2016 YEAR 6 PRELIMINARY EXAMINATION

MATHEMATICS PAPER 1 **9740/01**

Higher 2 **19 SEPTEMBER 2016**

Total Marks: 100 **3 hours**

Additional materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the test, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

[Turn over

1 It is given that $\mathrm{f}(x)=ax^3+bx^2+cx+d$, where a, b, c and d are constants. The graph of $y=\mathrm{f}(x)$ passes through the point with coordinates $(-1,-27)$ and has a turning point at $(2,27)$. Given also that $\mathrm{f}''(0)=0$, find $\mathrm{f}(x)$. **[4]**

2 In a laboratory experiment, an empty 10-litre tank is transported back and forth between station A and station B by a machine.

Starting at station A, 1000ml of water is added to the tank and on subsequent visits, 90% of the amount of water added previously is added to the tank, that is, 900ml on the 2nd visit, 810 ml on the 3rd visit and so on.

At station B, the machine removes 100ml of water from the tank and on subsequent visits, it removes 50ml more water than the previous visit, that is, 150ml on the 2nd visit, 200ml on the 3rd visit and so on.

(i) Show that the amount of water in the tank after the 3rd visit to station B is 2260ml. **[1]**

The machine stops when the amount of water to be removed exceeds the amount of water present in the tank.

(ii) Determine the amount of water in the tank when the machine stops. Leave your answer to the nearest millilitre. **[3]**

3 The even positive integers, starting at 2, are grouped into sets containing 1, 3, 5, 7, ... integers, as indicated below, so that the number of integers in each set is two more than the number of integers in the previous set.

$$\{2\},\ \{4,6,8\},\ \{10,12,14,16,18\},\ \{20,22,24,26,28,30,32\},\ \ldots$$

Find, in terms of r, an expression for

(i) the number of integers in the r^{th} set, **[1]**

(ii) the last integer in the r^{th} set. **[2]**

Given that the n^{th} set contains the integer 2016, find n. **[2]**

4 **(i)** Use integration by parts to show that for any real constant a,

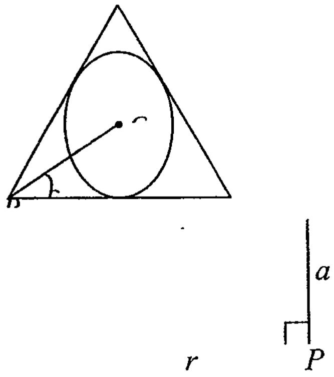
$$\int e^x \sin ax \, dx = \frac{e^x(\sin ax - a\cos ax)}{1+a^2} + c,$$

where c is an arbitrary constant. **[4]**

(ii) By expressing $\sin 2x \cos x$ in the form $A(\sin Px + \sin Qx)$, for real constants A, P and Q, find $\int e^x \sin 2x \cos x \, dx$. **[2]**

5 [Aright circular cone with base radius r, height h and slant height l has curved surface area πrl.]

A right circular cone of base radius r is designed to contain a sphere of fixed radius a. The sphere touches both the curved surface and the base of the cone. (See diagram for a cross-sectional view.)



The point O is the centre of the sphere, the point B is on the circumference of the base of the cone, the point P is the centre of the circular base of the coneand θ is the angle OB makes with the base.

(i) Show that $\cos 2\theta = \dfrac{r^2 - a^2}{r^2 + a^2}$. **[2]**

(ii) Use differentiation to find, in terms of a, the minimum total surface area of the cone (consisting of the curved surface area and the base area), proving that it is a minimum. **[6]**

6 **Do not use a calculator in answering this question.**

(i) For $y = 2\cos\left(\frac{2}{3}\cos^{-1} x\right)$, show that $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -\frac{4}{9}y$. Hence find the Maclaurin series for y, up to and including the term in x^2. **[4]**

(ii) Given that the first three terms found in part **(i)** are equal to the first three terms in the series expansion of $(1+bx)^n$, find the values of the constants b and n .**[4]**

7 The complex numbers z and w are such that

$$z = 1 - \mathrm{i}\sqrt{3} \text{ and } w = -\sqrt{2} + \mathrm{i}c$$

where$c$ is real and positive. It is given that $\left|\dfrac{z}{w}\right| = 1$.

(i) Find the exact value of c. **[2]**

(ii) Show that $\arg\left(\dfrac{z}{w}\right) = \dfrac{11\pi}{12}$. **[2]**

(iii) Express $\dfrac{z}{w}$ in the form $x + \mathrm{i}y$, where x and y are real, giving the exact values of x and y in non-trigonometrical form. **[2]**

(iv) Hence, by considering the complex number $\dfrac{z}{w}$ on an Argand diagram, show that $\tan\left(\dfrac{5\pi}{12}\right) = 2 + \sqrt{3}$. **[2]**

8 The function f is defined by

$$\mathrm{f} : x \to x^2 + \lambda x + 7, \quad x \in \mathbf{R},\ x \leq 3,$$

where λ is a constant.

(i) State the range of values that λ can take iff^{-1} exists. **[1]**

It is given that $\lambda = -6$.

(ii) Find$\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. **[3]**

(iii) Sketch on the same diagram the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$. **[3]**

(iv) Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$. Show algebraically that the solution to $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ satisfies the equation $x^2 - 7x + 7 = 0$.
Hence find the exact value of$x$ that satisfies the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. **[3]**

9 A curve C has parametric equations

$$x = t^2, \ y = 1 + 2t \text{ for } t > 0.$$

(i) Sketch C. **[2]**

(ii) Find the equations of the tangent and the normal to C at the point $P(p^2, 1+2p)$. **[4]**

(iii) The tangent and normal at P meet the y-axis at T and N respectively. Show that $\frac{PT^2}{TN} = p$. **[4]**

10 The curve C has equation $y = \frac{a(x-1)(x-2)}{2-3x}$.

(i) The curve C is scaled by a factor of 3 parallel to the x-axis to get the curve C'. Given that the point $(4,1)$ lies on C' show that $a = 9$. **[2]**

For the rest of the question, use $a = 9$.

(ii) Obtain the equations of the two asymptotes of C. **[2]**

(iii) Sketch C, stating the coordinates of any turning points and of the points where the curve crosses the axes. **[4]**

(iv) Without using a calculator, find the range of values of λ for which the line $y = 9x + \lambda$ and C have at least one point in common. **[3]**

11 The line l_1 passes through the point A, whose position vector is $-\mathbf{i} + 2\mathbf{j}$, and is parallel to the vector $\mathbf{i} + \mathbf{k}$. The line l_2 passes through the point B, whose position vector is $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, and is parallel to the vector $\mathbf{j} + \mathbf{k}$.

(i) Show that the lines l_1 and l_2 are skew. **[2]**

(ii) Find the position vector of the point N on l_2 such that AN is perpendicular to l_2. **[3]**

The plane Π contains l_2 and is perpendicular to AN.

(iii) Find a vector equation for Π in the form $\mathbf{r} = \mathbf{u} + \alpha\mathbf{v} + \beta\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are perpendicular vectors. **[3]**

(iv) The point X varies in such a way that the mid-point of AX is always in Π. Find a vector equation for the locus of X. Describe this locus and state its geometrical relationship with the plane Π. **[4]**

12 **(a)** By using the substitution $y = 2ux^2$, find the general solution of the differentialequation $2x^2\frac{\mathrm{d}y}{\mathrm{d}x} - 4xy + y^2 = 0$, where $x > 0$.

[4]

(b) A glass of water is taken from a refrigerator and placed in a room where the temperature is a constant 32°C. As the water warms up, the rate of increase of its temperature θ°C after t minutes is proportional to the temperature difference $(32-\theta)$°C. Initially the temperature of the water is 4°C and the rate of increase of the temperature is 2°C per minute.

By setting up and solving a differential equation, show that $\theta = 32 - 28\mathrm{e}^{-\frac{1}{14}t}$. **[6]**

(i) Find the time, to the nearest minute, it takes the water to reach a temperature of 20°C. **[1]**

(ii) State what happens to θ for large values of t. **[1]**

(iii) Sketch a graph of θ against t. **[2]**

******* **End of Paper** *******

2016 H2 Mathematics
Prelim Paper 1 Solutions

1 It is given that $\mathrm{f}(x)=ax^3+bx^2+cx+d$, where a, b, c and d are constants.
The graph of $y=\mathrm{f}(x)$ passes through the point with coordinates $(-1,-27)$ and has a turning point at $(2,27)$. Given also that $\mathrm{f}''(0)=0$, find $\mathrm{f}(x)$. [4]

Qn. [Marks]	Solution	Remarks
1 [4]	$\mathrm{f}(x)=ax^3+bx^2+cx+d$ $\mathrm{f}'(x)=3ax^2+2bx+c$ $\mathrm{f}''(x)=6ax+2b$ $\mathrm{f}''(0)=0+2b=0\Rightarrow b=0$ $\mathrm{f}(-1)=-27 \Rightarrow -a- c+d=-27$ $\mathrm{f}(2)=27 \Rightarrow 8a+2c+d=27$ $\mathrm{f}'(2)=0 \Rightarrow 12a+c =0$ Use GC to obtain $a=-2$, $c=24$ and $d=-5$. So $\mathrm{f}(x)=-2x^3+24x-5$.	**Extra Practice** **9740/2011/P1/2** **9740/2015/P1/1**

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Leave your answer to the nearest millilitre. [3]

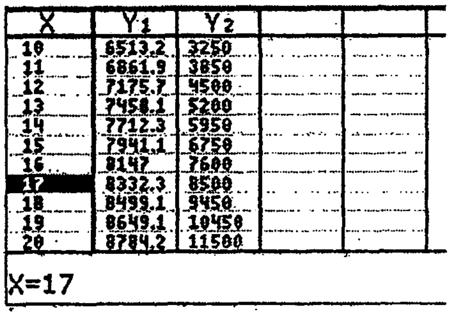
<table>
<tr><th>Qn.
[Marks]</th><th>Solution</th><th>Remarks</th></tr>
<tr><td>2(i)
[1]</td><td>$(1000-100)+(900-150)+(810-200)=2260$</td><td rowspan="2">Extra Practice
9740/2011/P1/9
9740/2014/P2/3
9740/2015/P1/8</td></tr>
<tr><td>(ii)
[3]</td><td>

Solution 1

Find the least n when

$$\frac{1000\left(1-0.9^n\right)}{1-0.9}<\frac{n}{2}\left(2(100)+(n-1)50\right)$$

$$10000\left(1-0.9^n\right)<25n(n+3)$$



<table>
<tr><th>X</th><th>Y₁</th><th>Y₂</th><th></th><th></th></tr>
<tr><td>10</td><td>6513.2</td><td>3250</td><td></td><td></td></tr>
<tr><td>11</td><td>6861.9</td><td>3850</td><td></td><td></td></tr>
<tr><td>12</td><td>7175.7</td><td>4500</td><td></td><td></td></tr>
<tr><td>13</td><td>7458.1</td><td>5200</td><td></td><td></td></tr>
<tr><td>14</td><td>7712.3</td><td>5950</td><td></td><td></td></tr>
<tr><td>15</td><td>7941.1</td><td>6750</td><td></td><td></td></tr>
<tr><td>16</td><td>8147</td><td>7600</td><td></td><td></td></tr>
<tr><td>17</td><td>8332.3</td><td>8500</td><td></td><td></td></tr>
<tr><td>18</td><td>8499.1</td><td>9450</td><td></td><td></td></tr>
<tr><td>19</td><td>8649.1</td><td>10450</td><td></td><td></td></tr>
<tr><td>20</td><td>8784.2</td><td>11500</td><td></td><td></td></tr>
</table>

X=17

Least n=17,

Amount of water is $8332.3-7600=732$ (nearest ml)

OR Solution 2

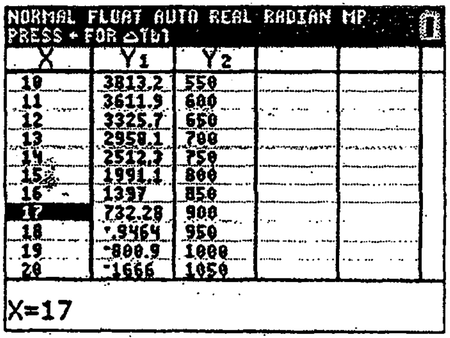
After nth visit to Station A, (so $n-1$ visits to Station B) the amount of water in tank is

$$\frac{1000\left(1-0.9^n\right)}{1-0.9}-\frac{n-1}{2}\left(2(100)+(n-2)50\right)$$

$$=10000\left(1-0.9^n\right)-25(n-1)(n+2)$$

Amount that is to be removed at nth visit to Station B is

$$100+(n-1)50=50(n+1)$$



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<table>
<tr><th>X</th><th>Y₁</th><th>Y₂</th><th></th><th></th></tr>
<tr><td>10</td><td>3813.2</td><td>550</td><td></td><td></td></tr>
<tr><td>11</td><td>3611.9</td><td>600</td><td></td><td></td></tr>
<tr><td>12</td><td>3325.7</td><td>650</td><td></td><td></td></tr>
<tr><td>13</td><td>2958.1</td><td>700</td><td></td><td></td></tr>
<tr><td>14</td><td>2512.3</td><td>750</td><td></td><td></td></tr>
<tr><td>15</td><td>1991.1</td><td>800</td><td></td><td></td></tr>
<tr><td>16</td><td>1397</td><td>850</td><td></td><td></td></tr>
<tr><td>17</td><td>732.28</td><td>900</td><td></td><td></td></tr>
<tr><td>18</td><td>⁻.9464</td><td>950</td><td></td><td></td></tr>
<tr><td>19</td><td>⁻800.9</td><td>1000</td><td></td><td></td></tr>
<tr><td>20</td><td>⁻1666</td><td>1050</td><td></td><td></td></tr>
</table>

X=17

Amount of water in the tank when machine stops is 732 ml.

OR Solution 3

Let u_n be the amount of water after nth visit to Station B

$u_n=u_{n-1}+1000\times0.9^{n-1}-50(n+1),\ u_0=0$

(i) $u_1=900,\ u_2=1650,\ u_3=2260$

(ii) From GC, $u_{16}=546.98,\ u_{17}=-167.7$

The required amount of water

$=u_{16}+1000\times0.9^{17-1}=732\,\text{ml}$

</td></tr>
</table>

3 The even positive integers, starting at 2, are grouped into sets containing 1, 3, 5, 7, ... integers, as indicated below, so that the number of integers in each set is two more than the number of integers in the previous set.

$$\{2\}, \{4,6,8\}, \{10,12,14,16,18\}, \{20,22,24,26,28,30,32\}, \ldots$$

Find, in terms of r, an expression for

(i) the number of integers in the r^{th} set, [1]

(ii) the last integer in the r^{th} set. [2]

Given that the n^{th} set contains the integer 2016, find n. [2]

Qn. [Marks]	Solution
3(a)(i) [1] (ii) [2]	Number of integers in rth set $= 1 + (r-1)2 = 2r - 1$ Total number of integers in r sets $= \frac{1+2r-1}{2}(r) = r^2$ Last integer in rth set $= 2 + (r^2 - 1)2 = 2r^2$
(iii)	$2(n-1)^2 < 2016 \le 2n^2$ $(n-1)^2 < 1008 \le n^2$ $n = 32$

4 (i) Use integration by parts to show that for any real constant a,

$$\int e^x \sin ax \, dx = \frac{e^x(\sin ax - a\cos ax)}{1+a^2} + c,$$

where c is an arbitrary constant. [4]

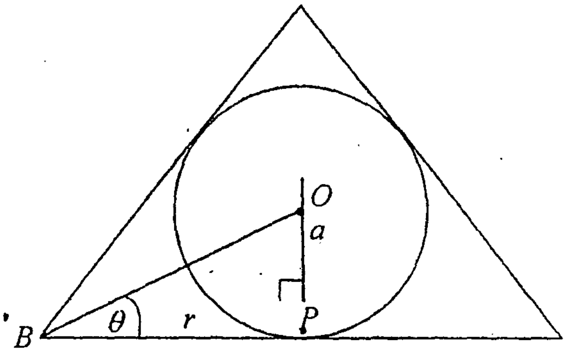
(ii) By expressing $\sin 2x \cos x$ in the form $A(\sin Px + \sin Qx)$, for real constants A, P and Q, find $\int e^x \sin 2x \cos x \, dx$. [2]

Qn. [Marks]	Solution	Remarks
4(i)	$\int e^x \sin ax \, dx = e^x \sin ax - a\int e^x \cos ax \, dx$ $= e^x \sin ax - a\left(e^x \cos ax + a\int e^x \sin ax \, dx\right)$ $= e^x \sin ax - ae^x \cos ax - a^2 \int e^x \sin ax \, dx$ $(1+a^2)\int e^x \sin ax \, dx = e^x \sin ax - ae^x \cos ax$ $\int e^x \sin ax \, dx = \frac{e^x(\sin ax - a\cos ax)}{1+a^2} + c$ (shown)	**Extra practice 9740/2011/P2/4 9740/2008/P1/5** In a **show question**, the derivation must be shown carefully

	OR: $\int e^x \sin ax \, dx = -\frac{e^x}{a}\cos ax + \frac{1}{a}\int e^x \cos ax \, dx$ $= -\frac{e^x}{a}\cos ax + \frac{1}{a}\left(\frac{e^x}{a}\sin ax - \frac{1}{a}\int e^x \sin ax \, dx\right)$ $= -\frac{e^x}{a}\cos ax + \frac{e^x}{a^2}\sin ax - \frac{1}{a^2}\int e^x \sin ax \, dx$ $a^2\int e^x \sin ax \, dx = -ae^x \cos ax + e^x \sin ax - \int e^x \sin ax \, dx$ $\left(1+a^2\right)\int e^x \sin ax \, dx = e^x \sin ax - ae^x \cos ax$ $\int e^x \sin ax \, dx = \frac{e^x\left(\sin ax - a\cos ax\right)}{1+a^2} + c$ (shown)	
(ii)	$\sin 2x \cos x = \frac{1}{2}\left(\sin 3x + \sin x\right)$ i.e. $A = \frac{1}{2}, P = 3, Q = 1$ $\int e^x \sin 2x \cos x \, dx = \frac{1}{2}\int e^x\left(\sin 3x + \sin x\right) dx$ $= \frac{1}{2}\int\left(e^x \sin 3x + e^x \sin x\right) dx$ $= \frac{1}{2}\left[\frac{e^x\left(\sin 3x - 3\cos 3x\right)}{1+3^2} + \frac{e^x\left(\sin x - \cos x\right)}{1+1^2}\right] + c$ $= \frac{e^x}{2}\left(\frac{\sin 3x - 3\cos 3x}{10} + \frac{\sin x - \cos x}{2}\right) + c$ $= \frac{e^x}{20}\left(\sin 3x - 3\cos 3x + 5\sin x - 5\cos x\right) + c$	Refer to MF15

5 [A right circular cone with base radius r, height h and slant height l has curved surface area $\pi r l$.]

A right circular cone of base radius r is designed to contain a sphere of fixed radius a. The sphere touches both the curved surface and the base of the cone. (See diagram for a cross-sectional view.)



The point O is the centre of the sphere, the point B is on the circumference of the base of the cone, the point P is the centre of the circular base of the cone and θ is the angle OB makes with the base.

(i) Show that $\cos 2\theta = \frac{r^2 - a^2}{r^2 + a^2}$. [2]

(ii) Use differentiation to find, in terms of a, the minimum total surface area of the cone (consisting of the curved surface area and the base area), proving that it is a minimum. [6]

<table>
<tr><th>Qn. [Marks]</th><th>Solution</th><th>Remarks</th></tr>
<tr><td>5(i)
[2]</td><td>

$\cos 2\theta = 2\cos^2\theta - 1$

$= 2\left(\frac{r}{\sqrt{r^2+a^2}}\right)^2 - 1 = \frac{2r^2}{r^2+a^2} - \frac{r^2+a^2}{r^2+a^2} = \frac{r^2-a^2}{r^2+a^2}$ [*shown*]

OR

$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{r}{\sqrt{r^2+a^2}}\right)^2 - \left(\frac{a}{\sqrt{r^2+a^2}}\right)^2 = \frac{r^2-a^2}{r^2+a^2}$ [*shown*]

</td><td>

Extra practice
9740/2014/P1/11
9740/2012/P1/10

</td></tr>
<tr><td>(ii)
[6]</td><td>

Let T be the total surface area of the cone.

$T = \pi r l + \pi r^2$

$= \pi r\left(\frac{r}{\cos 2\theta}\right) + \pi r^2$

$= \pi r^2\left(\frac{r^2+a^2}{r^2-a^2} + 1\right) = \frac{2\pi r^4}{r^2-a^2}$

$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{8\pi r^3\left(r^2-a^2\right) - 2\pi r^4(2r)}{\left(r^2-a^2\right)^2}$

$= \frac{4\pi r^3\left(r^2-2a^2\right)}{\left(r^2-a^2\right)^2}$ *or* $\frac{4\pi r^3}{\left(r^2-a^2\right)^2}\left(r-\sqrt{2}a\right)\left(r+\sqrt{2}a\right)$

For $\frac{\mathrm{d}T}{\mathrm{d}r} = 0$, $r = \sqrt{2}a$ since $r > 0$

By the First Derivative test, since $\frac{4\pi r^3}{\left(r^2-a^2\right)^2} > 0$, we have

<table>
<tr><td>r</td><td>$\left(\sqrt{2}a\right)^-$</td><td>$\left(\sqrt{2}a\right)$</td><td>$\left(\sqrt{2}a\right)^+$</td></tr>
<tr><td>r^2</td><td>$\left(2a^2\right)^-$</td><td>$\left(2a^2\right)$</td><td>$\left(2a^2\right)^+$</td></tr>
<tr><td>r^2-a^2</td><td>0^-</td><td>0</td><td>0^+</td></tr>
<tr><td>$\frac{\mathrm{d}T}{\mathrm{d}r}$</td><td>-ve</td><td>0</td><td>+ve</td></tr>
</table>

</td><td>

Sufficient explanation needs to be presented for the first derivative test!

</td></tr>
</table>

	Alternatively, by the Second Derivative Test, $\left.\frac{\mathrm{d}^2T}{\mathrm{d}r^2}\right\|_{r=\sqrt{2}a} = 4\pi\left.\frac{(r^2-a^2)^2(5r^4-6a^2r^2)-r^3(r^2-2a^2)[2(r^2-a^2)(2r)]}{(r^2-a^2)^4}\right\|_{r=\sqrt{2}a}$ $= 4\pi\frac{(a^2)^2(20a^4-12a^4)-0}{(a^2)^4} = 32\pi > 0$ Hence $r=\sqrt{2}a$ gives minimum $T = \frac{2\pi\left(\sqrt{2}a\right)^4}{\left(\sqrt{2}a\right)^2-a^2} = 8\pi a^2$.	

6 **Do not use a calculator in answering this question.**

(i) For $y = 2\cos\left(\frac{2}{3}\cos^{-1}x\right)$, show that $\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{9}y$. Hence find the Maclaurin series for y, up to and including the term in x^2. [4]

(ii) Given that the first three terms found in part (i) are equal to the first three terms in the series expansion of $(1+bx)^n$, find the values of the constants b and n.[4]

Qn. [Marks]	**Solution**	**Remarks**
6(i) [4]	$y = 2\cos\left(\frac{2}{3}\cos^{-1}x\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -2\sin\left(\frac{2}{3}\cos^{-1}x\right)\frac{2}{3}\left(-\frac{1}{\sqrt{1-x^2}}\right)$ $\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{3}\sin\left(\frac{2}{3}\cos^{-1}x\right)$ $\sqrt{1-x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{1}{2\sqrt{1-x^2}}(-2x)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{3}\cos\left(\frac{2}{3}\cos^{-1}x\right)\frac{2}{3}\left(-\frac{1}{\sqrt{1-x^2}}\right)$ $\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{9}y$ OR "implicit differentiation" $\cos^{-1}\frac{y}{2} = \frac{2}{3}\cos^{-1}x$ $\frac{-1}{\sqrt{1-\left(\frac{y}{2}\right)^2}}\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{3}\left(-\frac{1}{\sqrt{1-x^2}}\right)$ $\left(1-x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{16}{9}\left(1-\left(\frac{y}{2}\right)^2\right)$	**Extra practice** **9740/2015/P1/6** **9740/2010/P1/2** **9740/2009/P1/7** Try to simplify your expressions as you go along. Keeping four negative signs around is a good recipe for disaster. Brackets are your friends. Use them to keep your expressions clear and organized.

	$2\left(1-x^2\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)-2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{16}{9}\left(-\frac{y}{2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ $\left(1-x^2\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-\frac{4}{9}y$ At $x=0$, $y=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\sqrt{3}}{3}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{4}{9}$ The series is $y=1+\frac{2\sqrt{3}}{3}x-\frac{2}{9}x^2+\cdots$	
(ii) [4]	$(1+bx)^n=1+nbx+\frac{n(n-1)}{2}b^2x^2+\cdots$ $x:\quad nb=\frac{2\sqrt{3}}{3}\quad ---(1)$ $x^2:\quad \frac{n(n-1)}{2}b^2=-\frac{2}{9}\quad ---(2)$ $\frac{(2)}{(1)}:\quad \frac{n-1}{2}b=-\frac{\sqrt{3}}{9}\quad ---(3)$ $\frac{(1)}{(3)}:\quad \frac{2n}{n-1}=-6\ \Rightarrow\ n=\frac{3}{4}$ $(1):\quad b=\frac{2\sqrt{3}}{3}\cdot\frac{4}{3}=\frac{8\sqrt{3}}{9}$	When comparing the coefficients of corresponding powers of x in the series, exclude the power of x. If you square an expression in solving equations, check that all solutions obtained actually work in the original equations. It is not specified that n is a positive integer, so expressions like $n!$ and $\binom{n}{r}$ have no meaning.

7 The complex numbers z and w are such that

$$z=1-\mathrm{i}\sqrt{3}\ \text{and}\ w=-\sqrt{2}+\mathrm{i}c$$

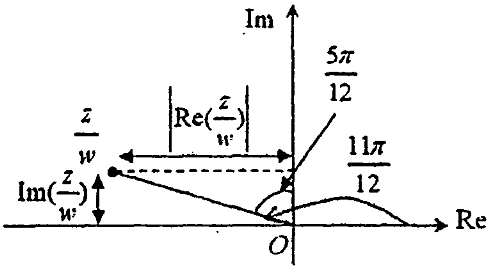
where c is real and positive. It is given that $\left|\frac{z}{w}\right|=1$.

(i) Find the exact value of c. [2]

(ii) Show that $\arg\left(\frac{z}{w}\right)=\frac{11\pi}{12}$. [2]

(iii) Express $\frac{z}{w}$ in the form $x+\mathrm{i}y$, where x and y are real, giving the exact values of x and y in non-trigonometrical form. [2]

(iv) Hence, by considering the complex number $\frac{z}{w}$ on an Argand diagram, show that $\tan\left(\frac{5\pi}{12}\right)=2+\sqrt{3}$. [2]

Qn. [Marks]	Solution	Remarks
7(i) [2]	Given $\left\|\frac{z}{w}\right\|=1$, we have $\frac{2}{\sqrt{2+c^2}}=1$ $c^2=2 \Rightarrow c=\sqrt{2}$ ($\because c$ is positive)	**Extra practice** **9740/2013/P1/8** **9740/2012/P1/6**
(ii) [2]	$\arg\left(\frac{z}{w}\right)=\arg(z)-\arg(w)+2\pi$ $=-\frac{\pi}{3}-\frac{3\pi}{4}+2\pi$ $=\frac{11\pi}{12}$ (shown)	$-\pi<\arg\left(\frac{z}{w}\right)\leq\pi$
(iii) [2]	$\frac{z}{w}=\frac{1-\mathrm{i}\sqrt{3}}{\sqrt{2}(-1+\mathrm{i})}\times\frac{(-1-\mathrm{i})}{(-1-\mathrm{i})}=\frac{-1-\sqrt{3}+\mathrm{i}(\sqrt{3}-1)}{2\sqrt{2}}$ $=\frac{-(\sqrt{3}+1)}{2\sqrt{2}}+\mathrm{i}\frac{(\sqrt{3}-1)}{2\sqrt{2}}$	
(iv) [2]	From the diagram,  $\tan\left(\frac{5\pi}{12}\right)=\frac{\left\|\mathrm{Re}(\frac{z}{w})\right\|}{\mathrm{Im}(\frac{z}{w})}=\frac{\sqrt{3}+1}{2\sqrt{2}}\div\frac{\sqrt{3}-1}{2\sqrt{2}}$ $=\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}=\frac{4+2\sqrt{3}}{2}$ $=2+\sqrt{3}$ (shown)	Question indicates to consider $\frac{z}{w}$ on an Argand diagram

8 The function f is defined by

$$\mathrm{f}: x \to x^2+\lambda x+7, \quad x \in \mathbb{R},\ x \leq 3,$$

where λ is a constant.

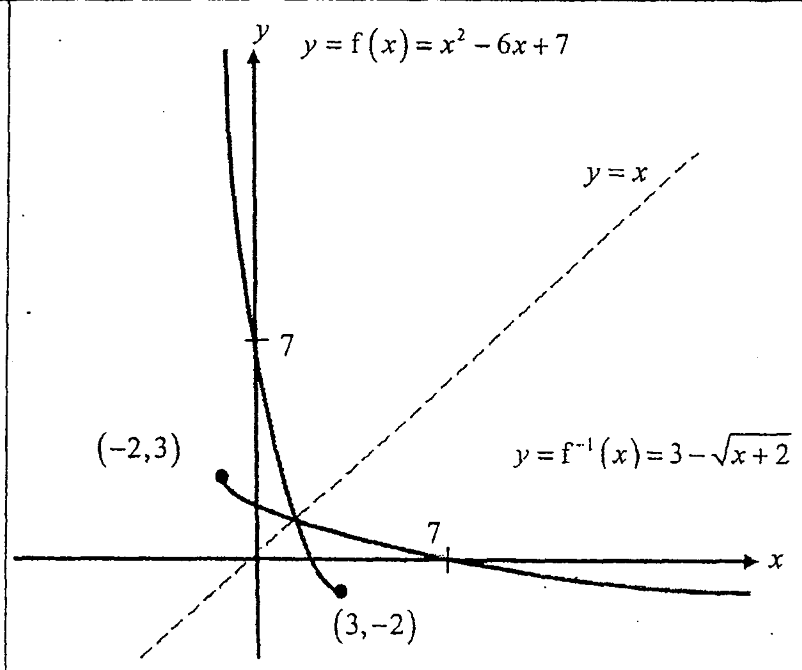
(i) State the range of values that λ can take if f^{-1} exists. [1]

It is given that $\lambda=-6$.

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(iii) Sketch on the same diagram the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$. [3]

(iv) Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$. Show algebraically that the solution to $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ satisfies the equation $x^2 - 7x + 7 = 0$.
Hence find the exact value of x that satisfies the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

Qn. [Marks]	**Solution**	**Remarks**
8(i) **[1]**	$\mathrm{f}(x) = x^2 + \lambda x + 7$ $\mathrm{f}'(x) = 2x + \lambda = 0$ $\Rightarrow x = -\frac{\lambda}{2}$ is the x-coodinate of the min. point Hence, f^{-1} exists when $-\frac{\lambda}{2} \geq 3 \Rightarrow \lambda \leq -6$ **Alternative solution:** $\mathrm{f}(x) = x^2 + \lambda x + 7 = \left(x + \frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{4} + 7$ Inverse of f exists when $-\frac{\lambda}{2} \geq 3 \Rightarrow \lambda \leq -6$.	**Extra practice 9740/2008/P2/4**
(ii) **[3]**	Given that $\lambda = -6$, $\mathrm{f} : x \rightarrow x^2 - 6x + 7$, $x \in \mathbb{R}, x \leq 3$ $R_f = [-2, \infty)$ For $\mathrm{f}^{-1}(x)$, let $y = x^2 - 6x + 7 = (x-3)^2 - 2$ $\Rightarrow y + 2 = (x-3)^2 \Rightarrow x = 3 \pm \sqrt{y+2}$ Since $D_f = (-\infty, 3] = R_{f^{-1}}$ $\mathrm{f}^{-1}(x) = 3 - \sqrt{x+2}$	
(iii) **[3]**		The scale on both axes has to be the same when drawing both f and its inverse on the same diagram. The end points, intersection with axes has to be labelled as well.

(iv) [2]	$y=x$ To solve $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$, we can use $\mathrm{f}(x)=x$ with $x\le 3$ $\Rightarrow x^2-6x+7=x\Rightarrow x^2-7x+7=0$ [shown] $x=\dfrac{7\pm\sqrt{49-28}}{2}$ $\because x\le 3,\ x=\dfrac{7-\sqrt{21}}{2}$	Include "±" when taking square root

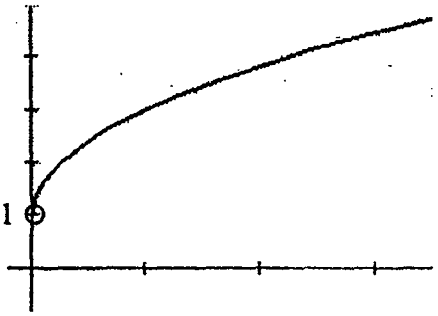
9 A curve C has parametric equations

$$x=t^2, \quad y=1+2t \quad \text{for } t>0.$$

(i) Sketch C. [2]

(ii) Find the equations of the tangent and the normal to C at the point $P(p^2, 1+2p)$. [4]

(iii) The tangent and normal at P meet the y-axis at T and N respectively.
Show that $\dfrac{PT^2}{TN}=p$. [4]

Qn. [Marks]	Solution	Remarks
9(i) [2]		**Extra practice** **9740/2014/P2/1** **9740/2011/P1/3** Indicate open circle at (0,1)
(ii) [4]	$x=t^2\Rightarrow\dfrac{\mathrm{d}x}{\mathrm{d}t}=2t$, $y=1+2t\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}t}=2$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}t}/\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{2}{2t}=\dfrac{1}{t}$ Equation of tangent at $P(p^2,1+2p)$: $y-(1+2p)=\dfrac{1}{p}(x-p^2)$ $y-1-2p=\dfrac{1}{p}x-p$ $y=\dfrac{1}{p}x+1+p$ Equation of normal at $P(p^2,1+2p)$: $y-(1+2p)=-p(x-p^2)$ $y-1-2p=-px+p^3$ $y=-px+p^3+1+2p$	Note that $t=p$ at the point P.

(iii) **[4]**	Subs. $x=0$ into $y=\frac{1}{p}x+1+p$, we have $y=1+p$ $\therefore T$ is $(0,1+p)$ Subs. $x=0$ into $y=-px+p^3+1+2p, y=p^3+1+2p$ $\therefore N$ is $(0, p^3+1+2p)$ Given that P is $(p^2,1+2p)$ $PT^2=(p^2-0)^2+(1+2p-1-p)^2$ $=p^4+p^2$ $TN=\left\|(p^3+1+2p)-(1+p)\right\|$ $=\left\|p^3+p\right\|$ $=p^3+p \quad (p\geq 0)$ $\frac{PT^2}{TN}=\frac{p^4+p^2}{p^3+p}=\frac{p^2(p^2+1)}{p(p^2+1)}=p$ (shown)	*PT* refers to the length of line joining points *P* and *T*. *TN* refers to the length of line joining *T* and *N*.

10 The curve C has equation $y=\frac{a(x-1)(x-2)}{2-3x}$.

(i) The curve C is scaled by a factor of 3 parallel to the x-axis to get the curve C'. Given that the point $(4,1)$ lies on C' show that $a=9$. [2]

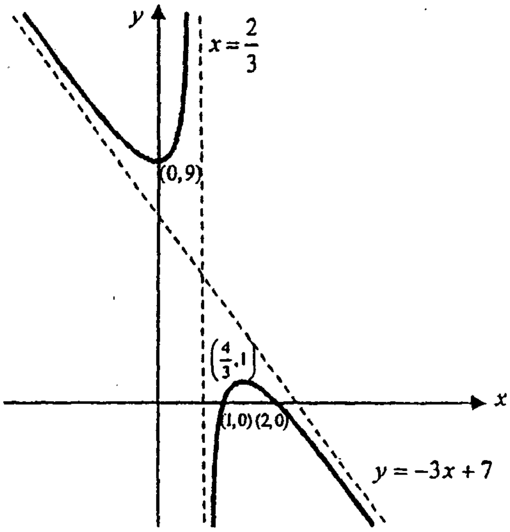
For the rest of the question, use $a=9$.

(ii) Obtain the equations of the two asymptotes of C. [2]

(iii) Sketch C, stating the coordinates of any turning points and of the points where the curve crosses the axes. [4]

(iv) Without using a calculator, find the range of values of λ for which the line $y=9x+\lambda$ and C have at least one point in common. [3]

Qn. [Marks]	Solution	Remark
10(i) **[2]**	If the point $(4,1)$ lies on the curve which is C transformed by a stretch with scale factor 3 parallel to the x-axis, then $\left(\frac{4}{3},1\right)$ is on C. Subs. $x=\frac{4}{3}$ and $y=1$ into C, $1=\frac{a\left(\frac{4}{3}-1\right)\left(\frac{4}{3}-2\right)}{2-3\left(\frac{4}{3}\right)} \Leftrightarrow 1=\frac{a\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{-2}$ $a=9$	**Extra practice** **9740/2015/P2/3** **9740/2013/P1/2**

(ii) [2]	$y=\frac{9x^2-27x+18}{2-3x}=-3x+7+\frac{4}{2-3x}$ Vertical asymptote, $x=\frac{2}{3}$ Oblique asymptote, $y=-3x+7$	
(iii) [4]		Asymptotes drawn with dotted line. Curve should appear to approach asymptotes Same scale along the same axis Label co-ordinates of intercepts and turning points, equation of asymptotes.
(iv) [3]	$\frac{9(x-1)(x-2)}{2-3x}=9x+\lambda$ $9x^2-27x+18=18x+2\lambda-27x^2-3\lambda x$ $36x^2+3(\lambda-15)x+2(9-\lambda)=0$ Since the line $y=9x+\lambda$ and C have at least one point in common, $b^2-4ac\geq 0$ $[3(\lambda-15)]^2-4(36)(2(9-\lambda))\geq 0$ $\lambda^2-30\lambda+225-288+32\lambda\geq 0$ $\lambda^2+2\lambda-63\geq 0$ $(\lambda+9)(\lambda-7)\geq 0$ $\lambda\leq -9$ or $\lambda\geq 7$	Question states "without use of calculator"

11 The line l_1 passes through the point A, whose position vector is $-\mathbf{i}+2\mathbf{j}$, and is parallel to the vector $\mathbf{i}+\mathbf{k}$. The line l_2 passes through the point B, whose position vector is $\mathbf{i}+\mathbf{j}+3\mathbf{k}$, and is parallel to the vector $\mathbf{j}+\mathbf{k}$.,

(i) Show that the lines l_1 and l_2 are skew. [2]

(ii) Find the position vector of the point N on l_2 such that AN is perpendicular to l_2. [3]

The plane Π contains l_2 and is perpendicular to AN.

(iii) Find a vector equation for Π in the form $\mathbf{r}=\mathbf{u}+\alpha\mathbf{v}+\beta\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are perpendicular vectors. [3]

(iv) The point X varies in such a way that the mid-point of AX is always in Π. Find a vector equation for the locus of X. Describe this locus and state its geometrical relationship with the plane Π. [4]

Qn. [Marks]	Solution	Remarks
11(i) [2]	$l_1 : \mathbf{r} = \begin{pmatrix}-1\\2\\0\end{pmatrix} + s\begin{pmatrix}1\\0\\1\end{pmatrix}, s \in \mathbb{R}$ $\qquad l_2 : \mathbf{r} = \begin{pmatrix}1\\1\\3\end{pmatrix} + t\begin{pmatrix}0\\1\\1\end{pmatrix}, t \in \mathbb{R}$ If l_1 and l_2 intersect, $-1+s=1 \quad \ldots(1)$ $2=1+t \ \ldots(2)$ $s=3+t \ \ldots(3)$ from (1), $s=2$ from (2), $t=1$ but for (3), LHS = 2 ≠ 3 + 1 = 4 = RHS Hence l_1 and l_2 are non-intersecting. Since $\begin{pmatrix}1\\0\\1\end{pmatrix} \neq k\begin{pmatrix}0\\1\\1\end{pmatrix}$ for any $k \in \mathbb{R}$, l_1 and l_2 are non-parallel. Thus the lines l_1 and l_2 are skew.	**Extra practice 9740/2011/P1/11** For skew lines, we need to show that two lines are non-parallel **AND** non-intersecting.
(ii) [3]	Since N is on l_2, $\overrightarrow{ON} = \begin{pmatrix}1\\1\\3\end{pmatrix} + t\begin{pmatrix}0\\1\\1\end{pmatrix}$ for some $t \in \mathbb{R}$. $\overrightarrow{AN} = \begin{pmatrix}1\\1\\3\end{pmatrix} + t\begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}-1\\2\\0\end{pmatrix} = \begin{pmatrix}2\\-1+t\\3+t\end{pmatrix}$ $\overrightarrow{AN}.\begin{pmatrix}0\\1\\1\end{pmatrix} = 0 \quad \Rightarrow \begin{pmatrix}2\\-1+t\\3+t\end{pmatrix}.\begin{pmatrix}0\\1\\1\end{pmatrix} = 0$ $\Rightarrow -1+t+3+t=0$ $\Rightarrow t=-1$ $\therefore \overrightarrow{ON} = \begin{pmatrix}1\\1\\3\end{pmatrix} - \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$	
(iii) [3]	Let $\mathbf{u} = \begin{pmatrix}1\\1\\3\end{pmatrix}, \mathbf{v} = \begin{pmatrix}0\\1\\1\end{pmatrix}$. $\mathbf{w}$ is perpendicular to $\mathbf{v} = \begin{pmatrix}0\\1\\1\end{pmatrix}$ and also to $\overrightarrow{AN} = \begin{pmatrix}2\\-2\\2\end{pmatrix}$.	It is required to have vectors v and w to be perpendicular and any vectors that are parallel to the plane.

	Consider $\begin{pmatrix}0\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$. Let $\mathbf{w}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$. $\therefore \Pi:\mathbf{r}=\begin{pmatrix}1\\1\\3\end{pmatrix}+\alpha\begin{pmatrix}0\\1\\1\end{pmatrix}+\beta\begin{pmatrix}2\\1\\-1\end{pmatrix}$, $\alpha,\beta\in\mathbb{R}$	
(iv) **[4]**	Let M be the mid-point of AX. By ratio theorem, $\overline{OM}=\frac{1}{2}\left(\overline{OA}+\overline{OX}\right)$. Since M lies on Π, $\overline{OM}=\begin{pmatrix}1\\1\\3\end{pmatrix}+\alpha\begin{pmatrix}0\\1\\1\end{pmatrix}+\beta\begin{pmatrix}2\\1\\-1\end{pmatrix}$ for some $\alpha,\beta\in\mathbb{R}$. $\frac{1}{2}\left[\begin{pmatrix}-1\\2\\0\end{pmatrix}+\overline{OX}\right]=\begin{pmatrix}1\\1\\3\end{pmatrix}+\alpha\begin{pmatrix}0\\1\\1\end{pmatrix}+\beta\begin{pmatrix}2\\1\\-1\end{pmatrix}$ $\overline{OX}=2\begin{pmatrix}1\\1\\3\end{pmatrix}+2\alpha\begin{pmatrix}0\\1\\1\end{pmatrix}+2\beta\begin{pmatrix}2\\1\\-1\end{pmatrix}-\begin{pmatrix}-1\\2\\0\end{pmatrix}$ $=\begin{pmatrix}3\\0\\6\end{pmatrix}+h\begin{pmatrix}0\\1\\1\end{pmatrix}+k\begin{pmatrix}2\\1\\-1\end{pmatrix}$ The equation of the locus of X is $\mathbf{r}=\begin{pmatrix}3\\0\\6\end{pmatrix}+h\begin{pmatrix}0\\1\\1\end{pmatrix}+k\begin{pmatrix}2\\1\\-1\end{pmatrix}$, $h,k\in\mathbb{R}$ The locus of X is a <u>plane</u> parallel to Π.	

12 (a) By using the substitution $y=2ux^2$, find the general solution of the differential equation $2x^2\frac{\mathrm{d}y}{\mathrm{d}x}-4xy+y^2=0$, where $x>0$. [4]

(b) A glass of water is taken from a refrigerator and placed in a room where the temperature is a constant 32°C. As the water warms up, the rate of increase of its temperature θ°C after t minutes is proportional to the temperature difference $(32-\theta)$°C. Initially the temperature of the water is 4°C and the rate of increase of the temperature is 2°C per minute.

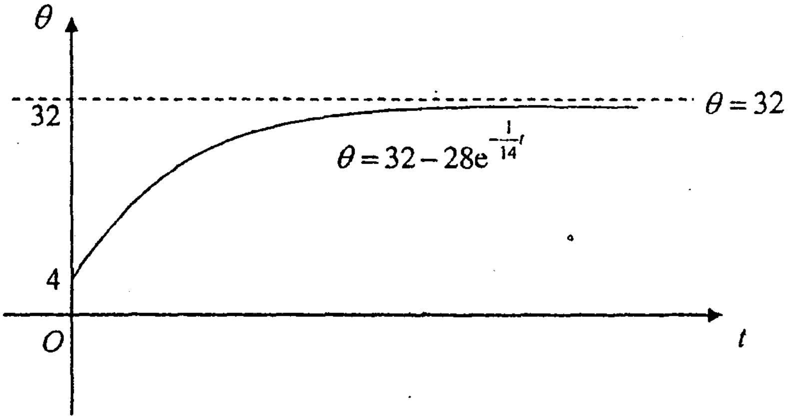
By setting up and solving a differential equation, show that $\theta=32-28e^{-\frac{1}{14}t}$. [6]

(i) Find the time, to the nearest minute, it takes the water to reach a temperature of 20°C. [1]

(ii) State what happens to θ for large values of t. [1]

(iii) Sketch a graph of θ against t. [2]

Qn. [Marks]	Solution	Remarks
12(a) [4]	$y = 2ux^2$ ------------(1) $\frac{\mathrm{d}y}{\mathrm{d}x} = 4ux + 2x^2\frac{\mathrm{d}u}{\mathrm{d}x}$ ------(2) Substituting (1) and (2) into $2x^2\frac{\mathrm{d}y}{\mathrm{d}x} - 4xy + y^2 = 0$, we have $2x^2\left(4ux + 2x^2\frac{\mathrm{d}u}{\mathrm{d}x}\right) - 4x\left(2ux^2\right) + \left(2ux^2\right)^2 = 0$ $8ux^3 + 4x^4\frac{\mathrm{d}u}{\mathrm{d}x} - 8ux^3 + 4u^2x^4 = 0$ $\frac{\mathrm{d}u}{\mathrm{d}x} = -u^2$ $\int -\frac{1}{u^2}\ \mathrm{d}u = \int \mathrm{d}x$ $\frac{1}{u} = x + c \Leftrightarrow u = \frac{1}{x+c}$ $\frac{y}{2x^2} = \frac{1}{x+c}$ $y = \frac{2x^2}{x+c}$	In expression $y = 2ux^2$, u is a function of x, so need to apply Chain rule when differentiating y wrt x. Express y in terms of x explicitly whenever possible.
(b) [6]	$\frac{\mathrm{d}\theta}{\mathrm{d}t} = k(32-\theta),\ k > 0$ $\int\frac{1}{32-\theta}\mathrm{d}\theta = \int k\ \mathrm{d}t$ $-\ln\lvert 32-\theta\rvert = kt + A$ $\lvert 32-\theta\rvert = \mathrm{e}^{-kt-A}$ $32-\theta = \pm\mathrm{e}^{-kt}\mathrm{e}^{-A} = C\mathrm{e}^{-kt}$ where $C = \pm\mathrm{e}^{-A}$ $\theta = 32 - C\mathrm{e}^{-kt}$ Given that $\theta = 4, \frac{\mathrm{d}\theta}{\mathrm{d}t} = 2$ when $t = 0$ $\theta = 32 - C\mathrm{e}^{-kt} \Rightarrow 4 = 32 - C \Rightarrow C = 28$ $\frac{\mathrm{d}\theta}{\mathrm{d}t} = k(32-\theta) \Rightarrow 2 = k(32-4) \Rightarrow k = \frac{1}{14}$ $\therefore \theta = 32 - 28\mathrm{e}^{-\frac{1}{14}t}$ (shown)	**Extra practice 9740/2010/P1/Q7**

(i) **[1]**	$\theta = 32 - 28e^{-\frac{1}{14}t}$ At $\theta = 20$, $20 = 32 - 28e^{-\frac{1}{14}t}$ $28e^{-\frac{1}{14}t} = 12$ $-\frac{1}{14}t = \ln\frac{3}{7}$ $t = -14\ln\frac{3}{7} = 11.86 \approx 12$ min	
(ii) **[1]**	As $t \to \infty$, $e^{-\frac{1}{14}t} \to 0$. $\therefore \theta \to 32$. i.e. the temperature of the water increases and approaches the room temperature, i.e. 32° C, for large values of *t*.	
(iii) **[2]**		Label the asymptote.

2016 YEAR 6 PRELIMININARY EXAMINATION

MATHEMATICS PAPER 2 **9740/02**

Higher 2 **21 SEPTEMBER 2016**

Total Marks: 100 **3 hours**

Additional materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the test, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of7 printed pages.

Section A: Pure Mathematics [40 Marks]

1 **(a)** Use the method of mathematical induction to prove that $\sum_{r=1}^{n} 4r^3 = \left[n(n+1)\right]^2$.

[4]

(b) It is given that $\mathrm{f}(r) = r^4 + 2r^3 + 2r^2 + r$.

Show that $\mathrm{f}(r) - \mathrm{f}(r-1) = ar^3 + 2r$, where a is a real constant to be determined.

Hence find a formula for $\sum_{r=1}^{n} r(2r^2+1)$. **[4]**

2 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on OA, between O and A, such that $OC : CA = 3 : 2$. Point D lies on OB, between O and B, such that $OD : DB = 1 : \mu$.

(i) It is given that the area of triangle ABD is twice the area of triangle ABC. Find μ. **[4]**

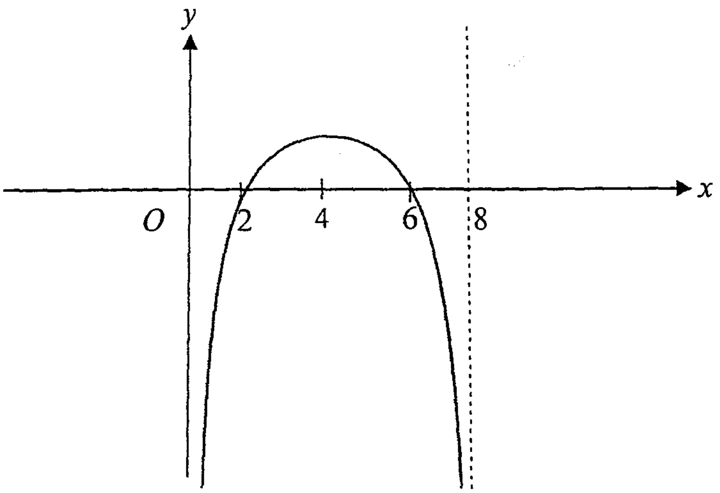
(ii) Show that the vector equation of the line BC can be written as $\mathbf{r} = \frac{3}{5}s\,\mathbf{a} + (1-s)\mathbf{b}$, where s is a parameter. By writing down the vector equation of the line AD in a similar form, in terms of a parameter t, find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point E where the lines BC and AD meet. **[3]**

It is further given that the angle AOB is 45° and O lies on the perpendicular bisector of the line segment AB.

(iii) Find the length of projection of $\mathbf{a}$ on $\mathbf{b}$, giving your answer in terms of $|\mathbf{b}|$. Hence find the position vector of the foot of the perpendicular from A to OB.

[3]

3 The diagram shows the curve with equation $y=2+\dfrac{24}{x(x-8)}$, for $0\le x\le 8$.The curve crosses the x-axis at $x=2$ and $x=6$, has a maximum turning point at $x=4$, andasymptotes at $x=0$ and $x=8$.



For $0\le x\le 8$, the region bounded by the curveswith equations $y=\left|2+\dfrac{24}{x(x-8)}\right|$ and $y=2+\dfrac{4}{x-8}$ is denoted by S.

(i) On the same diagram, for $0\le x\le 8$, sketch the curves with equations

$$y=\left|2+\frac{24}{x(x-8)}\right| \text{ and } y=2+\frac{4}{x-8}.$$

Indicate clearly on your diagram the region S, the coordinates of the points of intersection of the two curves and the equation(s) of any asymptote(s). [3]

(ii) Find the exact area of S.
Express your answer in the form $A+B\ln 2+C\ln 3+D\ln 7$ where A, B, C and D are constants to be determined. [5]

(iii) Find the volume of the solid of revolution formed when S is rotated about the x-axis through 360°. [2]

4 **(a)** Solve the equation

$$z^4 + 4\text{i} = 0,$$

giving the roots in the form $re^{\text{i}\alpha}$, where $r > 0$ and $-\pi < \alpha \leq \pi$. [3]

These roots correspond to fourpoints on an Argand diagram. Identify the quadrilateral that has these fourpoints as vertices. [1]

(b) The point A represents the fixed complex number a, which has modulus r and argument θ, where $0 < \theta < \frac{\pi}{2}$.

(i) On a single Argand diagram, sketch the loci

(a) $|z - a| = 2r$,

(b) $\arg(z + a) = \theta$,

making clear the relationship between the loci and the point A. [3]

(ii) Hence, or otherwise, find exactly the complex number(s) z that satisfy both equations in part **(i)**, giving your answer(s) in terms of r and θ. [2]

(iii) Given instead that $|z - a| \leq 2r$ and $-\pi < \arg(z + a) \leq \theta$, find exactly the minimum and maximum possible values of $|z - \text{i}a|$, giving your answers in terms of r. [3]

Sections B: Statistics [60 Marks]

5 The recreational committee of a large company is organizing a family day and would like to conduct a survey with 5% of its employees about their preferences for an outdoor or indoor based carnival as well as the activities involved.

Describe how the committee could obtain a sample using

(i) systematic sampling, [2]

(ii) quota sampling. [2]

6 A market stall sells rice in packets which have masses that are normally distributed. The stall owner claims that the mean mass of the packet of rice is at least 5 kg. Jane buys a random selection of 10 packets of rice from the stall. The 10 packets have masses, in kg, as follows:

4.9 4.7 5.1 4.8 4.5 5.3 5.0 4.8 4.6 5.2

Find unbiased estimates of the population mean and population variance of the mass of rice packets. **[2]**

A test, at $\lambda\%$ significance level, shows that there is insufficient evidence for Jane to doubt the stall owner. Find the set of possible values for λ. **[4]**

7 Simon owns a diecast car display case which has 4 shelves and 8 individual compartments on each shelf. Each of these compartments can only hold one diecast car. He arranged his collection of 32 different diecast cars in the display case.

Find the number of different selections that can be made by

(i) taking two cars, both from the same shelf, **[2]**

(ii) taking a total of six cars from the display case, **[1]**

(iii) taking a total of six cars from the display case with at least one from each shelf. **[3]**

8 **(a)** S and W are independent random variables with the distributions $\mathrm{N}(20, 25)$ and $\mathrm{N}(\mu, \sigma^2)$ respectively. It is known that $\mathrm{P}(W < 10) = \mathrm{P}(W > 13)$ and $\mathrm{P}(S > 2W) = 0.43$. Calculate the values of μ and σ correct to three significant figures. **[4]**

(b) A small hair salon has two hairstylists Joe and Joan attending to customers wanting an express haircut. For Joe, the time taken to attend to a customer follows a normal distribution with mean 10 minutes and standard deviation 42 seconds. For Joan, the time taken to attend to a customer follows a normal distribution with mean 10.2 minutes and standard deviation 45 seconds.

(i) Find the probability that among three randomly chosen customers attended to by Joe, one took more than 10.5 minutes while the other two each took less than 10 minutes. **[2]**

(ii) Joe and Joan each attended to two customers. Find the probability that the difference in the total time taken by Joe and Joan to attend to their two customers respectively is more than 3 minutes. State any assumption(s) that you have used in your calculation. **[4]**

[Turn over

9 The following table shows the marks (x) obtained in a mid-year examination and the marks (y) obtained in the year-end examination by a group of eight students. The year-end mark of the eighth student was accidentally deleted from the records after the marks were analyzed, and this is indicated by m below.

Mid-year mark (x)	70	31	68	73	46	78	79	55
Year-end mark (y)	80	39	70	80	48	94	98	m

It is given that the equation of the regression line of y on x is $y = 1.2x - 4$.
Show that $m = 59$. [2]

(i) Draw the scatter diagram for these values, labelling the axes clearly. Find the value of the product moment correlation coefficient between x and y. [2]

(ii) It is thought that a model of the form $\ln y = a + bx$ may also be a suitable fit to the data. Calculate least square estimates of a and b and find the value of the product moment correlation coefficient between x and $\ln y$. [3]

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of

$$y = 1.2x - 4 \text{ or } \ln y = a + bx$$

is the better model.

Hence, estimate the mark that a student who obtained a mark of 75 in the mid-year examination but was absent from the year-end examination would have obtained in the year-end examination. [3]

10 For events A and B, it is given that $P(A) = \frac{5}{8}$ and $P(B) = \frac{2}{3}$.

(i) Find the greatest and least possible values of $P(A \cap B)$. [2]

It is given in addition that $P(A' | B') = \frac{3}{8}$.

(ii) Find $P(A' \cap B')$. [1]

(iii) Find $P(A \cup B)$. [2]

(iv) Determine if A and B are independent events. [3]

(v) Given another event C such that $P(C) = \frac{3}{8}$, $P(A \cap C) = P(B \cap C) = \frac{1}{4}$ and $P(A \cup B \cup C) = \frac{11}{12}$, find $P(A \cap B \cap C)$. [2]

11 A chocolate shop puts gift vouchers at random into 7% of all their packets of mini chocolates produced. A customer must collect 3 vouchers to exchange for a gift.

(i) Adeline buys 8 packets of the mini chocolates. Find the probability that she gets exactly 2 gift vouchers. **[2]**

(ii) Aileen buys 31 packets of the mini chocolates. Find the probability that she is able to exchange for at least one gift. **[2]**

(iii) Angelina and Angeline buy 60 packets of the mini chocolates altogether. Use a suitable approximation to estimate the probability of them being able to exchange for exactly two gifts. **[4]**

(iv) Ashley buys n packets of the mini chocolates. Given that she already has 2 unused vouchers from her previous purchase, find the value of n forwhich the probability of her being able to exchange for exactly one gift is greatest. **[3]**

(v) The shopkeeper observes that the number of gifts exchanged in a day has a mean of 10 and variance of 25. Estimate the number of gifts the shop needs to stock if there is to be no more than a 5% chance of running out of gifts in a 40-day period. **[3]**

******* **End of Paper** *******

Prelim Paper 2 Solutions

Pure Mathematics [40 Marks]

1 (a) Use the method of mathematical induction to prove that $\sum_{r=1}^{n} 4r^3 = \left[n(n+1)\right]^2$. **[4]**

(b) It is given that $\mathrm{f}(r) = r^4 + 2r^3 + 2r^2 + r$.
Show that $\mathrm{f}(r) - \mathrm{f}(r-1) = ar^3 + 2r$, where a is a real constant to be determined. Hence find a formula for $\sum_{r=1}^{n} r(2r^2+1)$. **[4]**

Qn. [Marks]	Solution	Remarks
1(a) [4]	Let P_n be the statement $\sum_{r=1}^{n} 4r^3 = \left[n(n+1)\right]^2$ for $n \in \mathbb{Z}^+$. When $n=1$, LHS $=4$ and RHS$=\left[1\times 2\right]^2 = 4$ $\therefore P_1$ is true. Assume P_k is true for some $k \in \mathbb{Z}^+$. i.e. $\sum_{r=1}^{k} 4r^3 = \left[k(k+1)\right]^2$. To prove that P_{k+1} is true, i.e. $\sum_{r=1}^{k+1} 4r^3 = \left[(k+1)(k+2)\right]^2$. $\sum_{r=1}^{k+1} 4r^3 = \sum_{r=1}^{k} 4r^3 + 4(k+1)^3$ $= \left[k(k+1)\right]^2 + 4(k+1)^3$ $= (k+1)^2\left[k^2 + 4(k+1)\right]$ $= (k+1)^2(k^2+4k+4)$ $= (k+1)^2(k+2)^2$ $= \left[(k+1)(k+2)\right]^2$ $\therefore P_{k+1}$ is true. Hence P_k is true $\Rightarrow P_{k+1}$ is true, and since P_1 is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.	**Extra Practice: 9740/2013/P1/9** Define statement clearly. A proof by induction is an argument so it has to be presented carefully. Notation for the set of positive integers is $\mathbb{Z}^+$ whereas $\mathbb{R}$ refers to the set of real numbers. "$\Rightarrow$" is the symbol for implies that

(b) **[4]**	$f(r)-f(r-1)=(r^4+2r^3+2r^2+r)$ $-\left[(r-1)^4+2(r-1)^3+2(r-1)^2+(r-1)\right]$ $= r^4+2r^3+2r^2+r$ $-r^4+4r^3-6r^2+4r-1$ $-2r^3+6r^2-6r+2$ $-2r^2+4r-2$ $-r+1$ $=4r^3+2r,\ a=4$ $\sum_{r=1}^{n} r(2r^2+1)=\frac{1}{2}\sum_{r=1}^{n}(4r^3+2r)$ $=\frac{1}{2}\sum_{r=1}^{n}(\mathrm{f}(r)-\mathrm{f}(r-1))$ $=\frac{1}{2}\left[\begin{array}{l} \cancel{\mathrm{f}(1)}-\mathrm{f}(0) \\ +\cancel{\mathrm{f}(2)}-\cancel{\mathrm{f}(1)} \\ +\cancel{\mathrm{f}(3)}-\cancel{\mathrm{f}(2)} \\ \vdots \\ \vdots \\ +\cancel{\mathrm{f}(n-1)}-\cancel{\mathrm{f}(n-2)} \\ +\mathrm{f}(n)-\cancel{\mathrm{f}(n-1)} \end{array}\right]$ $=\frac{1}{2}(\mathrm{f}(n)-\mathrm{f}(0))$ $=\frac{n^4+2n^3+2n^2+n}{2}$ OR $\frac{n(n+1)(n^2+n+1)}{2}$	Required to "show" the result by Binomial expansion. It is not a valid approach to use the form ar^3+2r to find the value of "a" by substituting or comparing coefficients. Present Method of Difference clearly. Show cancellation.

2 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on OA, between O and A, such that $OC : CA = 3 : 2$. Point D lies on OB, between O and B, such that $OD : DB = 1 : \mu$.

(i) It is given that the area of triangle ABD is twice the area of triangle ABC. Find μ. [4]

(ii) Show that the vector equation of the line BC can be written as $\mathbf{r}=\frac{3}{5}s\,\mathbf{a}+(1-s)\mathbf{b}$, where s is a parameter. By writing down the vector equation of the line AD in a similar form, in terms of a parameter t, find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point E where the lines BC and AD meet. [3]

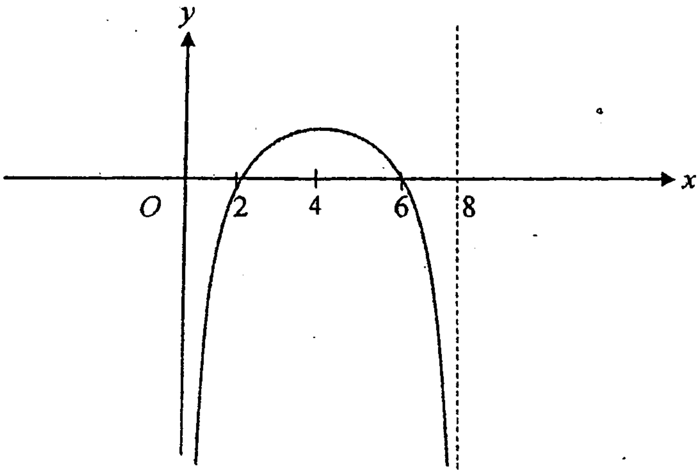
It is further given that the angle AOB is 45° and O lies on the perpendicular bisector of the line segment AB.

(iii) Find the length of projection of $\mathbf{a}$ on $\mathbf{b}$, giving your answer in terms of $\left|\mathbf{b}\right|$. Hence find the position vector of the foot of the perpendicular from A to OB. [3]

Qn. [Marks]	Solution	Remarks
2(i) [4]	Area of triangle $ABC = \frac{1}{2}\left\lvert\overrightarrow{CA}\times\overrightarrow{AB}\right\rvert = \frac{1}{2}\left\lvert\frac{2}{5}\mathbf{a}\times(\mathbf{b}-\mathbf{a})\right\rvert$ $=\frac{1}{5}\lvert\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{a}\rvert$ $=\frac{1}{5}\lvert\mathbf{a}\times\mathbf{b}\rvert$ Area of triangle $ABD = \frac{1}{2}\left\lvert\overrightarrow{DB}\times\overrightarrow{AB}\right\rvert$ $=\frac{1}{2}\left\lvert\frac{\mu}{1+\mu}\mathbf{b}\times(\mathbf{b}-\mathbf{a})\right\rvert$ $=\frac{\mu}{2(1+\mu)}\lvert\mathbf{b}\times\mathbf{b}-\mathbf{b}\times\mathbf{a}\rvert,\quad \mu>0$ $=\frac{\mu}{2(1+\mu)}\lvert\mathbf{a}\times\mathbf{b}\rvert$ Area of triangle $ABD = 2$ (area of triangle ABC) $\frac{\mu}{2(1+\mu)}\lvert\mathbf{a}\times\mathbf{b}\rvert=\frac{2}{5}\lvert\mathbf{a}\times\mathbf{b}\rvert$ $5\mu=4(1+\mu)$ $\mu=4$	**Extra Practice: 9740/2015/P1/7** Use proper vector "arrow" $\overrightarrow{AB}$ or "tilde" $\underset{\sim}{a}$ notation.
(ii) [3]	Line BC: $\mathbf{r}=\overrightarrow{OB}+s\overrightarrow{BC},\ s\in\mathbb{R}$ $=\mathbf{b}+s\left(\overrightarrow{OC}-\overrightarrow{OB}\right),\ s\in\mathbb{R}$ $=\mathbf{b}+s\left(\frac{3}{5}\mathbf{a}-\mathbf{b}\right),\ s\in\mathbb{R}$ $=\frac{3}{5}s\ \mathbf{a}+(1-s)\mathbf{b},\ s\in\mathbb{R}$ Line AD: $\mathbf{r}=\overrightarrow{OA}+t\overrightarrow{AD},\ t\in\mathbb{R}$ $=\mathbf{a}+t\left(\overrightarrow{OD}-\overrightarrow{OA}\right),\ t\in\mathbb{R}$ $=\mathbf{a}+t\left(\frac{1}{5}\mathbf{b}-\mathbf{a}\right),\ t\in\mathbb{R}$ $=\frac{1}{5}t\ \mathbf{b}+(1-t)\mathbf{a},\ t\in\mathbb{R}$ At point of intersection E, $\frac{3}{5}s\ \mathbf{a}+(1-s)\mathbf{b}=(1-t)\mathbf{a}+\frac{1}{5}t\ \mathbf{b}$ $\frac{3}{5}s=1-t\quad ...(1)$ $1-s=\frac{1}{5}t\quad ...(2)$	Expression of $\overrightarrow{OE}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is unique, so we can compare the coefficients of $\mathbf{a}$ and $\mathbf{b}$ to obtain simultaneous equations.

	Solving, $s=\frac{10}{11}$, $t=\frac{5}{11}$ and $\overline{OE}=\frac{3}{5}\left(\frac{10}{11}\right)\mathbf{a}+\left(1-\frac{10}{11}\right)\mathbf{b}=\frac{6}{11}\mathbf{a}+\frac{1}{11}\mathbf{b}$	
(iii) **[3]**	Length of projection of $\mathbf{a}$ on $\mathbf{b}$ = $\left\lvert \mathbf{a}\cdot\hat{\mathbf{b}} \right\rvert$ $=\lvert\mathbf{a}\rvert\lvert\hat{\mathbf{b}}\rvert\cos 45°$ $=\lvert\mathbf{a}\rvert\cos 45°$ $=\frac{1}{\sqrt{2}}\lvert\mathbf{b}\rvert$ $\overline{OF}=\left(\frac{1}{\sqrt{2}}\lvert\mathbf{b}\rvert\right)\hat{\mathbf{b}}=\frac{1}{\sqrt{2}}\mathbf{b}$	

3 The diagram shows the curve with equation $y=2+\frac{24}{x(x-8)}$, for $0\leq x\leq 8$. The curve crosses the x-axis at $x=2$ and $x=6$, has a maximum turning point at $x=4$, and asymptotes at $x=0$ and $x=8$.



For $0\leq x\leq 8$, the region bounded by the curves with equations $y=\left\lvert 2+\frac{24}{x(x-8)}\right\rvert$ and $y=2+\frac{4}{x-8}$ is denoted by S.

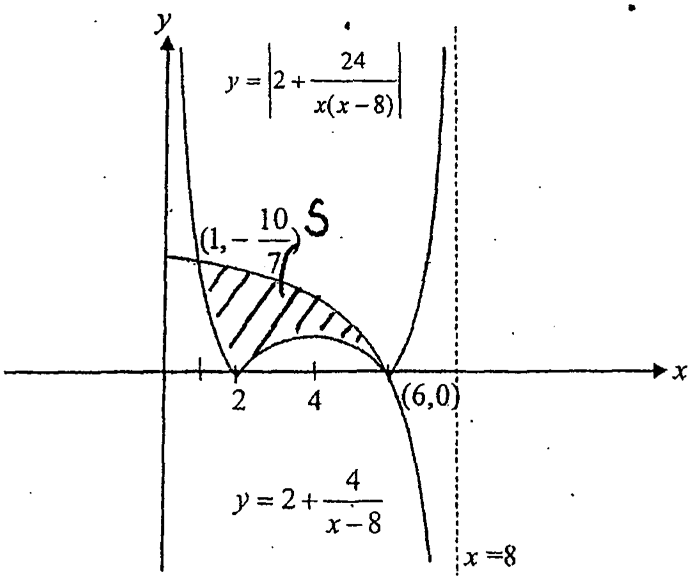
(i) On the same diagram, for $0\leq x\leq 8$, sketch the curves with equations

$$y=\left\lvert 2+\frac{24}{x(x-8)}\right\rvert \text{ and } y=2+\frac{4}{x-8}.$$

Indicate clearly on your diagram the region S, the coordinates of the points of intersection of the two curves and the equation(s) of any asymptote(s). [3]

(ii) Find the exact area of S.
Express your answer in the form $A+B\ln 2+C\ln 3+D\ln 7$ where A, B, C and D are constants to be determined. [5]

(iii) Find the volume of the solid of revolution formed when S is rotated about the x-axis through $360°$. [2]

Qn. [Marks]	Solution	Remarks
3(i) [3]	 	**Extra Practice:** **9740/2011/P1/Q5** Label the region S !
(ii) [5]	Area of S $=\int_1^2 \left(2+\frac{4}{x-8}\right)-\left[-\left(2+\frac{24}{x(x-8)}\right)\right]\,\mathrm{d}x$ $+\int_2^6 \left(2+\frac{4}{x-8}\right)-\left[2+\frac{24}{x(x-8)}\right]\,\mathrm{d}x$ $=\int_1^2 \left(2+\frac{4}{x-8}\right)-\left[-\left(2+\frac{3}{x-8}-\frac{3}{x}\right)\right]\,\mathrm{d}x$ $+\int_2^6 \left(2+\frac{4}{x-8}\right)-\left[2+\frac{3}{x-8}-\frac{3}{x}\right]\,\mathrm{d}x$ $=\int_1^2 \left(4+\frac{7}{x-8}-\frac{3}{x}\right)\,\mathrm{d}x+\int_2^6 \left[\frac{1}{x-8}+\frac{3}{x}\right]\,\mathrm{d}x$ $=\left[4x+7\ln\lvert x-8\rvert-3\ln\lvert x\rvert\right]_1^2+\left[\ln\lvert x-8\rvert+3\ln\lvert x\rvert\right]_2^6$ $=\left[4+7\ln 6-7\ln 7-3\ln 2\right]+\left[\ln 2-\ln 6+3\ln 6-3\ln 2\right]$ $=4+4\ln 2+9\ln 3-7\ln 7 \quad \therefore A=4,\ B=4,\ C=9,\ D=-7$	**Extra Practice** **9740/2014/P2/Q2** Include modulus sign!
(iii) [2]	Required Volume $=\pi\int_1^6 \left(2+\frac{4}{x-8}\right)^2-\left(2+\frac{24}{x(x-8)}\right)^2\,\mathrm{d}x$ $\approx 4.63989\pi = 14.577$ units^2 (5 s.f)=14.6 units^2 (3 s.f)	

4 (a) Solve the equation

$$z^4 + 4\mathrm{i} = 0,$$

giving the roots in the form $r\mathrm{e}^{\mathrm{i}\alpha}$, where $r > 0$ and $-\pi < \alpha \le \pi$. [3]

These roots correspond to four points on an Argand diagram. Identify the quadrilateral that has these four points as vertices. [1]

(b) The point A represents the fixed complex number a, which has modulus r and argument θ, where $0 < \theta < \frac{\pi}{2}$.

(i) On a single Argand diagram, sketch the loci

(a) $|z-a| = 2r$,

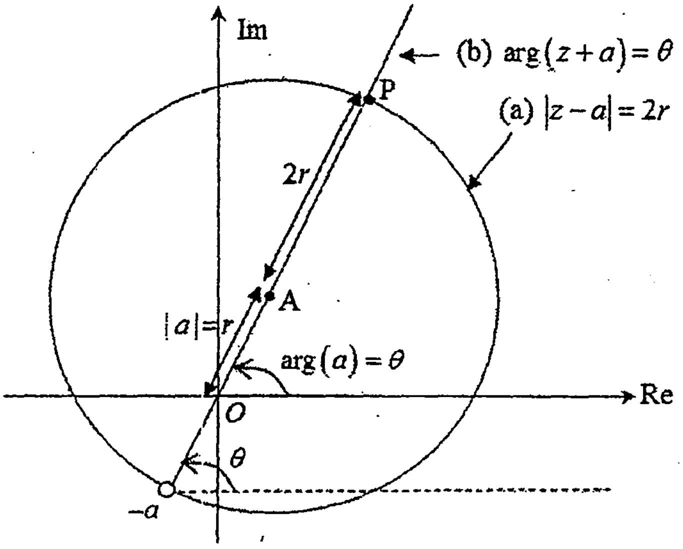
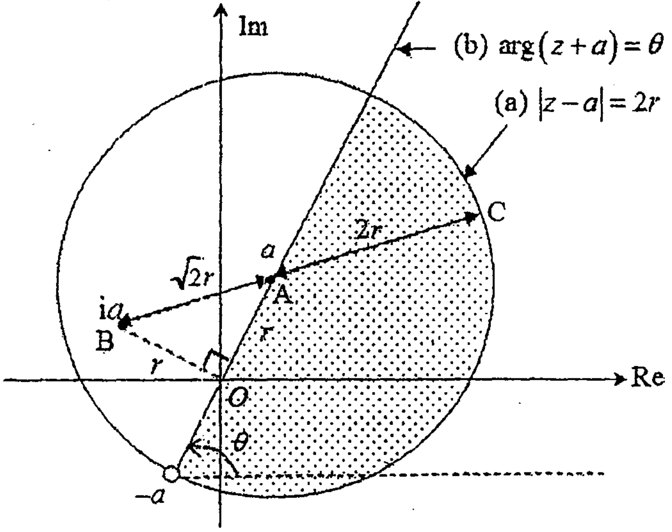
(b) $\arg(z+a) = \theta$,

making clear the relationship between the loci and the point A. [3]

(ii) Hence, or otherwise, find exactly the complex number(s) z that satisfy both equations in part (i), giving your answer(s) in terms of r and θ. [2]

(iii) Given instead that $|z-a| \le 2r$ and $-\pi < \arg(z+a) \le \theta$, find exactly the minimum and maximum possible values of $|z-\mathrm{i}a|$, giving your answers in terms of r. [3]

Qn. [Marks]	Solution	Remarks
4(a) [4]	$z^4 = -4\mathrm{i} = 4\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}} \times \mathrm{e}^{2k\pi\mathrm{i}}$ where $k \in \mathbb{Z}$ By De Moivre's Theorem, $z = \left[4\mathrm{e}^{\left(\frac{4k-1}{2}\right)\pi\mathrm{i}}\right]^{\frac{1}{4}}$ $= \sqrt{2}\mathrm{e}^{\left(\frac{4k-1}{8}\right)\pi\mathrm{i}}, \quad k = 0, \pm 1, 2$ The quadrilateral is a square.	**Extra Practice:** **9740/2009/P1/9** A sketch is not necessary as the n^{th} roots of a complex number form the vertices of a regular n-sided polygon. In this case, it is a square with diagonals $2\sqrt{2}$ and sides 2 units.

(b)(i) **[3]**		**Extra Practice:** **9740/2010/P1/8** Point A cannot be on the axes as $0<\theta<\frac{\pi}{2}$. Ensure that circle has radius twice the length of OA. $-a$ is in the 3rd quadrant (since $a=x+\mathrm{i}y \Rightarrow -a=-x-\mathrm{i}y$) and lies on the circumference of the circle since $\lvert -a\rvert=r$. Half-line passes through O and A (since OA makes an angle of θ with the positive real axis). Exclude starting point of half-line, i.e. open circle at $-a$.
(ii) **[2]**	At point of intersection of the loci, P, $z=3re^{i\theta}$ or $3r(\cos\theta+\mathrm{i}\sin\theta)$	only 1 point of intersection (at P) as $-a$ is excluded from the half-line. Length $OP=OA+AP=3r$. Note that OP makes an angle of θ with the positive real axis.
(iii) **[3]**	 Minimum $\lvert z-\mathrm{i}a\rvert=BO=r$ Maximum $\lvert z-\mathrm{i}a\rvert=BC=BA+AC=\sqrt{r^2+r^2}+2r=(\sqrt{2}+2)r$	Shaded region represents the given relations. Multiplying a by i is an anti-clockwise rotation of A by $\frac{\pi}{2}$ radians about O. In this case, shortest distance = perpendicular distance. Note that $\lvert \mathrm{i}a\rvert=r$. Point C is furthest away from B. Note that BC passes through the centre of the circle.

Statistics [60 Marks]

5 The recreational committee of a large company is organizing a family day and would like to conduct a survey with 5% of its employees about their preferences for an outdoor or indoor based carnival as well as the activities involved.

Describe how the committee could obtain a sample using

(i) systematic sampling, [2]

(ii) quota sampling. [2]

<table>
<tr><th>Qn.
[Marks]</th><th>Solution</th><th>Remarks</th></tr>
<tr><td>5(i)
[2]</td><td>Note: For the interval number, k
$k=\frac{\text{total population}}{\text{Sample Size}}=\frac{\text{total population}}{5\% \text{ of total population}}=\frac{1}{5\%}=\frac{1}{5/100}=20$
Use a computer to generate a random number from 1 to 20, say 5. Select the 5th employee to arrive at the office, thereafter every 20th employee that arrives, ie 5th, 25th, 45th, etc.</td><td>Extra Practice:
9740/2014/P2/5
9740/2015/P2/5</td></tr>
<tr><td>(ii)
[2]</td><td>Any reasonable choice of strata: Age/Gender/Marital status etc.
<table>
<tr><td>Strata</td><td>1</td><td>2</td><td>3</td><td>total</td></tr>
<tr><td>Sample Size (n)</td><td>x</td><td>y</td><td>z</td><td>x+y+z = 5% of population</td></tr>
</table>
Choose the individuals from each strata according to some convenient/non-random scheme such as the first x, y, z individuals committee encounters belong to strata 1,2,3 respectively such that sample size, $x+y+z$ is 5% of the population.
OR
Choose the first "n" employee arrived at the company on a particular morning from each stratum as shown in the table.</td><td>May be clearer to use a table with numerical values to illustrate</td></tr>
</table>

6 A market stall sells rice in packets which have masses that are normally distributed. The stall owner claims that the mean mass of the packet of rice is at least 5 kg. Jane buys a random selection of 10 packets of rice from the stall. The 10 packets have masses, in kg, as follows:

4.9 4.7 5.1 4.8 4.5 5.3 5.0 4.8 4.6 5.2

Find unbiased estimates of the population mean and population variance of the mass of rice packets. [2]

A test, at λ% significance level, shows that there is insufficient evidence for Jane to doubt the stall owner. Find the set of possible values for λ. [4]

Qn. [Marks]	Solution	Remarks
6 [2][4]	From GC, unbiased estimate of population mean, $\bar{x} = 4.89$ unbiased estimate of population variance, $s^2 = 0.2601281735^2 \left(or \ \dfrac{203}{3000} \right)$ $H_0: \mu = 5$ $\quad$ $H_1: \mu < 5$ Perform 1-tail **t-test** at $\lambda\%$ significance level Under H_0, Test statistic , $T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1)$ where $\mu_0 = 5$ and $n = 10$ For the sample, $\bar{x} = 4.89$, $s = 0.2601281735$. Using a t-test, p-value $= 0.1069800566$ Since there is **insufficient evidence** for Jane to doubt the stall owner, we **DO NOT** reject H_0, $p\text{-value} > \lambda\% = \dfrac{\lambda}{100}$ $\Rightarrow \lambda < 10.7$ (3sf)	**Extra Practice:** **9740/2015/P2/8** Hypothesis on μ. Use the correct symbols and notation. Write the correct test statistic.

7 Simon owns a diecast car display case which has 4 shelves and 8 individual compartments on each shelf. Each of these compartments can only hold one diecast car. He arranged his collection of 32 different diecast cars in the display case.

Find the number of different selections that can be made by

(i) taking two cars, both from the same shelf, [2]

(ii) taking a total of six cars from the display case, [1]

(iii) taking a total of six cars from the display case with at least one from each shelf. [3]

Qn. [Marks]	Solution	Remark
7(i) [2]	No. of selections = ${}^4C_1 \times {}^8C_2 = 112$ First select the shelf before selecting the car and apply the multiplication principle.	**Extra Practice** **9740/2009/P2/10** **9740/2014/P2/6**
(ii) [1]	No. of selections = ${}^{32}C_6 = 906192$	

(iii) **[3]**	Case 1: Choose 2 shelves from which we select 1 car. Remaining 2 shelves we choose 2 cars. (6 cars in total) Case 2: Choose 3 shelves to select 1 car. Remaining shelf choose 3 cars. (6 cars in total) No. of selections = ${}^4C_2\left({}^8C_1\right)^2\left({}^8C_2\right)^2+{}^4C_3\left({}^8C_1\right)^3\left({}^8C_3\right)$ $=301056+114688$ $=415744$ <u>Alternative Solutions</u> **(I) By complement** No. of selections if All from 1 shelf = ${}^8C_4\times 4=112$ All from 2 shelves = ${}^4C_2[\left({}^8C_3\right)^3+({}^8C_2\times{}^8C_4\times 2)+({}^8C_1\times{}^8C_5\times 2)]$ All from 3 shelves = ${}^4C_3[\left({}^8C_2\right)^3+(({}^8C_1)^2\times{}^8C_4\times 3)+({}^8C_1\times{}^8C_2\times{}^8C_3\times 3!)]$ Answer = $906192-112-47712-442624=415744$ **(II)** No. of selections if No cars are taken from 1 of the 4 shelves = ${}^4C_1\times{}^{24}C_6=538384$ No cars are taken from 2 of the 4 shelves = ${}^4C_2\times{}^{16}C_6=48048$ No cars are taken from 3 of the 4 shelves = 112 Answer = $906192-538384+48048-112=415744$	Be careful to consider the cases carefully and watch for double-counting.

8 (a) S and W are independent random variables with the distributions $\mathrm{N}(20, 25)$ and $\mathrm{N}(\mu, \sigma^2)$ respectively. It is known that $\mathrm{P}(W<10)=\mathrm{P}(W>13)$ and $\mathrm{P}(S>2W)=0.43$. Calculate the values of μ and σ correct to three significant figures. [4]

(b) A small hair salon has two hairstylists Joe and Joan attending to customers wanting an express haircut. For Joe, the time taken to attend to a customer follows a normal distribution with mean 10 minutes and standard deviation 42 seconds. For Joan, the time taken to attend to a customer follows a normal distribution with mean 10.2 minutes and standard deviation 45 seconds.

(i) Find the probability that among three randomly chosen customers attended to by Joe, one took more than 10.5 minutes while the other two each took less than 10 minutes. [2]

(ii) Joe and Joan each attended to two customers. Find the probability that the difference in the total time taken by Joe and Joan to attend to their two customers respectively is more than 3 minutes. State any assumption(s) that you have used in your calculation. [4]

Qn. [Marks]	**Solution**	**Remark**
8(a) **[4]**	Since $P(W < 10) = P(W > 13)$, $\mu = \frac{10+13}{2} = 11.5$. $2W - S \sim N(3,\ 25 + 4\sigma^2)$ $P(S > 2W) = 0.43$ $P(2W - S < 0) = 0.43$ $P(Z < \frac{0 - -3}{\sqrt{25+4\sigma^2}}) = 0.43$ From GC, $\frac{-3}{\sqrt{25+4\sigma^2}} = -0.176374$ Solving, $\sigma \approx 8.13$.	**Extra Practice: 9740/2008/P2/11** Normal distribution is symmetric about μ
8(b) **(i)** **[2]**	Let X be the time taken by Joe to attend to a customer. $X \sim N(10, 0.7^2)$ Required probability = ${}^3C_2 \times [P(X < 10)]^2\, P(X > 10.5) = 0.178$	Note the difference in units for mean (in mins) and standard deviation (in secs)
(ii) **[4]**	Let Y be the time taken by Joan to attend to a customer. $Y \sim N(10.2, 0.75^2)$ $(X_1 + X_2) - (Y_1 + Y_2) \sim N(-0.4, 2.105)$ $P(\lvert (X_1 + X_2) - (Y_1 + Y_2) \rvert > 3)$ $= 1 - P(-3 < (X_1 + X_2) - (Y_1 + Y_2) < 3)$ $= 0.0461$ (i) The time taken by Joe to attend to a customer is independent of the time taken by Joan to attend to a customer. (ii) The time taken by Joe (Joan) to attend to one customer is independent of the time taken by Joe (Joan) to attend to another customer.	X and Y are independent X_1 and X_2 are independent AND Y_1 and Y_2 are independent

9 The following table shows the marks (x) obtained in a mid-year examination and the marks (y) obtained in the year-end examination by a group of eight students. The year-end mark of the eighth student was accidentally deleted from the records after the marks were analyzed, and this is indicated by m below.

Mid-year mark (x)	70	31	68	73	46	78	79	55
Year-end mark (y)	80	39	70	80	48	94	98	m

It is given that the equation of the regression line of y on x is $y = 1.2x - 4$. Show that $m = 59$. [2]

(i) Draw the scatter diagram for these values, labelling the axes clearly. Find the value of the product moment correlation coefficient between x and y. [2]

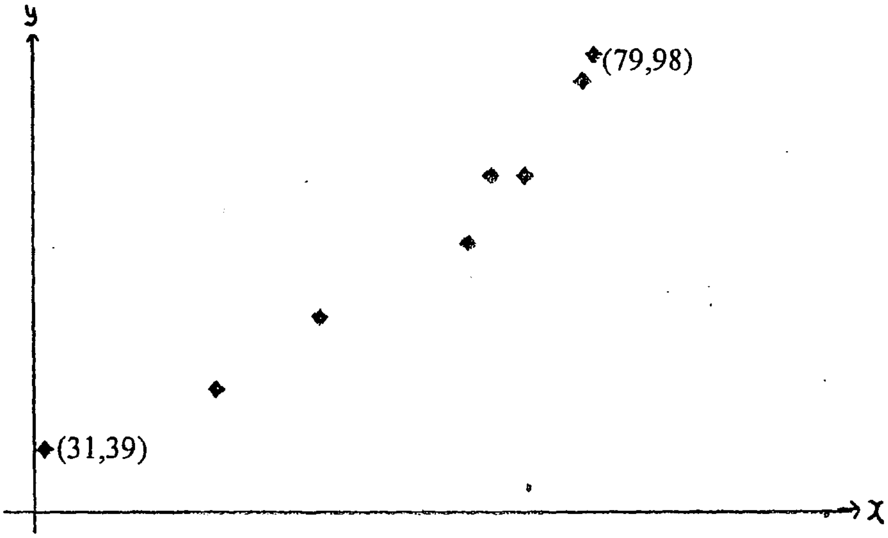
(ii) It is thought that a model of the form $\ln y = a + bx$ may also be a suitable fit to the data. Calculate least square estimates of a and b and find the value of the product moment correlation coefficient between x and $\ln y$. [3]

(iii) Use your answers to parts (i) and (ii) to explain which of

$$y = 1.2x - 4 \text{ or } \ln y = a + bx$$

is the better model.

Hence, estimate the mark that a student who obtained a mark of 75 in the mid-year examination but was absent from the year-end examination would have obtained in the year-end examination. [3]

Qn. [Marks]	**Solution**	**Remarks**
9 **[2]**	$\sum x = 500, \ \sum y = 509 + m, \ n = 8$ Since $(\bar{x}, \bar{y})$ lies on the regression line of y on x, $\bar{y} = 1.2\bar{x} - 4$ $\frac{509+m}{8} = 1.2\left(\frac{500}{8}\right) - 4$ $m = 59$	**Extra Practice: 9740/2015/P2/10**
(i) **[2]**	 **From GC,** $r = 0.96826 \approx 0.968$	Label axes. Label range of data. Note two data points with the same y value. Vertical and horizontal differences appear relatively correct trend of the data appears exponential
(ii) **[3]**	**From GC,** $\ln y = 3.0351 + 0.018945x$ $a = 3.0351 \approx 3.04$ $b = 0.018945 \approx 0.0189$ $r = 0.990499 \approx 0.990$	Do not "truncate", always "Round off"

(iii) [1]	The scatter plot of x and y shows a non-linear relationship as when x increases, y appears to be increasing at an increasing rate. Since the product moment correlation coefficient **between x and $\ln y$** of 0.990 is closer to 1 than the product moment correlation coefficient between x and y of 0.968, $\ln y = a + bx$ is the better model.	Proper phrasing!
[2]	When $x = 75$, $\ln y = 3.0351 + 0.018945(75) = 4.455975$ $y = 86.140$ The student would have obtained 86.1 marks in the year-end examination.	Accuracy!

10 For events A and B, it is given that $\text{P}(A) = \frac{5}{8}$ and $\text{P}(B) = \frac{2}{3}$.

(i) Find the greatest and least possible values of $\text{P}(A \cap B)$. [2]

It is given in addition that $\text{P}(A' \mid B') = \frac{3}{8}$.

(ii) Find $\text{P}(A' \cap B')$. [1]

(iii) Find $\text{P}(A \cup B)$. [2]

(iv) Determine if A and B are independent events. [3]

(v) Given another event C such that $\text{P}(C) = \frac{3}{8}$, $\text{P}(A \cap C) = \text{P}(B \cap C) = \frac{1}{4}$ and $\text{P}(A \cup B \cup C) = \frac{11}{12}$, find $\text{P}(A \cap B \cap C)$. [2]

Qn. [Marks]	Solution	Remarks
10(i) [2]	$\text{P}(A) + \text{P}(B) - 1 \le \text{P}(A \cap B) \le \min\{\text{P}(A), \text{P}(B)\}$ $\frac{7}{24} \le \text{P}(A \cap B) \le \frac{5}{8}$ Greatest value of $\text{P}(A \cap B)$ is $\frac{5}{8}$. Least value of $\text{P}(A \cap B)$ is $\frac{7}{24}$.	**Extra Practice:** **9740/2015/P2/9** **9740/2010/P2/7** Draw a Venn Diagram
(ii) [1]	$\text{P}(A' \cap B') = \text{P}(A' \mid B')\text{P}(B') = \frac{3}{8} \times \frac{1}{3} = \frac{1}{8}$	
(iii) [2]	$\text{P}(A \cup B) = 1 - \text{P}(A' \cap B') = \frac{7}{8}$	

(iv) **[3]**	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $\frac{7}{8} = \frac{5}{8} + \frac{2}{3} - P(A \cap B)$ $P(A \cap B) = \frac{5}{12}$ $= \frac{5}{8} \times \frac{2}{3} = P(A)P(B)$ Hence, A and B are independent events. *OR* $\because P(A' \mid B') = \frac{3}{8} = 1 - \frac{5}{8} = P(A')$ A' and B' are independent events $\Rightarrow A$ and B are independent events	Present argument clearly to show A and B are independent events.
(v) **[2]**	$P(A \cup B \cup C) = P(A) + P(B) + P(C)$ $-P(A \cap B) - P(A \cap C) - P(B \cap C)$ $+P(A \cap B \cap C)$ $\frac{11}{12} = \frac{5}{8} + \frac{2}{3} + \frac{3}{8} - \frac{5}{12} - \frac{1}{4} - \frac{1}{4} + P(A \cap B \cap C)$ $P(A \cap B \cap C) = \frac{1}{6}$	Alternatively use a Venn diagram.

11 A chocolate shop puts gift vouchers at random into 7% of all their packets of mini chocolates produced. A customer must collect 3 vouchers to exchange for a gift.

(i) Adeline buys 8 packets of the mini chocolates. Find the probability that she gets exactly 2 gift vouchers. [2]

(ii) Aileen buys 31 packets of the mini chocolates. Find the probability that she is able to exchange for at least one gift. [2]

(iii) Angelina and Angeline buy 60 packets of the mini chocolates altogether. Use a suitable approximation to estimate the probability of them being able to exchange for exactly two gifts. [4]

(iv) Ashley buys n packets of the mini chocolates. Given that she already has 2 unused vouchers from her previous purchase, find the value of n for which the probability of her being able to exchange for exactly one gift is greatest. [3]

(v) The shopkeeper observes that the number of gifts exchanged in a day has a mean of 10 and variance of 25. Estimate the number of gifts the shop needs to stock if there is to be no more than a 5% chance of running out of gifts in a 40-day period. [3]

Qn. [Marks]	Solution	Remarks
11 (i) [2]	Let X denote the number of gift vouchers obtained (out of 8). $X \sim \mathrm{B}(8, 0.07)$ $\mathrm{P}(X=2) = 0.0888$ (3 sf)	**Extra Practice 9740/2013/P2/7** Define all random variables clearly !
(ii) [2]	Let Y denote the number of gift vouchers obtained (out of 31). $Y \sim \mathrm{B}(31, 0.07)$ $\mathrm{P}(Y \geq 3) = 1 - \mathrm{P}(Y \leq 2) = 0.371$ (3 sf)	
(iii) [4]	Let V denote the number of gift vouchers obtained (out of 60). $V \sim \mathrm{B}(60, 0.07)$ Since $n = 60$ is large, $p = 0.07$ is small such that $np = 4.2 < 5$, $V \sim \mathrm{Po}(4.2)$ approximately $\mathrm{P}(6 \leq V \leq 8) = \mathrm{P}(V \leq 8) - \mathrm{P}(V \leq 5) = 0.219$ (3 sf)	State Poisson approximation conditions explicitly. 6, 7 or 8 vouchers can all lead to the desired scenario.
(iv) [3]	Let W denote the number of gift vouchers obtained (out of n). $W \sim \mathrm{B}(n, 0.07)$ For the condition of the question to be satisfied, the probability is $p = \mathrm{P}(1 \leq W \leq 3)$ From GC, when $n = 25, p = 0.74343$ when $n = 26, p = 0.74365$ when $n = 27, p = 0.74250$ $\therefore$ Value of n for the probability p to be maximum = 26.	To obtain exactly 1 gift, Ashley can have 1, 2 or 3 vouchers
(v) [3]	Let T denote the number of gifts exchanged in a day. $E(T) = 10, \quad Var(T) = 25$ Let $T_1, T_2, \ldots, T_{40}$ be a random sample from distribution X $n = 40$ large, by the Central Limit Theorem, $T_1 + T_2 + \cdots + T_{40} \sim N(40(10), 40(25))$ approximately $\mathrm{P}(T_1 + T_2 + \cdots + T_{40} > \mathrm{k}) \leq 0.05$ From GC, $\mathrm{k} \geq 452.01$ Least $\mathrm{k} = 453$. The shop needs to stock at least 453 gifts.	Distribution of T is not given here, and CANNOT be assumed to follow a normal distribution, otherwise we have positive probability of P(T<0).