) = $\tan^{-1}(3/2) = 0.983$ radians

Min arg(z) = arg(2 – 5i) = $-\tan^{-1}(5/2) = -1.19$ radians

So range required is $-1.19 \le \arg(z) \le 0.983$

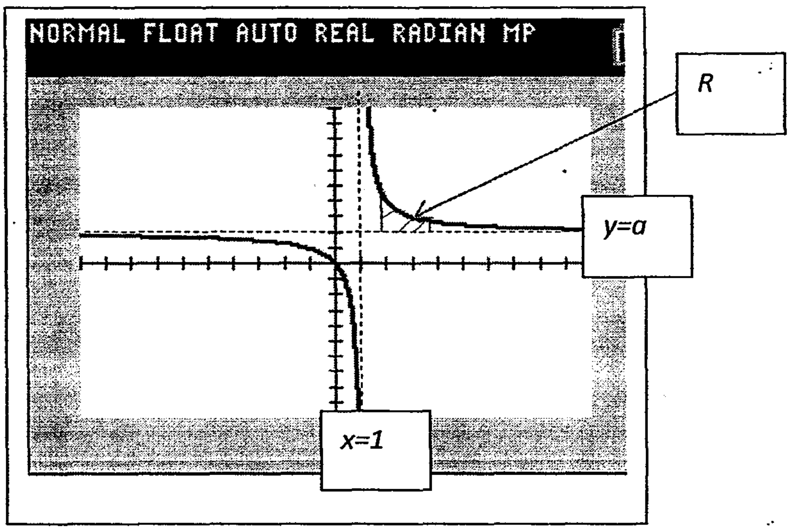
2

(i) $y=\dfrac{ax}{x-1}=a+\dfrac{a}{x-1}$

$y=\dfrac{1}{x}$ is translated 1 unit in the direction of x-axis, followed by a scaling of a units parallel to the y axis and is translated a units in the direction of y-axis

(ii) The equations of asymptotes are $x = 1$ and $y = a$

The intercepts are (0,0)



(iii) The volume required $=\pi\int_2^4\left(a+\dfrac{a}{x-1}\right)^2 \mathrm{d}x-\pi(a)^2(2)$

$$=a^2\pi\int_2^4\left(1+\frac{2}{x-1}+\frac{1}{(x-1)^2}\right)\mathrm{d}x-2\pi a^2$$

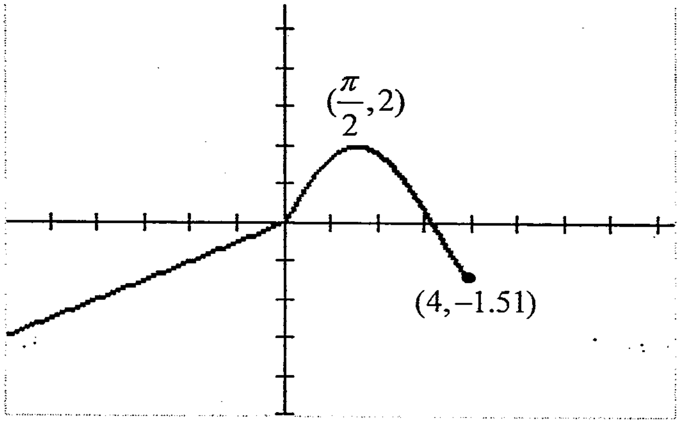
$$=a^2\pi\left[x+2\ln|x-1|-\frac{1}{x-1}\right]_2^4-2\pi a^2$$

$$=\left(\frac{2}{3}+2\ln 3\right)\pi a^2$$

(iv) Using part (i), the area S is the same as the area R found in (iii). To rotate S about the line $y = -a$ is the same as to rotate R about the x-axis. So the volume obtained is $\left(\dfrac{2}{3}+2\ln 3\right)\pi a^2$

3

(i)



(ii) From the graph, take $k=\frac{\pi}{2}$

(iii)

$$f(x)=\begin{cases}\frac{x}{2} & \text{if } x\le 0\\ 2\sin x & \text{if } 0<x\le\frac{\pi}{2}\end{cases}$$

Let $y_1=\frac{x}{2}$

$x=2y_1$

Let $y_2=2\sin x$

$x=\sin^{-1}\left(\frac{y_2}{2}\right)$

$$f^{-1}(x)=\begin{cases}2x & \text{if } x\le 0\\ \sin^{-1}\left(\frac{x}{2}\right) & \text{if } 0<x\le 2\end{cases}$$

(iv) $f^{-1}(x)=f(x)$ is the same as solving $f(x)=x$

$\frac{x}{2}=x \Rightarrow x=0 \quad \text{if } x\le 0$

$2\sin x=x$, $x=1.90>\frac{\pi}{2}$, so only solution is $x=0$

(v) Using $R_g=(-\infty,0)$, $fg(x)=\frac{-x^3}{2}$

4

(i)

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{\sqrt{3}}{2}x\right)\right)=\frac{1}{1+\left(\frac{\sqrt{3}}{2}x\right)^2}\left(\frac{\sqrt{3}}{2}\right)$$

$$=\left(\frac{\sqrt{3}}{2}\right)\frac{1}{1+\frac{3}{4}x^2}$$

$$=\left(\frac{\sqrt{3}}{2}\right)\frac{4}{3x^2+4}$$

$$=\frac{2\sqrt{3}}{3x^2+4}$$

(ii)

$$\frac{1}{3x^2+4}=(3x^2+4)^{-1}=4^{-1}\left(1+\frac{3x^2}{4}\right)^{-1}$$

$$=\frac{1}{4}\left[1+(-1)\left(\frac{3x^2}{4}\right)+\frac{-1(-2)}{2}\left(\frac{3x^2}{4}\right)^2+\frac{-1(-2)(-3)}{6}\left(-\frac{3x^2}{4}\right)^3+\ldots\right]$$

$$=\frac{1}{4}\left(1-\frac{3}{4}x^2+\frac{9}{16}x^4-\frac{27}{64}x^6+\ldots\right)$$

$$\approx\frac{1}{4}-\frac{3}{16}x^2+\frac{9}{64}x^4-\frac{27}{256}x^6$$

Range of validity: $\left|\frac{3x^2}{4}\right|<1\Rightarrow x^2<\frac{4}{3}\Rightarrow -\frac{2}{\sqrt{3}}<x<\frac{2}{\sqrt{3}}$

(iii)

$$y=\tan^{-1}\left(\frac{\sqrt{3}}{2}x\right)=2\sqrt{3}\int\frac{1}{3x^2+4}\,dx$$

$$=2\sqrt{3}\int\left(\frac{1}{4}-\frac{3}{16}x^2+\frac{9}{64}x^4-\frac{27}{256}x^6\right)dx$$

$$=2\sqrt{3}\left(\frac{1}{4}x-\frac{1}{16}x^3+\frac{9}{320}x^5-\frac{27}{1792}x^7\right)+C$$

when $x=0$, $y=0$, $C=0$

$$\therefore \tan^{-1}\left(\frac{\sqrt{3}}{2}x\right)=\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{8}x^3+\frac{9\sqrt{3}}{160}x^5-\frac{27\sqrt{3}}{896}x^7$$

5 (i) Number the list of patients from 1 to *N*. $k = N/0.05N = 20$ (Randomly select a number from 1 to 20, and let every 20th patient after first patient chosen try the new drug. For example, if a number 5 is chosen, then survey every 5th, 25th, 45th patient and so on, until the sample size of 5% patients is obtained.

(ii) Disadvantage: The sample is not representative of the population of diabetic patients as age and gender may affect the drug. More appropriate method is stratified sampling.

6

(i) $P(B')=4P(B)$

$P(B')=4\left[1-P(B')\right]$

$P(B')=4-4P(B')$

$5P(B')=4$

$P(B')=\frac{4}{5}$ (shown)

Alternatively,

$1-P(B)=4P(B)$

$5P(B)=1$

$P(B)=\frac{1}{5}$

$P(B')=\frac{4}{5}$ (shown)

(ii) $P(A\mid B')=3P(A\mid B)$

$\frac{P(A\cap B')}{P(B')}=\frac{3P(A\cap B)}{P(B)}$

$\frac{P(A\cap B')}{P(B')}=\frac{3P(A\cap B)}{1-P(B')}$

$\frac{P(A\cap B')}{4/5}=\frac{3P(A\cap B)}{1/5}$

$P(A\cap B')=12P(A\cap B)$----(*)

$P(A)-P(A\cap B)=12P(A\cap B)$

$P(A)=13P(A\cap B)$----(**)

$P(B'\mid A)=\frac{P(A\cap B')}{P(A)}$

$=\frac{12P(A\cap B)}{13P(A\cap B)}$

$=\frac{12}{13}$ or 0.923 (3 s.f.)

7 (i)Number of teams $= {}^{2}C_{1}\,{}^{15}C_{5} = 6006$

(i) 4 cases: 4F3M, 5F2M, 6F1M and 7F

Number of teams $= {}^{10}C_{4}\,{}^{8}C_{3} + {}^{10}C_{5}\,{}^{8}C_{2} + {}^{10}C_{6}\,{}^{8}C_{1} + {}^{10}C_{7} = 20616$

(ii) Total − teams from *A* and *B* − teams from *B* and *C* − teams from *A* and *C*

$= {}^{18}C_{7} - {}^{12}C_{7}\times 3$

$= 29448$

8 (i) X – mass in kilograms of an Atlantic salmon

$X \sim N(\mu, \sigma^2)$

$P(X < 22) = 0.159$

$P\left(Z < \frac{22-\mu}{\sigma}\right) = 0.159$

$\frac{22-\mu}{\sigma} = -0.99858$

$\mu - 0.99858\sigma = 22$ ---(1)

$P(X > 31) = 0.106$

$P\left(Z > \frac{31-\mu}{\sigma}\right) = 0.106$

$\frac{31-\mu}{\sigma} = 1.2481$

$\mu + 1.2481\sigma = 31$ ---(2)

Solving (1) and (2):

$\mu = 26.00022 \approx 26.0$ (shown)

$\sigma = 4.00591 \approx 4.01$ (shown)

(ii) Let W be the number of Atlantic salmon with more than 31 kg, out of 40

$W \sim B(40, 0.106)$

$n = 40$ large, $np = 40(0.106) = 4.24 < 5$

so $W \sim Po(4.24)$ approx.

Required prob = $P(W \leq 5) = 0.74659 \approx 0.747$

(iii) Y – mass in kilograms of an Bluefin tuna.

$Y \sim N(380, 10^2)$

Let T be the mass of 2 Bluefin tuna and 3 Atlantic salmon

$T = X_1 + X_2 + X_3 + Y_1 + Y_2 \sim N(838, 248.2403)$

$\frac{X_1 + X_2 + X_3 + Y_1 + Y}{5} = \overline{T} \sim N\left(\frac{838}{5}, \frac{248.2403}{25}\right)$ exactly

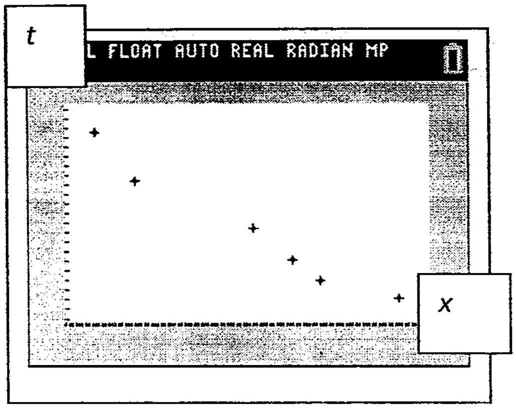
$P(\overline{T} \leq 170) = 0.777$ (3 s.f.)

Alternatively

$P(T \leq 170 \times 5) = 0.777$

9

(i)



(ii) The time for swimming cannot decrease forever as there is a limit on how fast a swimmer can swim and from the scatter diagram, as x increases, t decreases with decreasing amount, so linear model is not appropriate.

(iii)

$\ln t = a + bx$: $r = -0.9851$

$\frac{1}{t} = a + bx : r = 0.9877$

(iv) Since $|r|$ for $\frac{1}{t} = a + bx$ is higher than that of $\ln t = a + bx$, $\frac{1}{t} = a + bx$ is the preferred model.

(v) Let the timing be t

$$\frac{1}{t} = -0.09836 + \left(5.96846 \times 10^{-5}\right)x$$

Only value that satisfies the equation is $\left(\bar{x}, \overline{\left(\frac{1}{t}\right)}\right)$.

$$\overline{\frac{1}{t}} = -0.09836 + \left(5.96846 \times 10^{-5}\right)\bar{x} = -0.09836 + 0.0000596846(1956) = 0.018383$$

So $\frac{1}{7}\left(\frac{1}{65.6} + \frac{1}{60.4} + \frac{1}{55.4} + \frac{1}{52.2} + \frac{1}{49.99} + \frac{1}{48.18} + \frac{1}{t}\right) = 0.018383$

$t = 52.87 \approx 52.9$

So the timing at 1964 is 52.9 second

10

(i) Let X be the carbon emission of "Green Leaf".
From GC, unbiased estimate of population mean $= \bar{x} = 80.35$,
Unbiased estimate of population variance $= s^2 = (1.3089)^2 = 1.7132$

(ii) Since n is small and σ^2 is unknown, we use the t-test.
Assumption: The carbon emission of the "Green Leaf" is normally distributed.

$H_0 : \mu = 80$ vs $H_1 : \mu > 80$
Test Statistic, $t = 1.1959$
From GC, p-value $= 0.12323 > 0.1$

Since the p-value is more than the level of significance, we do not reject H_0. There is insufficient evidence, at the 10% level, to indicate that the manufacturer's claim is not true.

(iii) Since n is small and σ^2 is unknown, we use the t-test.
For H_0 to be rejected, Test Statistic > 2.8214
Unbiased estimate of population variance $s^2 = \frac{10}{9}m^2$

Test Statistic, $t = \frac{80.6-80}{m/3} > 2.8214$

$m < 0.638$

11 (i) X – number of T4-cells in 0.01 mm^3 of blood

$X \sim \text{Po}(5)$
Y – number of T8-cells in 0.01 mm^3 of blood
$Y \sim \text{Po}(1.5)$

$$\text{P}(\text{healthy}) = \text{P}(X \geq 4)\text{P}(Y \geq 1)$$
$$= [1-\text{P}(X \leq 3)][1-\text{P}(Y=0)]$$
$$= 0.57098 \approx 0.571 \text{ (3 s.f.)}$$

(ii) Req prob $= [\text{P}(\text{healthy})][\text{P}(\text{unhealthy})]^2 \times \frac{3!}{2!}$

$$= (0.57098)(1-0.57098)^2 \times \frac{3!}{2!} = 0.315 \text{ (3 s.f.)}$$

Alternatively,
A – number of patients who are healthy out of 3 patients
$A \sim \text{B}(3, 0.57098)$
$\text{P}(A=1) = 0.31528 \approx 0.315$ (3 s.f.)

(iii) $\text{P}(\text{susceptible})$
$= \text{P}(X < 3) = \text{P}(X \leq 2)$
$= 0.12465$
W – number of patients who are susceptible to infection out of 100 patients
$W \sim \text{B}(100, 0.12465)$

Since n is large and
$np = (100)(0.12465) = 12.465 > 5$ and $n(1-p) = (100)(1-0.12465) = 87.535 > 5$
$W \sim \text{N}(12.465, 10.911)$ approx

$\mathrm{P}(20 \le W \le 50)$
$= \mathrm{P}(19.5 < W < 50.5)$ (c.c)
$= 0.016595 \approx 0.0166$ (3 s.f.)

12 (i)The occurrences of faulty gearbox must be independent of one another

The probability of a faulty gearbox is constant

(ii) C – number of cars that has gearbox issues out of 20 cars

$C \sim \mathrm{B}(20, 0.02)$

$\mathrm{P}(2 < C < 6) = \mathrm{P}(C \le 5) - \mathrm{P}(C \le 2)$

$= 0.0070667 = 0.00707$ (3 s.f.)

Alternatively,

$\mathrm{P}(2 < C < 6) = \mathrm{P}(C = 3) + \mathrm{P}(C = 4) + \mathrm{P}(C = 5)$

$= 0.0070667 = 0.00707$ (3 s.f.)

(iii) $C \sim B(n, 0.02)$

$\mathrm{P}(C < 2) \le 0.95$

$\mathrm{P}(C \le 1) \le 0.95$

$\mathrm{P}(C = 0) + \mathrm{P}(C = 1) \le 0.95$

${}^nC_0 (0.02)^0 (0.98)^{n-0} + {}^nC_1 (0.02)^1 (0.98)^{n-1} \le 0.95$

$(0.98)^n + n\left(\frac{1}{49}\right)(0.98)^n \le 0.95$

$(0.98)^n \left(1 + \frac{n}{49}\right) \le 0.95$

$(0.98)^n \left(1 + \frac{n}{49}\right) - 0.95 \le 0$

$n \ge 18.0977$

Hence the least number of cars Mr Ouyang has to sample is 19

Alternatively, By GC using table

n	$(0.98)^n\left(1+\frac{n}{49}\right)$
17	0.95541
18	0.95049
19	0.94538

Hence the least number of cars Mr Ouyang has to sample is 19.

(iv) C – number of cars that has gearbox issues out of 20 cars

$C \sim \mathrm{B}(20, 0.02)$

$\mathrm{E}(C) = 0.4$ $\mathrm{Var}(C) = 0.392$

Since $n = 100$ large, by CLT, $\overline{C} \sim N\left(0.4, \frac{0.392}{100}\right)$ approx.

$\mathrm{P}(\overline{C} \ge 0.3) = 0.945$