1. (i) The first three terms of a sequence are given by $u_1 = -4.4$, $u_2 = -4.1$ and $u_3 = -2.6$. Given that u_n is a quadratic polynomial in n, find u_n in terms of n. [3]

(ii) Find the set of values of n for which u_n is more than 15. [2]

2. Let $\mathrm{f}(x) = \dfrac{3x}{(x+4)(x+1)}$.

(i) Express f(x) in partial fractions. [1]

(ii) Find the series expansion, in ascending powers of x, of f(x) up to and including the term x^3. State the values of x for which this expansion is valid. [3]

(iii) The first two terms of the expansion of f(x) are identical to that of the expansion of $ax(1-bx)^{-\frac{1}{2}}$. Find the exact values of a and b. [2]

3. (a) Differentiate e^{-x^2} with respect to x. Hence find $\int x^3 \mathrm{e}^{-x^2}\,\mathrm{d}x$. [4]

(b) Find the exact value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \left|\sin x\right| \mathrm{d}x$. [3]

4. On the first of January 2011, John opened a new special savings account and deposited $\$x$ into his account. On the first day of each subsequent month, he would deposit another $\$x$ into the account. A compound interest of 0.3% per month would be paid at the end of each month.

   (i) Given that $x = 600$, find the total amount of money in the savings account on first of January 2013 just before John deposited money into his account. [3]

   On first January 2013, instead of going to the bank to make a deposit into his savings account, John went to a yatch club and decided to purchase a yatch. For the yatch that he was interested to buy, he had to make a down payment of $23000.

   (ii) What should his minimum monthly deposit be (in multiples of $10) so that his savings account would have sufficient amount for him to make the down payment? [2]

   (iii) Assume that upon making the withdrawal for the down payment, there was $190 left in his savings account. John transferred this amount from the savings account to a checking account which charges no interest. With the $190 as the initial savings, he decided to make a monthly contribution of 90% of the amount he contributed in the previous month at the start of every subsequent month to the checking account. He claimed that he would be able to save $2000 eventually. Do you agree with him? Explain. [3]

5. With reference to the origin *O*, the points *A* and *B* have position vectors **a** and**b** respectively and **b** is a unit vector.

   (i) Give a geometric description of $\left|\mathbf{b}\square\mathbf{a}\right|$. [1]

   (ii) Given that *S* denotes the area of triangle *OAB* , show that $4S^2 = \left|\mathbf{a}\right|^2 - \left(\mathbf{a}\square\mathbf{b}\right)^2$. [3]

   (iii) Given that *C* is a point on the line *AB* with position vector **c**, explain why $(\mathbf{c}-\mathbf{a})\times(\mathbf{c}-\mathbf{b}) = \mathbf{0}$. [2]

   (iv) Suppose **a** is perpendicular to **b** and that the angle between **c** and**b**is $30°$,find in terms of $|\mathbf{a}|$ and $|\mathbf{c}|$, the ratio of the area of triangle *OAB* to that of triangle *OCB*. [3]

6. **(a)** The complex numbers z and w are such that $z=\dfrac{3a-5\text{i}}{1+2\text{i}}$ and $w=1+13b\text{i}$, where a and b are real numbers. Given that $z^*=w$, find the exact values of a and b. [4]

(b) Without using a graphic calculator, find the modulus and argument of the complex number $z=\dfrac{(1-\text{i})^2}{(-1+\sqrt{3}\text{i})^4}$, giving your answers in exact form. Hence evaluate z^6 exactly. [5]

7. A sequence u_1, u_2, u_3, ... is such that $u_0=1$ and $u_{n+1}=u_n-\dfrac{2}{(2n+1)(2n+3)}$, for $n\geq 0$.

(i) Prove by induction that $u_n=\dfrac{1}{2n+1}$. [4]

(ii) Find $\displaystyle\sum_{n=1}^{N}\frac{1}{(2n+1)(2n+3)}$ in terms of N. [3]

(iii) Hence find $\displaystyle\sum_{n=0}^{N}\frac{1}{(2n+3)(2n+5)}$. [2]

(iv) State the value of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)}$. [1]

8. In a certain fish farm, the growth of the population of garoupa is studied. The population of garoupa at time $t$days is denoted by x (in thousands). It was found that the rate of birth per day is twice of x, and the rate of death per day is proportional to x^2.

(i) Given that there is no change in the population of garoupa when its population hits 10000, write down a differential equation relating $\frac{dx}{dt}$ and x. [2]

(ii) Its owner decides to sell away 1800 garoupa daily. Modify the differential equation in part **(i)** and show that the resulting differential equation can be written as $\frac{dx}{dt} = -\frac{1}{5}\left[(x-5)^2 - a^2\right]$, where a is a constant to be determined. [2]

Given that the initial population of garoupa is 13000, solve this modified differential equation, expressing x in terms of t. [4]

Deduce the long term implication on the population of garoupa in the farm, and sketch the curve of x against t. [2]

9. The parametric equations of a curve C,are

$$x = 4\cos\theta,\ y = \sin 2\theta, \text{ where } 0 \le \theta \le \pi.$$

(i) Sketch C, stating the exact coordinates of any points of intersection with the axes. [2]

(ii) The point P on the curve has parameter p. Show that the equation of the tangent at $P$is $y = -\left(\frac{\cos 2p}{2\sin p}\right)x + 2\cot p \cos 2p + \sin 2p$. [4]

(iii) The region S is the area bounded by C, the tangent at $\theta = \frac{\pi}{2}$ and the line $x = 1$.

Show that the area of S can be expressed in the form $a + b\int_c^d \sin^2\theta \cos\theta \, d\theta$,

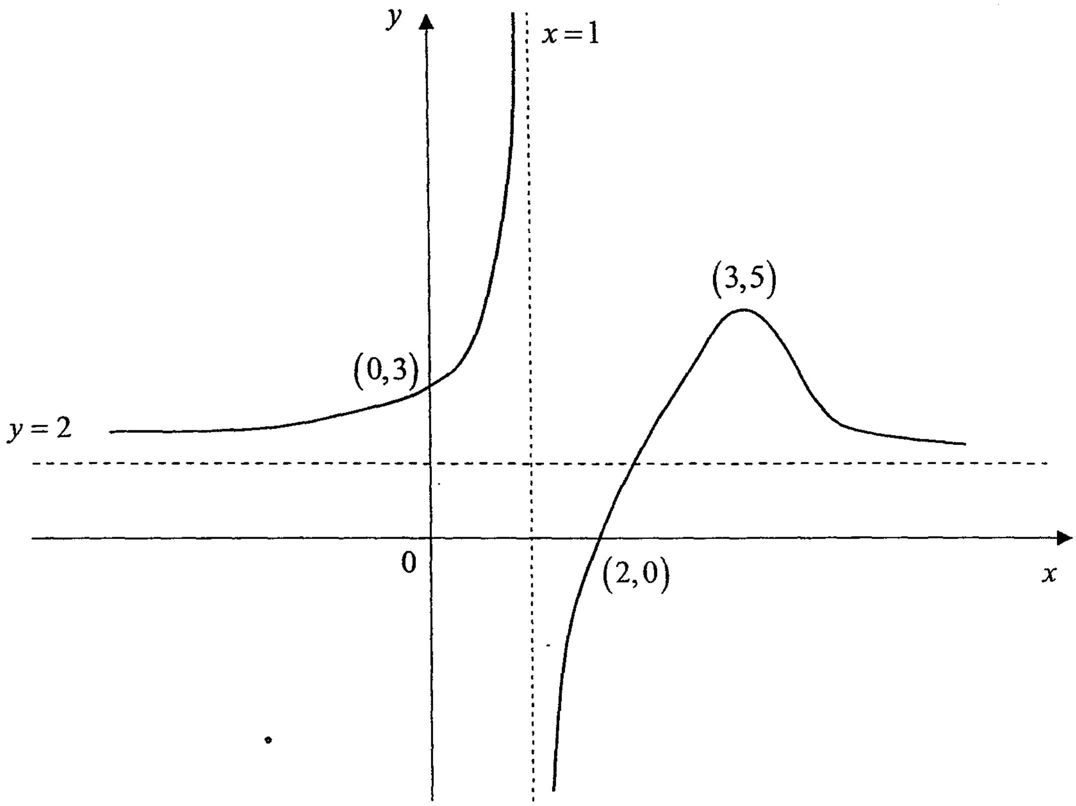
where a, b, c and d are exact constants to be determined. Hence evaluate the area of S numerically. [4]

(iv) C is transformed to give the curve C' given by

$$x = 2\cos\theta,\ y = \sin 2\theta - 1, \text{ where } 0 \le \theta \le \pi.$$

Describe the sequence of transformations from $C$to C'. [2]

10. The diagram shows the graph of $y = 2\mathrm{g}(x+1)$. The graph intersects the axes at $(0,3)$ and $(2,0)$ and has a turning point at $(3, 5)$. The asymptotes of the graph are at $x=1$ and $y=2$.



On separate diagrams, sketch the following graphs indicating the points corresponding to the axial intercepts, turning point and asymptotes where necessary.

(i) $y = \mathrm{g}(x)$ [3]

(ii) $y = \dfrac{1}{2\mathrm{g}(x+1)}$ [3]

(iii) $y = -\sqrt{2\mathrm{g}(x+1)}$ [3]

Suppose the above diagram shows the graph for $y = \mathrm{h}(x)$. Sketch the graph of $y = \mathrm{h}'(x)$. [3]

11. The function f is defined by

$$\mathrm{f}: x \mapsto -2x^2 + 12x - 19,\ x \in \mathbb{R}.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$, indicating clearly the coordinates of any turning point(s) and axial intercept(s). [1]

(ii) Give a reason why f does not have an inverse. [1]

(iii) If the domain of f is further restricted to $x \le k$, state the greatest value of k for which the function f^{-1} exists. [1]

(iv) Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. [3]

For the rest of the question, the domain of f is as found in part **(iii)**.

(v) On the same diagram, sketch the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$, showing clearly the geometrical relationship among the graphs. [3]

The function g is defined by

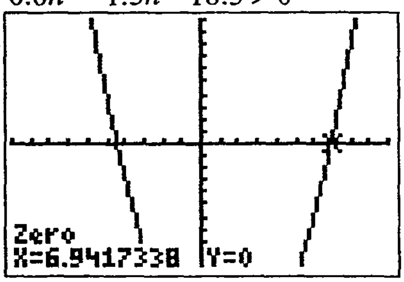
$$\mathrm{g}: x \mapsto \mathrm{e}^{x^2-3},\ x \in \mathbb{R}.$$

(vi) Explain why the composite function gf^{-1} exists. [1]

(vii) Find the range of gf^{-1}, giving your answer in the exact form. [2]

END OF PAPER

Solutions to 2016 Y6 H2 Maths Preliminary Exam II (Paper 1)

Question 1 [5 Marks]		
(i)	Let $u_n = an^2 + bn + c$ $u_1 = -4.4$ $a + b + c = -4.4$ -------------- (1) $u_2 = -4.1$ $4a + 2b + c = -4.1$ ----------- (2) $u_3 = -2.6$ $9a + 3b + c = -2.6$ ----------- (3) Solving (1), (2) and (3) $a = 0.6$, $b = -1.5$ and $c = -3.5$ $u_n = 0.6n^2 - 1.5n - 3.5$	
(ii)	$u_n > 15$ $0.6n^2 - 1.5n - 3.5 > 15$ $0.6n^2 - 1.5n - 18.5 > 0$  Since n is a positive integer, the set of values of n is $n \geq 7$.	

Question 2 [6 Marks]		
(i)	$\mathrm{f}(x) = \dfrac{3x}{(x+4)(x+1)}$ $= \dfrac{A}{(x+4)} + \dfrac{B}{(x+1)}$ $3x = A(x+1) + B(x+4)$ When $x = -1$, $-3 = 3B \Rightarrow B = -1$ When $x = -4$, $-12 = -3A \Rightarrow A = 4$ $\mathrm{f}(x) = \dfrac{4}{(x+4)} - \dfrac{1}{(x+1)}$	
(ii)	$\mathrm{f}(x) = 4(4+x)^{-1} - (1+x)^{-1}$ $= 4(4+x)^{-1} - (1+x)^{-1}$ $= 4\left[4^{-1}\left(1+\dfrac{x}{4}\right)^{-1}\right] - (1+x)^{-1}$ $= 1 + (-1)\left(\dfrac{x}{4}\right) + \dfrac{(-1)(-2)}{2!}\left(\dfrac{x}{4}\right)^2 + \dfrac{(-1)(-2)(-3)}{3!}\left(\dfrac{x}{4}\right)^3 + \ldots$ $-\left[1 + (-1)(x) + \dfrac{(-1)(-2)}{2!}x^2 + \dfrac{(-1)(-2)(-3)}{3!}x^3 + \ldots\right]$	

	$=1-\frac{x}{4}+\frac{x^2}{16}-\frac{x^3}{64}-1+x-x^2+x^3+...$ $=\frac{3}{4}x-\frac{15}{16}x^2+\frac{63}{64}x^3+...$ Range of validity for $\left(1+\frac{x}{4}\right)^{-1}$ is $-4<x<4$ Range of validity for $(1+x)^{-1}$ is $-1<x<1$ Therefore, overall range of validity is $-1<x<1$.	
(iii)	$ax(1-bx)^{-\frac{1}{2}}$ $=ax\left[1+\left(-\frac{1}{2}\right)(-bx)+...\right]$ $=ax+\frac{ab}{2}x^2+...$ Comparing the coefficient of x terms: $a=\frac{3}{4}$ Comparing the coefficients of x^2 terms: $\frac{ab}{2}=-\frac{15}{16} \quad \Rightarrow \quad b=-\frac{5}{2}$	

Question 3 [7 Marks]		
(a)	$\frac{d}{dx}e^{-x^2}=-2xe^{-x^2}$ $\int x^3e^{-x^2}dx$ $=\int\left(-\frac{1}{2}x^2\right)(-2xe^{-x^2})dx$ $=-\frac{1}{2}x^2e^{-x^2}+\int xe^{-x^2}dx$ $=-\frac{1}{2}x^2e^{-x^2}-\frac{1}{2}\int(-2x)e^{-x^2}dx$ $=-\frac{1}{2}x^2e^{-x^2}-\frac{1}{2}e^{-x^2}+c$	
(b)	$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}}\sin x\lvert\sin x\rvert dx$ $=\int_{-\frac{\pi}{4}}^{0}\sin x(-\sin x)dx+\int_0^{\frac{\pi}{2}}\sin x(\sin x)dx$ $=-\int_{-\frac{\pi}{4}}^{0}\sin^2x\,dx+\int_0^{\frac{\pi}{2}}\sin^2x\,dx$ $=-\int_{-\frac{\pi}{4}}^{0}\frac{1-\cos 2x}{2}dx+\int_0^{\frac{\pi}{2}}\frac{1-\cos 2x}{2}dx$	

	$= -\left[\frac{1}{2}x - \frac{\sin 2x}{4}\right]_{-\frac{\pi}{4}}^{0} + \left[\frac{1}{2}x - \frac{\sin 2x}{4}\right]_{0}^{\frac{\pi}{2}}$ $= -\left[(0) - \left(-\frac{\pi}{8} + \frac{1}{4}\right)\right] + \left[\left(\frac{\pi}{4}\right) - (0)\right]$ $= \frac{\pi}{8} + \frac{1}{4}$ $= \frac{\pi + 2}{8}$	

Question 4 [8 Marks]

(i)

Mth	Start of Month	End of Month
1	600	1.003×600
2	$1.003 \times 600 + 600$	$(1.003 \times 600 + 600) \times 1.003$ $= 1.003^2 \times 600 + 1.003 \times 600$
3	$1.003^2 \times 600 + 1.003 \times 600 + 600$	$1.003^3 \times 600 + 1.003^2 \times 600$ $+ 1.003 \times 600$
n		$1.003^n \times 600 + 1.003^{n-1} \times 600$ $+ \cdots + 1.003 \times 600$

At the end of December 2012, there are 24 months.

Amount in the account at the end of December 2012 after interest paid

$= 1.003^{24} \times 600 + 1.003^{23} \times 600 + \cdots + 1.003 \times 600$

$= 600\left(1.003 + 1.003^2 + \cdots 1.003^{24}\right)$

$= 600\left[\frac{1.003\left(1.003^{24} - 1\right)}{1.003 - 1}\right]$

$= 14952.62754$

$\approx \$14952.63$

(ii) Let the monthly contribution be $\$x$.

$x\left[\frac{1.003\left(1.003^{24} - 1\right)}{1.003 - 1}\right] \geq 23000$

$x \geq 922.9147161$

Since his monthly contributions are in multiple of \$10, he should deposit \$930 monthly into his savings account.

(iii)

Months	Amount contributed ($)
1	190
2	190×0.9
3	190×0.9^2
n	$190\times0.9^{n-1}$

Sum of the savings eventually

$=190\left(1+0.9+0.9^2+0.9^3+\cdots+0.9^{n-1}+\cdots\right)$

$=190\left(\dfrac{1}{1-0.9}\right)$

$=\$1900$, which is less than \$2000

Hence, John would not be able to save \$2000 in the long run.

Question 5 [9 Marks]

(i) $|\mathbf{b}\square\mathbf{a}|=|\mathbf{a}\square\mathbf{b}|$, where **b** is a unit vector, which gives the length of projection of the vector $\overrightarrow{\mathbf{OA}}$ onto the line OB (or on vector $\overrightarrow{\mathbf{OB}}$).

(ii)

$S=\dfrac{1}{2}|\mathbf{a}\times\mathbf{b}|$

$2S=|\mathbf{a}||\mathbf{b}|\sin\theta$

$4S^2=|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta$

$4S^2=|\mathbf{a}|^2|\mathbf{b}|^2\left(1-\cos^2\theta\right)$

$=|\mathbf{a}|^2|\mathbf{b}|^2\left(1-\left(\dfrac{(\mathbf{a}\square\mathbf{b})}{|\mathbf{a}||\mathbf{b}|}\right)^2\right)$

$=|\mathbf{a}|^2|\mathbf{b}|^2-(\mathbf{a}\square\mathbf{b})^2$

$=|\mathbf{a}|^2-(\mathbf{a}\square\mathbf{b})^2$ **since** $|\mathbf{b}|=1$

(iii) As C lies on AB, A, C and B are collinear.

$\Rightarrow \overrightarrow{AC}$ and $\overrightarrow{BC}$ are parallel

$\therefore (\mathbf{c}-\mathbf{a})\times(\mathbf{c}-\mathbf{b})=\overrightarrow{AC}\times\overrightarrow{BC}$

$=\left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{BC}\right|\sin 0\ \hat{\mathbf{n}}$

$=\mathbf{0}$, since $\sin 0=0$

(iv)	**a** is perpendicular to **b** $\Rightarrow \mathbf{a}\cdot\mathbf{b}=0$ $\therefore S=\frac{\lvert\mathbf{a}\rvert}{2}$ Area of triangle OCB, S_1 $=\frac{1}{2}\lvert\mathbf{c}\times\mathbf{b}\rvert$ $=\frac{1}{2}\lvert\mathbf{c}\rvert\lvert\mathbf{b}\rvert\sin(30^\circ)$ $=\frac{1}{4}\lvert\mathbf{c}\rvert$, since $\lvert\mathbf{b}\rvert=1$ Hence, $S:S_1=\frac{\lvert\mathbf{a}\rvert}{2}:\frac{\lvert\mathbf{c}\rvert}{4}=2\lvert\mathbf{a}\rvert:\lvert\mathbf{c}\rvert$	

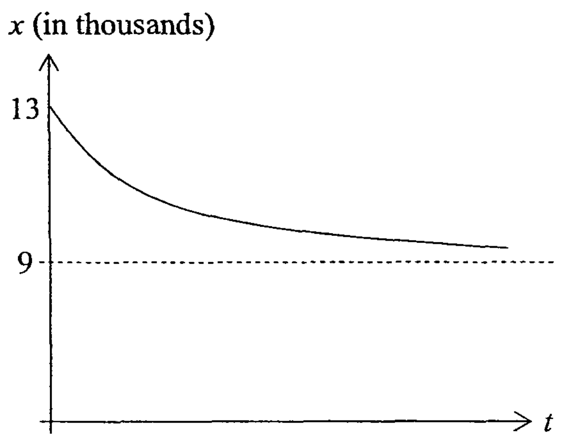
Question 6 [9 Marks]		
(a)	$z=\frac{3a-5\text{i}}{1+2\text{i}}$ $=\frac{3a-5\text{i}}{1+2\text{i}}\times\frac{1-2\text{i}}{1-2\text{i}}$ $=\frac{3a-10-(6a+5)\text{i}}{5}$ Therefore, $z^*=\frac{3a-10+(6a+5)\text{i}}{5}$ $z^*=w$ $\frac{3a-10+(6a+5)\text{i}}{5}=1+13b\text{i}$ Comparing real parts: $\frac{3a-10}{5}=1 \quad\Rightarrow\quad a=5$ Comparing imaginary parts: $\frac{6a+5}{5}=13b \quad\Rightarrow\quad b=\frac{7}{13}$	
(b)	$\lvert z\rvert=\frac{\lvert 1-\text{i}\rvert^2}{\lvert -1+\sqrt{3}\text{i}\rvert^4}$ $=\frac{\sqrt{2}^2}{2^4}=\frac{1}{8}$ $\arg(z)=\arg\left(\frac{(1-\text{i})^2}{(-1+\sqrt{3}\text{i})^4}\right)$ $=2\arg(1-\text{i})-4\arg(-1+\sqrt{3}\text{i})$ $=2\left(-\frac{\pi}{4}\right)-4\left(\frac{2\pi}{3}\right)$ $=-\frac{\pi}{2}-\frac{8\pi}{3}=-\frac{19\pi}{6}=\frac{5\pi}{6}$ (principal range)	

	$\therefore z = \frac{1}{8}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$ $\Rightarrow z^6 = \frac{1}{8^6}\left[\cos(5\pi) + i\sin(5\pi)\right]$ $= \frac{1}{8^6}[-1+0]$ $= -\frac{1}{262144}$	

Question 7 [10 Marks]		
(i)	Let P_n be the statement $u_n = \frac{1}{2n+1}$ for $n \geq 0$. When $n = 0$, $LHS = u_0 = 1$ (Given) $RHS = \frac{1}{2(0)+1} = 1 = LHS$ $\therefore P_0$ is true. Assume P_k is true for some $k \in □$, $k \geq 0$, i.e. $u_k = \frac{1}{2k+1}$ Show that P_{k+1} is also true, i.e. $u_{k+1} = \frac{1}{2k+3}$ Using the recurrence relation, $u_{k+1} = u_k - \frac{2}{(2k+1)(2k+3)}$ $= \frac{1}{2k+1} - \frac{2}{(2k+1)(2k+3)}$ $= \frac{2k+3-2}{(2k+1)(2k+3)}$ $= \frac{2k+1}{(2k+1)(2k+3)} = \frac{1}{2k+3}$ P_{k+1} is true. Since P_0 is true and P_k is true, implies P_{k+1} is also true. By mathematical induction, P_n is true for $n \geq 0$.	

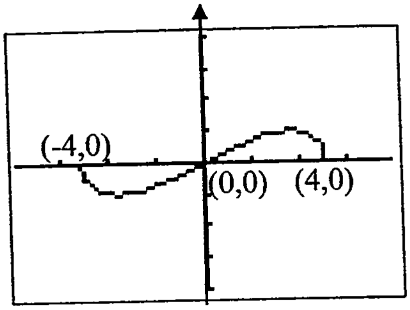
(ii)	$\sum_{n=1}^{N}\frac{1}{(2n+1)(2n+3)}$ $=\frac{1}{2}\sum_{n=1}^{N}(u_n-u_{n+1})$ $=\frac{1}{2}[u_1-u_2$ $+u_2-u_3$ $+u_3-u_4$ $\vdots$ $+u_{N-1}-u_N$ $+u_N-u_{N+1}]$ $=\frac{1}{2}(u_1-u_{N+1})$ $=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2N+3}\right)$	
(iii)	Let $n=r-1$, $\sum_{n=0}^{N}\frac{1}{(2n+3)(2n+5)}$ $=\sum_{r-1=0}^{r-1=N}\frac{1}{(2r-2+3)(2r-2+5)}$ $=\sum_{r=1}^{N+1}\frac{1}{(2r+1)(2r+3)}$ $=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2N+5}\right)$	
(iv)	As $N\to\infty$, $\frac{1}{2N+3}\to 0$, $\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)}=\frac{1}{6}$	

Question 8 [10 marks]		
(i)	Let $\frac{\mathrm{d}A}{\mathrm{d}t}$ and $\frac{\mathrm{d}B}{\mathrm{d}t}$ be the rate of birth and rate of death respectively. $\frac{\mathrm{d}A}{\mathrm{d}t}=2x,\quad \frac{\mathrm{d}B}{\mathrm{d}t}\propto x^2 \quad\Rightarrow\quad \frac{\mathrm{d}B}{\mathrm{d}t}=kx^2$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}t}-\frac{\mathrm{d}B}{\mathrm{d}t}=2x-kx^2$ When $x=10$, $\frac{\mathrm{d}x}{\mathrm{d}t}=0$: $0=2(10)-k(10)^2 \quad\Rightarrow\quad k=\frac{1}{5}$ $\therefore \frac{\mathrm{d}x}{\mathrm{d}t}=2x-\frac{x^2}{5}$	

(ii)	$\therefore \frac{dx}{dt} = 2x - \frac{x^2}{5} - 1.8$ $= -\frac{1}{5}(x^2 - 10x + 9)$ $= -\frac{1}{5}\left[(x-5)^2 - 4^2\right]$, where $a = 4$ (shown)	
	$\frac{dx}{dt} = -\frac{1}{5}\left[(x-5)^2 - 4^2\right]$ $\int \frac{1}{(x-5)^2 - 4^2} dx = \int -\frac{1}{5} dt$ $\frac{1}{2(4)} \ln\left\|\frac{(x-5)-4}{(x-5)+4}\right\| = -\frac{1}{5}t + c$ $\ln\left\|\frac{x-9}{x-1}\right\| = -\frac{8}{5}t + 8c$ $\frac{x-9}{x-1} = e^{-\frac{8}{5}t + 8c} = Ae^{-\frac{8}{5}t}$ When $t = 0, x = 13$: $\frac{13-9}{13-1} = A \quad \Rightarrow \quad A = \frac{1}{3}$ $\frac{x-9}{x-1} = \frac{1}{3}e^{-\frac{8}{5}t}$ $3(x-9) = (x-1)e^{-\frac{8}{5}t}$ $x(3 - e^{-\frac{8}{5}t}) = 27 - e^{-\frac{8}{5}t}$ $x = \frac{27 - e^{-\frac{8}{5}t}}{3 - e^{-\frac{8}{5}t}}$	
	As $t \to \infty$, $e^{-\frac{8}{5}t} \to 0$ $x \to \frac{27}{3} = 9$ In the long term, the population of garoupa will stabilise at 9000.  	

Question 9 [12 Marks]

(i)



(ii)

$x = 4\cos\theta \qquad y = \sin 2\theta$

$\dfrac{dx}{d\theta} = -4\sin\theta \qquad \dfrac{dy}{d\theta} = 2\cos 2\theta$

$$\frac{dy}{dx} = \frac{2\cos 2\theta}{-4\sin\theta} = -\frac{\cos 2\theta}{2\sin\theta}$$

Equation of tangent at P,

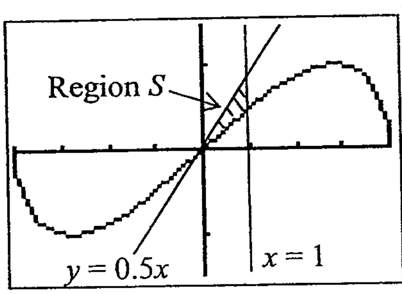
$$y - \sin 2p = -\frac{\cos 2p}{2\sin p}(x - 4\cos p)$$

$$y = -\left(\frac{\cos 2p}{2\sin p}\right)x + \frac{4\cos p\cos 2p}{2\sin p} + \sin 2p$$

$$= -\left(\frac{\cos 2p}{2\sin p}\right)x + 2\cot p\cos 2p + \sin 2p$$

(iii)

Equation of the tangent at $\theta = \dfrac{\pi}{2}$, $y = \dfrac{1}{2}x$



Area of S

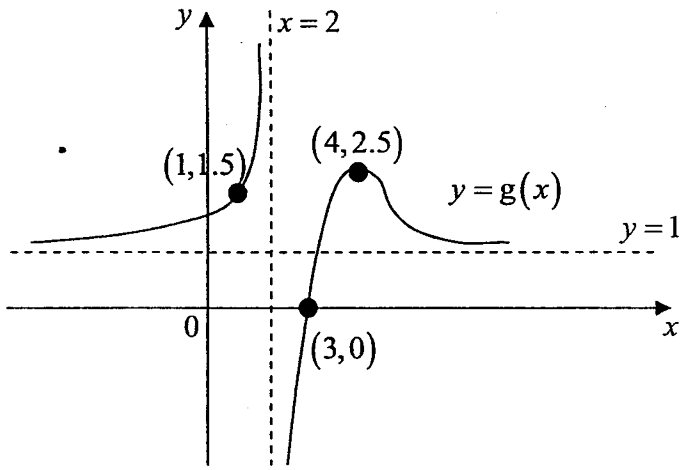
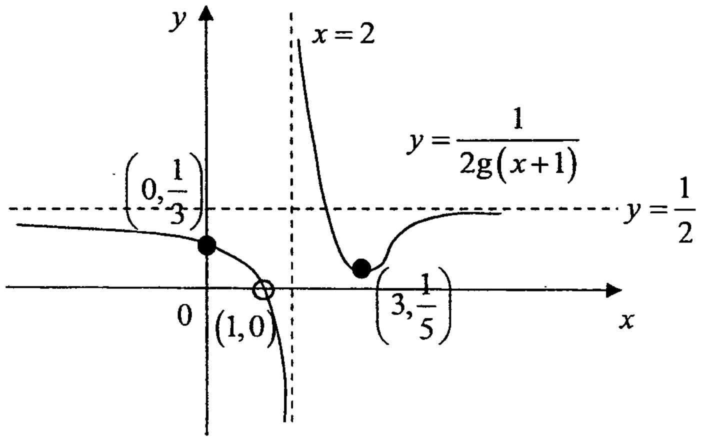
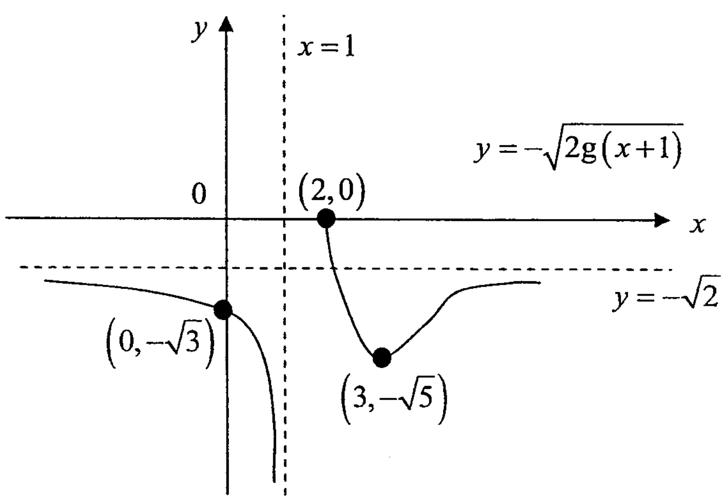
$$= \int_0^1 \frac{x}{2}\,dx - \int_0^1 y\,dx$$

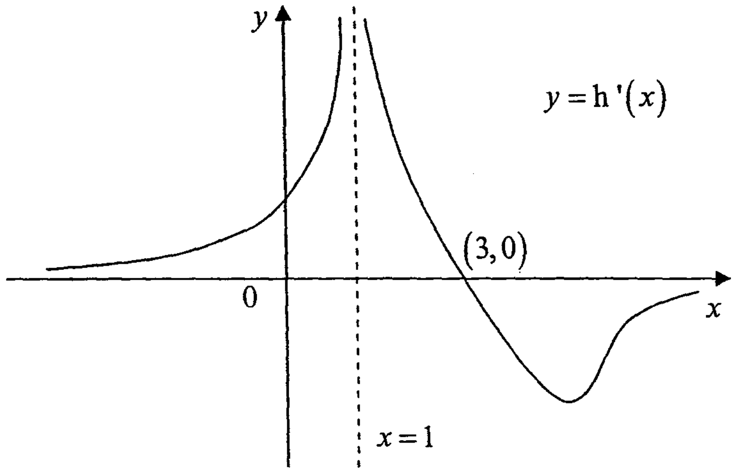
$$= \left[\frac{x^2}{4}\right]_0^1 - \int_{\pi/2}^{\cos^{-1}\left(\frac{1}{4}\right)} \sin 2\theta(-4\sin\theta\,d\theta)$$

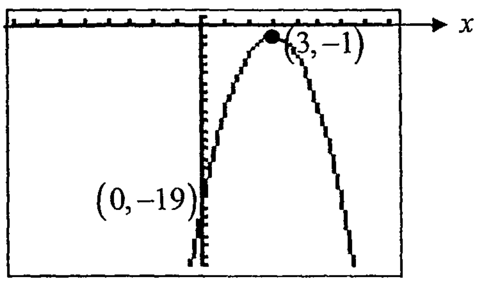
$$= \frac{1}{4} + 8\int_{\pi/2}^{\cos^{-1}\left(\frac{1}{4}\right)} \sin^2\theta\cos\theta\,d\theta \quad \text{(shown)}$$

$= 0.00395$ units^2 (to 3 s.f.)

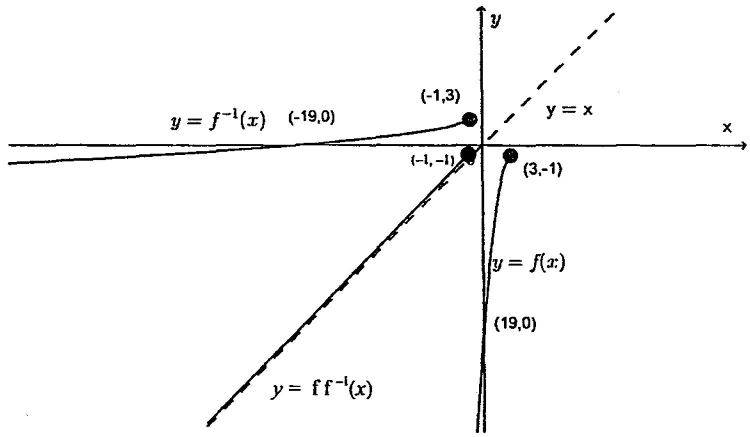
(iv)	$C \to C': \begin{cases} x \to 2x \\ y \to y+1 \end{cases}$ Sequence of transformations: - Scaling by factor 0.5 parallel to x-axis - Translation of 1 unit in the negative y-direction	

Question 10 [12 marks]		
(i)	 	
(ii)	 	
(iii)	 	



Question 11 [12 marks]		
(i)		
(ii)	The horizontal line, $y=-5$, cuts the graph more than once, hence f is not one-one $\Rightarrow$ f^{-1} does not exist.	
(iii)	Turning point of $y=f(x)$ occurs at $(3,-1)$. Hence greatest value of $k=3$	
(iv)	$y=-2x^2+12x-19$ $=-2(x-3)^2-1$ $(x-3)^2=\frac{y+1}{-2}$ $x-3=\pm\sqrt{\frac{y+1}{-2}}$ $x=3\pm\sqrt{\frac{y+1}{-2}}$ Since $x\le 3, x=3-\sqrt{\frac{y+1}{-2}}$ $f^{-1}(x)=3-\sqrt{\frac{x+1}{-2}}$ $D_{f^{-1}}=R_f=(-\infty,-1]$	

(v)



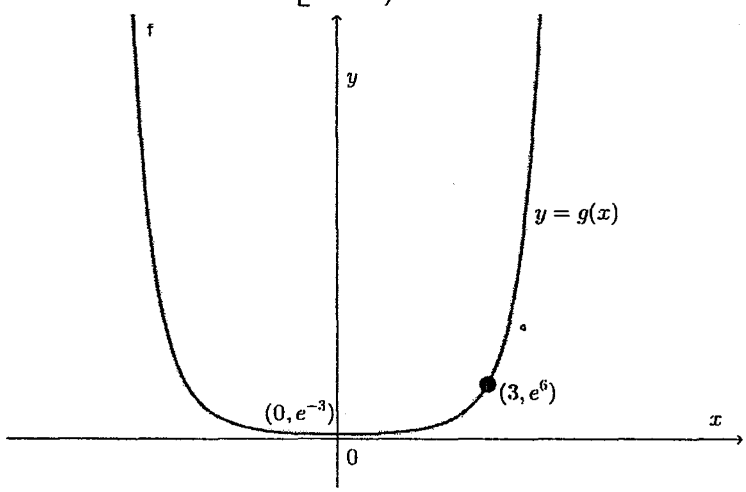
(vi)

$R_{f^{-1}} = (-\infty, 3]$ and $D_g = \mathbb{R}$

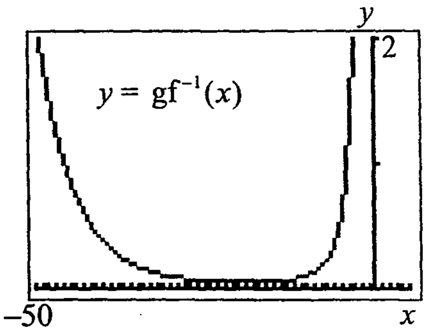
Since $R_{f^{-1}} \subseteq D_g$, hence gf^{-1} exists.

(vii)

$$\underset{(-\infty,-1]}{D_{f^{-1}}} \xrightarrow{} \underset{(-\infty,3]}{R_{f^{-1}}} \xrightarrow{} \underset{[e^{-3},\infty)}{R_{gf^{-1}}}$$

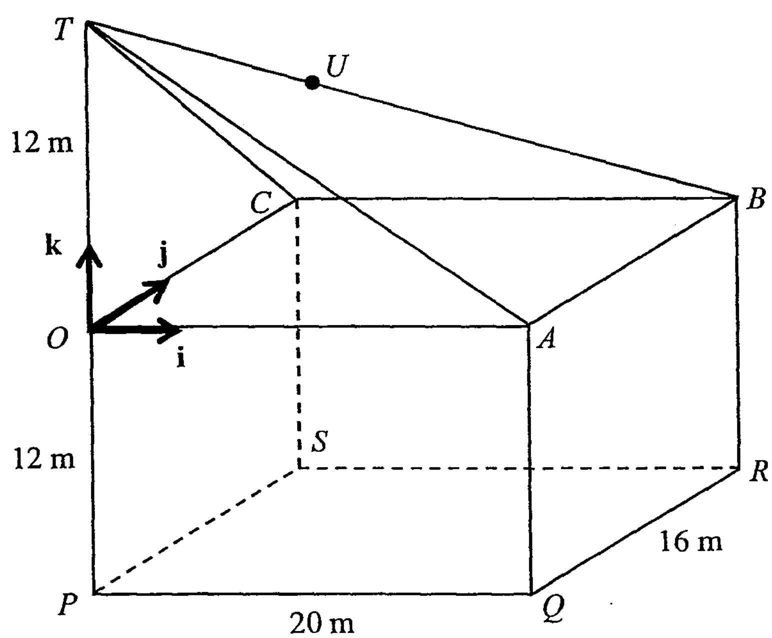


<u>ALTERNATIVELY:</u>

	(ALTERNATIVELY) $\mathrm{gf}^{-1}(x) = \mathrm{g}\left(3 - \sqrt{\frac{x+1}{-2}}\right)$ $= \mathrm{e}^{\left(3-\sqrt{\frac{x+1}{-2}}\right)^2 - 3}, \quad x \le -1$  From GC, minimum point is at $x = -19 \Rightarrow y = \mathrm{e}^{-3}$ $\therefore R_{\mathrm{gf}^{-1}} = \left[\mathrm{e}^{-3}, \infty\right)$	

Section A: Pure Mathematics [40 Marks]

1. The diagram below shows the structure of a military building which consists of two parts. For the first part, the rectangular plane $OABC$ is on the ground level and the point T is 12 m vertically above O. The second part of the building is an underground store room in the form of a cuboid with sides $OP = AQ = BR = CS = 12$ m, $OA = CB = PQ = SR = 20$ m and $OC = AB = PS = QR = 16$ m.



To better study the structure using vector method, the point O is taken as the origin and vectors **i, j, k**, each of length 1 m, are taken along OA, OC and OT respectively.

(i) The military officials decide to install a surveillance camera U along the edge TB such that $TU : UB = 1:3$ for better coverage. Determine the position vector of U and hence the vector equation of the plane UCB in scalar product form. [3]

(ii) Another camera is to be placed at the point P for effective coverage. Find the acute angle between the planes UCB and PCB. [3]

(iii) As highly explosive items that need to be kept at low temperature are present in the underground store room, the military officials also place an infra-red sensor device at the point T. Determine the shortest distance of T from the plane PCB.

[3]

2. (i) Solve the equation $z^5 - 16 - 16\sqrt{3}\mathrm{i} = 0$, giving the roots in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [3]

(ii) The roots of the equation $z^5 - 16 - 16\sqrt{3}\mathrm{i} = 0$ are represented by z_1, z_2, z_3, z_4 and z_5, where $-\pi < \arg(z_1) < \arg(z_2) < \arg(z_3) < \arg(z_4) < \arg(z_5) \le \pi$. Show all these roots on an Argand diagram. [2]

(iii) Given that the complex number v satisfies the equation $|v - z_3| = |v - z_4|$, sketch the locus of the points which represent v on the same Argand diagram. Determine if this locus passes through the point which represent the complex number z_1. [2]

(iv) Another complex number w is such that $|w| \le 1$ and $|w - z_3| \le |w - z_4|$. Shade on the same diagram, the region representing w. Determine the range of value of $\arg(w - z_2)$. [3]

3. (i) It is given that $y\cos x = \mathrm{e}^x$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 + \tan x)$. [2]

(ii) By further differentiation, find the Maclaurin series for y, up to and including the term in x^2. [5]

(iii) Hence, find the set of values of x for which the value of y in part(i) is within $\pm\ 0.05$ of the value found by its Maclaurin series in part(ii). [3]

4. The equation of a curve C is given by $y=\sqrt{\dfrac{3x}{4-x}}$.

(i) State the largest range of values of x for$C$ to be defined. [1]

For the rest of this question, define$C$ for the range of values found in part **(i)**.

(ii) Sketch$C$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [2]

(iii) The regionbounded by the curve, the line $y=3$ and the y-axis is denoted by R. Find the exact value of the area of$R$. [5]

(iv) The region$R$is rotated 2πradians about the line $y=3$ to form a solid.Write down the equation of the curve obtained when C is translated by 3 units in the negative y-direction. Hencefind the volume of the solid formed. [3]

Section B: Statistics [60 Marks]

5. A school has 200 teachers of whom 5% are in the 21 – 30 age group, 60% are in the 31 – 40 age group and the rest are in the age group of 41 and above. During a meeting held in a Lecture Theatre, the principal intends to obtain a sample of 20 teachers for a survey. She decides to select 20 teachers from the last occupied row for the survey.

(i) Name the sampling method described. State a reason, in the context of the question, why this sampling method is not desirable. [2]

(ii) Suggest a method of obtaining a representative sample and describe how it may be carried out. [3]

6. For events A and B, it is given that $\mathrm{P}(A\mid B')=\frac{4}{7}$, $\mathrm{P}(B'\mid A')=\frac{2}{3}$ and $\mathrm{P}(A)=\frac{11}{20}$. Give a reason why events A and B are not independent. [1]

Find

(i) $\mathrm{P}(A\cup B)$, [3]

(ii) $\mathrm{P}(A\cap B')$. [2]

7. Find the number of ways to arrange the nine letters of the word PERMUTATE,

(i) in a circle; [2]

(ii) in a row with exactly one pair of identical letters together; [3]

(iii) in a rowwith identical letters separated (for example "PERMUATET", "PERTUTAEM" etc...). [3]

8. In a college Mathematics examination with a large candidature, the percentage mark X obtained by each male candidates was found to follow a Normal distribution with mean μ and standard deviation σ. It is further found that $\mathrm{P}(X > 45) = 0.85$ and $\mathrm{P}(X > 85) = 0.15$.

(i) Find the value of μ and σ. [3]

(ii) Find the least integral value of a such that the probability of a male candidate scoring less than a is at least 0.75. [2]

In the same examination, the percentage mark obtained by each female candidate was found to follow a Normal distribution with mean of 67 and standard deviation of 22.

(iii) Find the probability that the total percentage mark of two randomly selected female candidates is more than three times that of a randomly selected male candidate. [3]

One Mathematics teacher of the college claimed that the percentage mark of each candidate in the examination followed a Normal distribution of mean of 66 and standard deviation of 45. Comment on the validity of this claim. [1]

9. Company Apacks and supplies salt in small packets. The mass of salt in one packet is denoted by $x$grams(g). The company claims that the mean mass of salt per packet is at least μ_0 g. Arandom sample of 12 packets of salt is takenand its mean and standard deviation are found to be 9.81g and 0.217g respectively.

Find the range of value of μ_0 for which Company$A$'s claim will not be rejected at the 5% significance level. State any assumption that you have made. [6]

Another company, Company B, claims that the meanmass of salt supplied by them per packet is 10 g. In a test against Company B's claim at the α % significance level, the hypotheses are as follows:

Null hypothesis: $\mu_B = 10$,
Alternative hypothesis: $\mu_B \neq 10$,

where μ_B is the population mean mass of salt in a packet of salt from Company B.

The p-value is found to be 0.0438 (corrected to 3 significant figures). Explain the meaning of this p-value in the context of the question. [1]

State the range of values of α for which Company B's claim is not rejected. For the range of values of α found,explain if Company B's claim is still valid under a one-tailed test. [3]

10. During the Arts Festival period, JC students in a college are encouraged to view the arts exhibits put up by the Arts Talent Programme students in the canteen during lunch time from 12 noon to 2pm daily. Given that the JC students do not influence each other in their decision to view the exhibits, state a condition for the random variable X, which denotes the number of students who view the exhibits in independent 15-minute interval, to be well modelled by a Poisson distribution. [1]

Assuming that X has a Poisson distribution with mean 7.5, find

(i) the probability that less than 12 students view the arts exhibits from 12:30 pm to 1 pm on a particular day during the Arts Festival; [2]

(ii) the probability that there are three 30-minute intervals in which at least 12 students view the arts exhibits on a particular day during the Festival, [2]

(iii) the probability that the mean number of students who view the art exhibits daily from 12 noon to 2 pm on 65 school days is more than 60.5. [3]

The organising committee decides to study the students' support for the arts exhibits for the Arts Festival. By using a suitable approximation, determine the probability that for the first 50 days, there are more than 25 days whereby there is only one 30-minute interval among the four possible intervals daily from 12 noon to 2 pm, in which there are less than 12 students who view the arts exhibits. [3]

11. The table below gives the time t, in minutes, taken by Andy to complete a particular stage of a computer game x weeks after he has started playing the game.

x	1	2	3	4	5	6
t	48.2	32.5	22.7	18.0	16.4	14.6

(i) Draw a scatter diagram for the data, labelling the axes clearly. [2]

(ii) Calculate the product moment correlation coefficient and comment on why its value does not necessarily mean that the best model for the relationship between x and t is linear. [2]

(iii) Determine which of the following would be a better model for the set of data, justifying your choice clearly:

(A) $t = a(x-10)^2 + b$,

(B) $t = ce^{-x} + d$,

where a, b, c and d are positive constants. [2]

(iv) For the better model identified in part **(iii)**, calculate the product moment correlation coefficient. [1]

(v) Andy would like to estimate the time taken by himself to complete the stage of the game 10 weeks after he has started playing the game. Find the equation of a suitable regression line and use it to obtain the estimate. Give two reasons why the estimate is not reliable. [4]

END OF PAPER

Solutions to 2016 Y6 H2 Maths Preliminary Exam II (Paper 2)

Section A: Pure Mathematics [40 marks]

Question 1 [9 Marks]

(i) We first have the following position vectors:

$$\overrightarrow{OT}=\begin{pmatrix}0\\0\\12\end{pmatrix},\ \overrightarrow{OA}=\begin{pmatrix}20\\0\\0\end{pmatrix},\ \overrightarrow{OB}=\begin{pmatrix}20\\16\\0\end{pmatrix}\text{ and }\overrightarrow{OC}=\begin{pmatrix}0\\16\\0\end{pmatrix}$$

Then by Ratio Theorem:
Since U divides TB in the ratio 1:3,

$$\overrightarrow{OU}=\frac{1}{4}\overrightarrow{OB}+\frac{3}{4}\overrightarrow{OT}$$

$$=\frac{1}{4}\begin{pmatrix}20\\16\\0\end{pmatrix}+\frac{3}{4}\begin{pmatrix}0\\0\\12\end{pmatrix}=\begin{pmatrix}5\\4\\9\end{pmatrix}$$

To find the vector equation of the plane UCB, we first have:

$$\overrightarrow{UB}=\overrightarrow{OB}-\overrightarrow{OU}=\begin{pmatrix}20\\16\\0\end{pmatrix}-\begin{pmatrix}5\\4\\9\end{pmatrix}=\begin{pmatrix}15\\12\\-9\end{pmatrix}$$

and

$$\overrightarrow{CB}=\overrightarrow{OB}-\overrightarrow{OC}=\begin{pmatrix}20\\0\\0\end{pmatrix}.$$

We then apply cross product of the vectors to obtain normal vector to the plane UCB:

$$\overrightarrow{UB}\times\overrightarrow{CB}=\begin{pmatrix}15\\12\\-9\end{pmatrix}\times\begin{pmatrix}20\\0\\0\end{pmatrix}=\begin{pmatrix}0\\-180\\-240\end{pmatrix}=-60\begin{pmatrix}0\\3\\4\end{pmatrix}$$

Thus, the vector equation of the plan UCB in scalar product form is then $\mathbf{r}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}=\begin{pmatrix}0\\16\\0\end{pmatrix}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}=48$

(ii) For the plane PCB, we first need to find the normal vector to the plane. We have:

$$\overrightarrow{PC}\times\overrightarrow{CB}=\left(\begin{pmatrix}0\\16\\0\end{pmatrix}-\begin{pmatrix}0\\0\\-12\end{pmatrix}\right)\times\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$=\begin{pmatrix}0\\16\\12\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\12\\-16\end{pmatrix}=4\begin{pmatrix}0\\3\\-4\end{pmatrix}$$

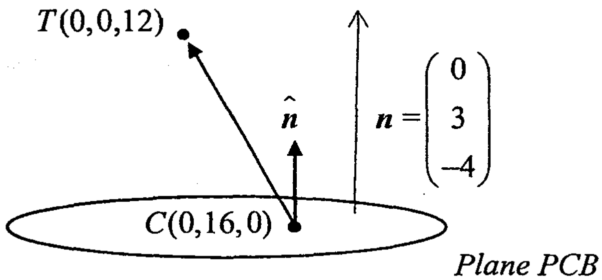
Therefore a vector normal to the plane PCB will be $\begin{pmatrix}0\\3\\-4\end{pmatrix}$.

Thus, the acute angle θ between plane *UCB* and *PCB* will be given by:

$$\cos\theta = \left|\frac{\begin{pmatrix}0\\3\\4\end{pmatrix}\cdot\begin{pmatrix}0\\3\\-4\end{pmatrix}}{\left|\begin{pmatrix}0\\3\\4\end{pmatrix}\right|\left|\begin{pmatrix}0\\3\\-4\end{pmatrix}\right|}\right| = \left|\frac{9-16}{\sqrt{3^2+4^2}\sqrt{3^2+(-4)^2}}\right| = \frac{7}{25}$$

That is, $\theta = 73.7^\circ$ (3 s.f.)

(iii) <u>Method 1:</u>



We note that $C = (0,16,0)$ is a point on the plane *PCB* and thus we projected the vector $\overrightarrow{CT}$ onto $\hat{\boldsymbol{n}}$.

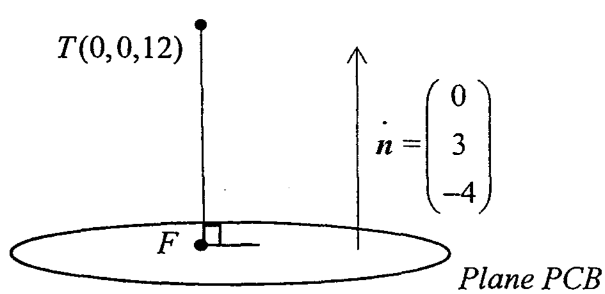
$$\overrightarrow{CT} = \overrightarrow{OT} - \overrightarrow{OC} = \begin{pmatrix}0\\-16\\12\end{pmatrix}$$

The shortest distance of *T* to the plane *PCB* is given by:

$$d = \left|\overrightarrow{CT}\cdot\hat{n}\right| = \frac{\left|\begin{pmatrix}0\\-16\\12\end{pmatrix}\cdot\begin{pmatrix}0\\3\\-4\end{pmatrix}\right|}{\sqrt{3^2+(-4)^2}}$$

$$= \frac{|-48-48|}{5} = \frac{96}{5} \text{ units}$$

<u>Method 2:</u>



Let *F* be the foot of the perpendicular from point *T* to the plane *PCB*.

Then $\overrightarrow{TF} = \lambda\mathbf{n} = \lambda\begin{pmatrix}0\\3\\-4\end{pmatrix}$

$$\Rightarrow \overline{OF} - \overline{OT} = \lambda \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} \Rightarrow \overline{OF} = \overline{OT} + \lambda \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3\lambda \\ 12-4\lambda \end{pmatrix}$$

Since F is also a point on the plane PCB, $\overline{OF}$ will also satisfy the vector equation of plane PCB.

We then have $\begin{pmatrix} 0 \\ 3\lambda \\ 12-4\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 16 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = 48$

Thus,

$9\lambda - 4(12-4\lambda) = 48$

$\Rightarrow 25\lambda = 96$

$\Rightarrow \lambda = \dfrac{96}{25}$

Therefore, the perpendicular distance is

$$\left|\overline{TF}\right| = \frac{96}{25}\left|\begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}\right| = \frac{96}{25}\sqrt{3^2 + (-4)^2} = \frac{96}{5} \text{ units}$$

Question 2 [10 Marks]		
(i)	Given that $z^5 - 16 - 16\sqrt{3}i = 0$, we have $z^5 = 16 + 16\sqrt{3}i = 32e^{i\left(\frac{\pi}{3}\right)}$. Then by solving, the roots are given by: $z = 32^{\frac{1}{5}} e^{i\frac{1}{5}\left(\frac{\pi}{3}+2k\pi\right)}$ $= 2e^{i\left(\frac{1+6k}{15}\right)\pi}$ where $k = -2, -1, 0, 1, 2$.	

(ii) Given that

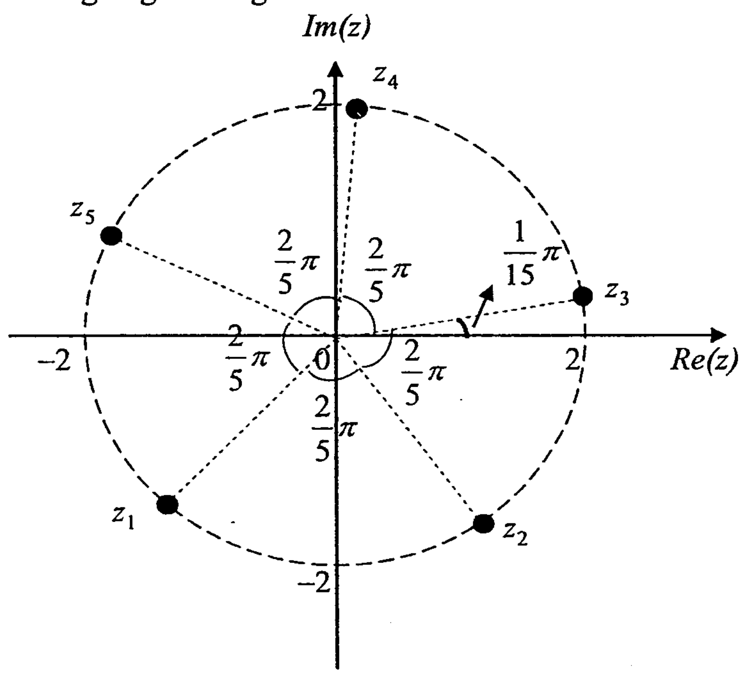
$-\pi < \arg(z_1) < \arg(z_2) < \arg(z_3) < \arg(z_4) < \arg(z_5) \le \pi$

$\Rightarrow$ we have the following roots:

$z_1 = 2e^{i\left(-\frac{11}{15}\pi\right)}$, $z_2 = 2e^{i\left(-\frac{5}{15}\pi\right)}$, $z_3 = 2e^{i\left(\frac{1}{15}\pi\right)}$, $z_4 = 2e^{i\left(\frac{7}{15}\pi\right)}$

and $z_5 = 2e^{i\left(\frac{13}{15}\pi\right)}$.

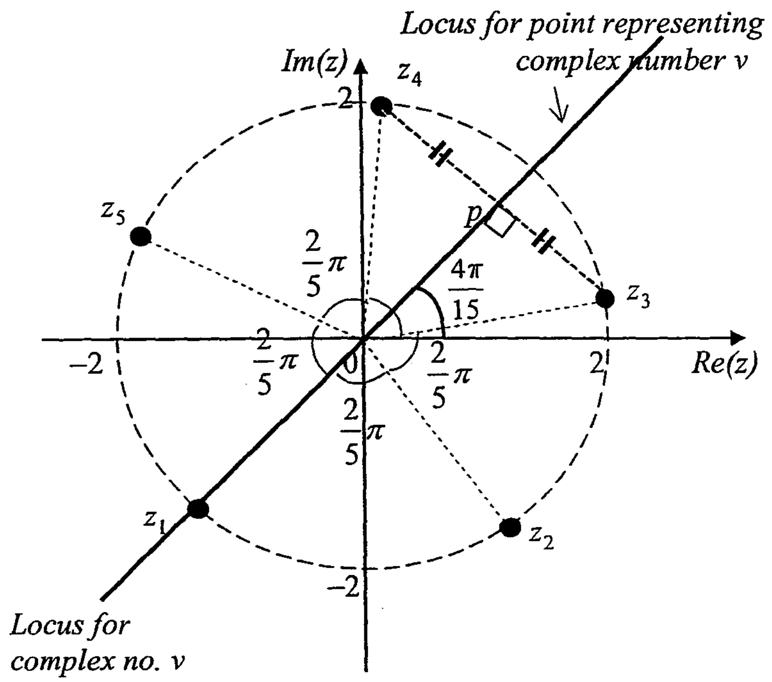
The roots z_1, z_2, z_3, z_4 and z_5 are then represented in the following Argand diagram:



Note that if the points Z_1, Z_2, Z_3, Z_4 and Z_5 represent the complex number z_1, z_2, z_3, z_4 and z_5 respectively on the Argand diagram, then

$$\angle Z_1OZ_2 = \angle Z_2OZ_3 = \angle Z_3OZ_4 = \angle Z_4OZ_5 = \angle Z_5OZ_1 = \frac{2\pi}{5}$$

(iii)

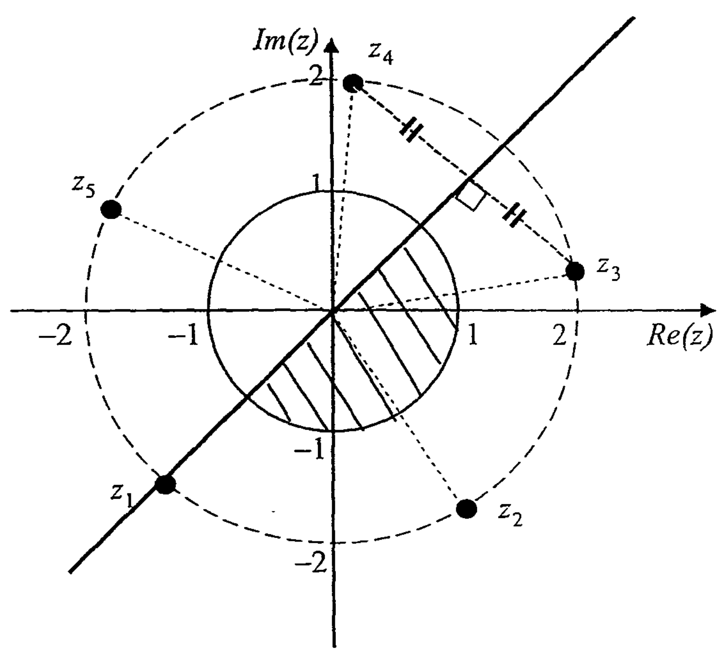


We note that if p is the complex number represented by the midpoint of the line joining Z_3 and Z_4, then

$$\arg(p) = \frac{\frac{1}{15}\pi + \frac{7}{15}\pi}{2} = \frac{4}{15}\pi .$$

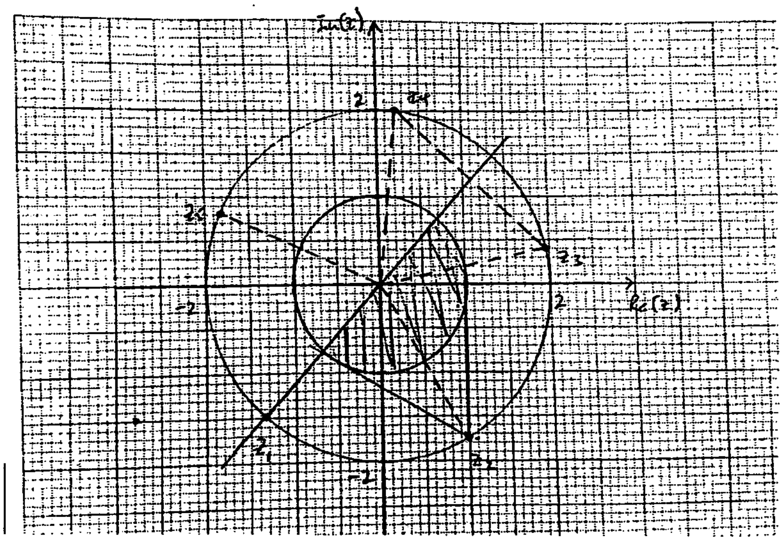
Since $\arg(p) + |\arg(z_1)| = \frac{4}{15}\pi + \frac{11}{15}\pi = \pi$, we can conclude that the point representing z_1 lies on the locus of v.

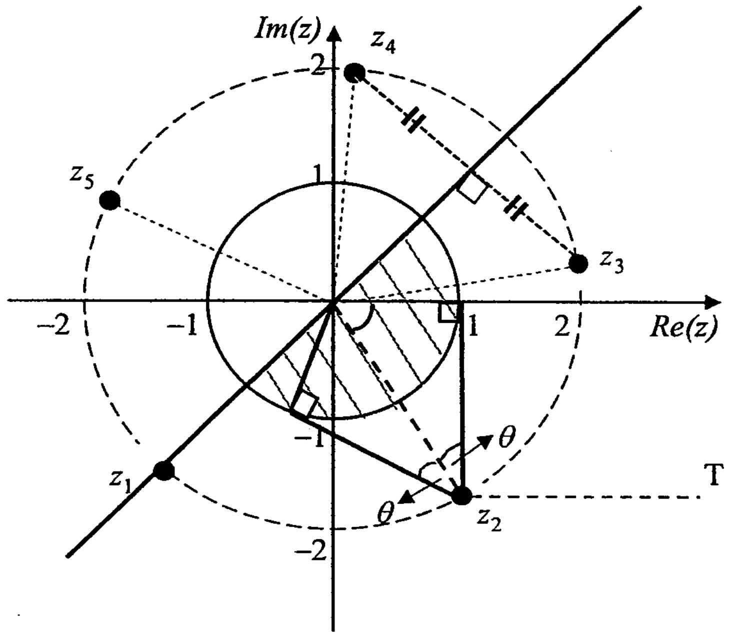
(iv)



Shaded region represents possible region for complex number w.

Q2 . (iii) & (iv)





To find range of $\arg(w-z_2)$, we note that :

$$\angle OZ_2T = \pi - \frac{5}{15}\pi = \frac{2}{3}\pi$$

and $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$.

Thus the range of value of $\arg(w-z_2)$ is

$$\frac{2}{3}\pi - \frac{1}{6}\pi \le \arg(w-z_2) \le \frac{2}{3}\pi + \frac{1}{6}\pi$$

i.e. $\frac{1}{2}\pi \le \arg(w-z_2) \le \frac{5}{6}\pi$ or $1.57 \le \arg(w-z_2) \le 2.62$

Question 3 [10 Marks]

(i)

Let $y\cos x = e^x$ ------(1)

$$\left(\frac{dy}{dx}\right)\cos x + y(-\sin x) = e^x$$

$$\left(\frac{dy}{dx}\right)\cos x = y\sin x + y\cos x$$

$$\frac{dy}{dx} = y\frac{\cos x + \sin x}{\cos x}$$

$$\frac{dy}{dx} = y(1+\tan x) \text{------(2)} \quad \text{(shown)}$$

(ii)

Differentiate with respect to x:

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)(1+\tan x) + y(\sec^2 x) \text{------(3)}$$

At $x = 0$,

From (1): $y = f(0) = \dfrac{e^0}{\cos 0} = 1$

From (2): $\dfrac{dy}{dx} = (1)(1+\tan 0) \Rightarrow \dfrac{dy}{dx} = f'(0) = 1$

From (3):

$$\frac{d^2y}{dx^2} = (1)(1+\tan 0) + (1)\sec^2 0 \Rightarrow \frac{d^2y}{dx^2} = f''(0) = 2$$

$$\therefore y = f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots\ldots\ldots$$

$$= 1 + x + x^2 + \ldots\ldots\ldots\ldots \quad \text{(up to } x^2 \text{ term)}$$

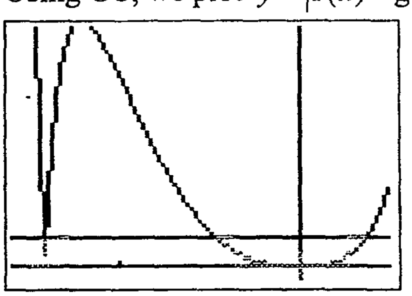
(iii)

Let $f(x) = \dfrac{e^x}{\cos x}$, where $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$,

and $g(x) = 1 + x + x^2$ (from (ii)).

We thus solve $|f(x) - g(x)| < 0.05$.

Using GC, we plot $y = |f(x) - g(x)|$ and $y = 0.05$:



x-values of the intersection points are:

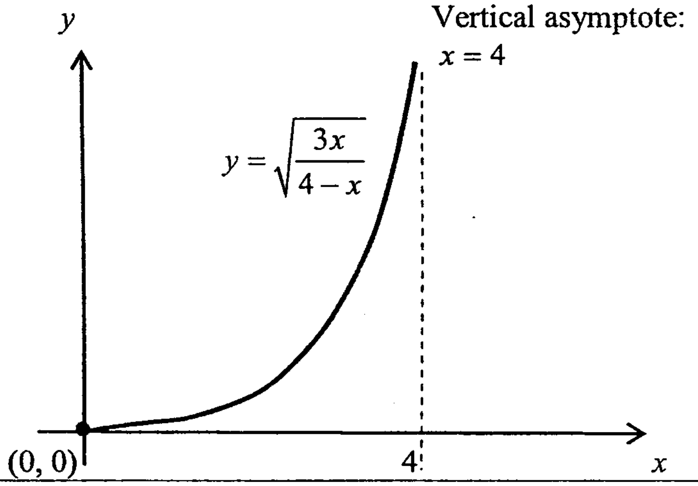
–1.425019, –1.410526, –0.4704015 and 0.3795259

$\therefore$ Ans: $\{x : x \in □, -1.43 < x < -1.41 \text{ or } -0.470 < x < 0.380\}$

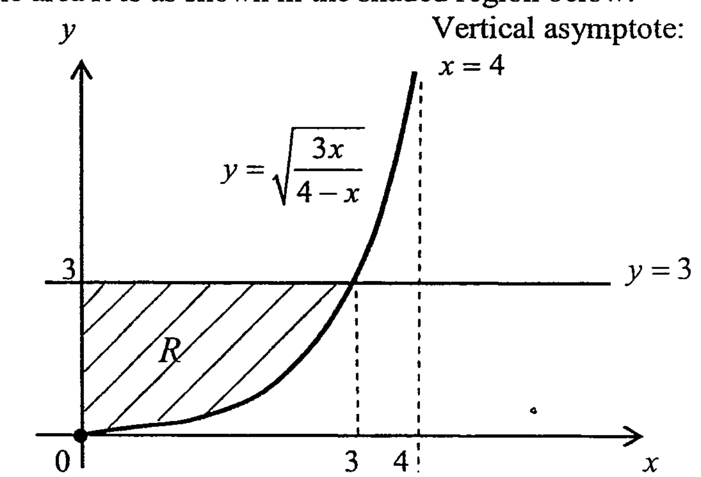
Question 4 [11 Marks]

(i) The largest range of values of x for curve C to be defined is $0 \le x < 4$.

(ii)



(iii) The area R is as shown in the shaded region below:



$y = \sqrt{\dfrac{3x}{4-x}}$ then $y^2 = \dfrac{3x}{4-x}$

$\Rightarrow 4y^2 - xy^2 = 3x$

$\Rightarrow x = \dfrac{4y^2}{3+y^2}$

Thus, the area of the region R

$= \int_0^3 x \, \mathrm{d}y$

$= \int_0^3 \dfrac{4y^2}{3+y^2} \, \mathrm{d}y$

$= \int_0^3 4 - \dfrac{12}{3+y^2} \, \mathrm{d}y$

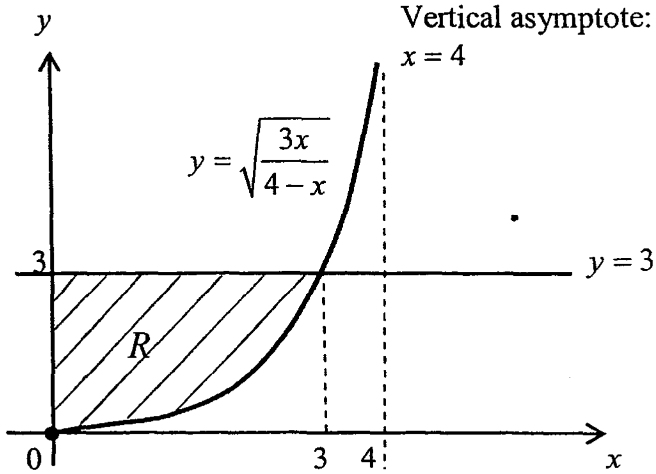
$= \left[4y - \dfrac{12}{\sqrt{3}} \tan^{-1}\left(\dfrac{y}{\sqrt{3}}\right) \right]_0^3$

$= \left(4 \times 3 - \dfrac{12}{\sqrt{3}} \tan^{-1}\left(\sqrt{3}\right) \right) - 0$

$= 12 - \frac{12}{\sqrt{3}} \times \frac{\pi}{3}$

$= 12 - \frac{4\sqrt{3}}{3}\pi$ unit^2

(iv) The area R is rotated 2π radian about the line $y = 3$:



The equation of curve when C is translated by –3 units in the y-direction is $y = \sqrt{\frac{3x}{4-x}} - 3$.

Next, let

$y = 3 \Rightarrow \sqrt{\frac{3x}{4-x}} = 3 \Rightarrow \frac{3x}{4-x} = 9 \Rightarrow x = 3$

The volume V generated when R is rotated completely about $y = 3$ is the same as the volume generated when the new curve is rotated about the x-axis from $x = 0$ to $x = 3$.

Therefore,

$V = \int_0^3 \pi y^2 \, dx$

$= \pi \int_0^3 \left(\sqrt{\frac{3x}{4-x}} - 3 \right)^2 dx$

$= 28.6$ unit^3

Section B: Statistics [60 marks]

Question 5 [5 Marks]

(i) Quota Sampling.
This method is non-random as not every teacher has an equal chance of being selected. <u>OR</u>
This method does not give a representative sample.

(ii) Stratified Sampling.
Principal can draw **random** samples from each stratum as follows:

Age group	21 - 30	31 - 40	41 and above
No. of teachers selected	$0.05 \times 20 = 1$	$0.6 \times 20 = 12$	$20 - 1 - 12 = 7$

Question 6 [6 marks]		
	Since $\mathrm{P}(A) \neq \mathrm{P}(A \mid B')$, events A and B' are not independent. Thus A and B are not independent.	
(i)	$\mathrm{P}(B' \mid A') = \frac{2}{3} \Rightarrow \frac{\mathrm{P}(B' \cap A')}{\mathrm{P}(A')} = \frac{2}{3}$ $\Rightarrow \mathrm{P}(B' \cap A') = \frac{2}{3} \times \frac{9}{20} = \frac{3}{10}$ $\mathrm{P}(A \cup B) = 1 - \mathrm{P}(B' \cap A') = \frac{7}{10}$	
(ii)	$\mathrm{P}(A \mid B') = \frac{4}{7} \Rightarrow \frac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')} = \frac{4}{7}$ $\Rightarrow \frac{\mathrm{P}(A \cap B')}{\mathrm{P}(A \cap B') + \mathrm{P}(A' \cap B')} = \frac{4}{7}$ $\Rightarrow \frac{\mathrm{P}(A \cap B')}{\mathrm{P}(A \cap B') + \frac{3}{10}} = \frac{4}{7}$ $\Rightarrow \mathrm{P}(A \cap B') = \frac{2}{5}$	

Question 7 [8 Marks]		
(i)	$\frac{(9-1)!}{2!2!} = 10080$	
(ii)	For the case where <u>EE</u> are together and T, T separated: Arrange <u>EE</u>, P, R, M, U, A first, then slot in T, T to separate them. No. of ways = $6! \times \binom{7}{2} = 15120$ Hence, required no. of ways = $15120 \times 2 = 30240$	
(iii)	No. of ways without restrictions = $\frac{9!}{2!2!} = 90720$ No. of ways with 'E's together & 'T's together = 7! = 5040 Hence, required no. of ways = No. of ways without restrictions – n(each of the pair of identical letters together) – n(exactly one of the pairs is together) = 90720 – 5040 – 30240 = 55440	
	<u>ALTERNATIVELY</u> <u>Case 1</u>: E, E, T, T all separated No. of ways = $5! \times \binom{6}{4} \times \frac{4!}{2!2!} = 10800$	

Case 2: with exactly 1 E and 1 T together (i.e. ET, E, T separated)

No. of ways $= 5! \times \binom{6}{3} \times 3! \times 2 = 28800$

Case 3: 2 pairs of ET, but the pairs separated

No. of ways $= 5! \times \binom{6}{2} \times (2!)^2 = 7200$

Case 4: ETE or TET together

No. of ways $= \left[5! \times \binom{6}{2} \times 2!\right] \times 2 = 7200$

Case 5: ETET or TETE

No. of ways $= 6! \times 2 = 1440$

Hence, required no. of ways

$= 10800 + 28800 + 7200 + 7200 + 1440 = 55440$

Question 8 [9 Marks]

(i) $P(X > 45) = 0.85 \Rightarrow P(X < 45) = 0.15$

Since $P(X > 85) = 0.15$, by symmetry property of normal distribution curve, $\mu = \frac{45+85}{2} = 65$

Next,

$P(X < 45) = 0.15$

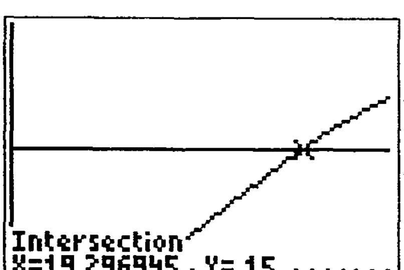
$\Rightarrow P\left(Z < \frac{45-\mu}{\sigma}\right) = 0.15$

$\Rightarrow P\left(Z < \frac{45-65}{\sigma}\right) = 0.15$

$\Rightarrow \frac{45-65}{\sigma} = -1.03643338$

$\Rightarrow \sigma = 19.296947 = 19.3$ (3 s.f.)

ALTERNATIVE METHOD



$y = P(X < 45)$

$y = 0.15$

$\therefore \sigma = 19.296947 = 19.3$ (3 s.f.)

(ii)	If the probability of a male candidate scoring less than a is at least 0.75, then we have $P(X < a) \geq 0.75$. $X \sim N(65, 19.296947^2)$ $P(X < a) \geq 0.75$ $\Rightarrow a \geq 78.01559295$ Thus, the least integral value of a is 79.	
(iii)	Let Y denotes the random variable that represents the percentage marks obtained by a randomly selected female candidate in the Maths exam. Then $Y \square N(67, 22^2)$. We first note that $P(Y_1 + Y_2 > 3X) = P(Y_1 + Y_2 - 3X > 0)$ Since $X \square N(65, 19.296947^2)$, we have $E(Y_1 + Y_2 - 3X) = 67 + 67 - 3 \times 65 = -61$ $Var(Y_1 + Y_2 - 3X) = 22^2 + 22^2 + 3^2 \times 19.296947^2$ $= 4319.349472$ Then $Y_1 + Y_2 - 3X \square N(-61, 4319.349472)$ Thus, $P(Y_1 + Y_2 > 3X)$ $= P(Y_1 + Y_2 - 3X > 0)$ $= 0.177$ (3 s.f.)	
	We note that for a random variable W with normal distribution of mean 66 and standard deviation 45, $P(W < 0) = 0.0712334139$ which is non negligible. As $W \geq 0$, we conclude that the teacher's claim is not valid. OR: If $W \square N(66, 45)$, then 99.7% of the values would lie within $66 \pm 3 \times 45$, which contains a significant range of negative values. Thus, the teacher's claim is not valid.	

Question 9 [10 Marks]		
	Let μ be the mean mass of salt in a packet of salt from Company A. Test $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$ at 5% significance level Test statistic: Under H_0, $T = \dfrac{\overline{X} - \mu_0}{s/\sqrt{12}} \sim t(11)$, where $s^2 = \dfrac{12}{11}(0.217^2) = 0.0513698182$ Critical region: Reject H_0 if $t_{calc} < -1.795884781$	

	Since H_0 is not rejected, $\dfrac{9.81-\mu_0}{\sqrt{\dfrac{0.0513698182}{12}}} \geq -1.795884781$ $\mu_0 \leq 9.927501081$ i.e. $\mu_0 \leq 9.93$ (to 3s.f.) (OR $\mu_0 \leq 9.92$) We assume that mass of salt follows a normal distribution.	
	p-value of 0.0438 is the lowest level of significance for which Company B's claim that the mean mass of salt per packet is 10 g is rejected.	
	To not reject Company B's claim (i.e. to not reject H_0), $p\text{-value} \geq \dfrac{\alpha}{100}$ $\Rightarrow \alpha \leq 0.0438 \times 100$ i.e. $\alpha \leq 4.38$ Supposed a one-tailed test is conducted instead, i.e. testing $H_0 : \mu_B = 10$ against $H_1 : \mu_B < 10$ $p\text{-value} = \dfrac{0.0438}{2} = 0.0219$. Then we would not be certain if p-value < α since $\alpha \leq 4.38$. (no conclusion)	

Question 10 [11 marks]		
	X can be well modelled by a Poisson distribution if (i) the mean number of JC students who viewed the arts exhibits in each 15-minute interval remains **constant**, *or* (ii) the mean number of JC students who viewed the arts exhibits in other time intervals is **proportional** to the mean number for each 15-minute interval.	
(i)	Let X' denotes the random variable that represents the number of students who view the arts exhibits in a 30-minute period. Then $X' \sim \text{Po}(15)$ since $X \sim \text{Po}(7.5)$ Then the required probability $= \text{P}(X' < 12) = \text{P}(X' \leq 11)$ $= 0.184751799$ $= 0.185$ (3 s.f.)	

(ii) The required probability

$= \binom{4}{3}(1-0.184751799)^3(0.184751799)^1$

$= = \frac{4!}{3!1!}(0.815248201)^3(0.184751799)^1$

$= 0.400422262$

$= 0.400$ (3 s.f.)

OR:

Let Y denotes the number of 30-minute intervals among the 4 possible ones from 12 noon to 2 pm in which there are at least 12 students who view the arts exhibits.

Then $Y \sim B(4, 1-0.184751799) = B(4, 0.815248201)$

Then required probability $= P(Y = 3) = 0.400422262$

$= 0.400$ (3 s.f.)

(iii) Let W denotes the random variable that represents the number of students who view the arts exhibits from 12 noon to 2 pm on a randomly chosen day during the Arts Festival.

Since $X \sim Po(7.5)$, we have $W \sim Po(7.5 \times 8)$ i.e. Po(60)

For the 65-day period, $n = 65$ (> 50) is large.

By the Central Limit Theorem (CLT),

$\overline{W} \sim N\left(60, \frac{60}{65}\right)$ approximately.

Thus, $P(\overline{W} > 60.5) = 0.3013866388$

$= 0.301$ (3 s.f.)

Next, let V be the random variable that denotes the number of days among 50 whereby there is only one 30 minutes interval among the 4 possible intervals daily, in which there are less than 12 students who view the arts exhibits.

Then $V \sim B(n, p)$ where $n = 50$, $p = 0.400422262$ (part ii)

Then upon checking,

$n = 50$ is large,

$np = 50 \times 0.40042262 = 20.0211131 > 5$,

$nq = 50 \times (1 - 0.40042262) = 29.9788869 > 5$

We conclude that $V \sim N(np, npq)$ approximately.

That is, $V \sim N(20.0211131, 12.00421727)$ approximately.

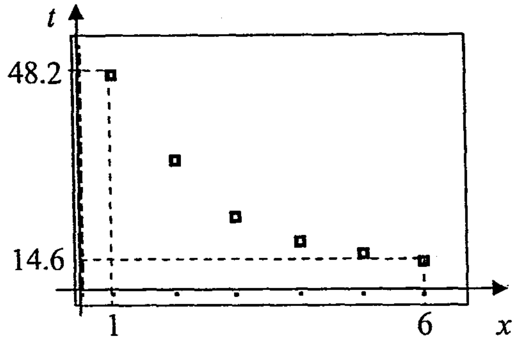
Then, $P(V > 25)$

$= P(V \geq 26)$

$= P(V \geq 25.5)$ (continuity correction)

$= 0.0569000711$

$= 0.0569$ (3 s.f.)

Question 11 [11 marks]		
(i)		
(ii)	$r = -0.9168956355 \approx -0.917$ Although $\lvert r \rvert$ is close to 1, suggesting a linear relationship between x and t (i.e $t = mx + k$), it does appear that the points follow a curvilinear trend from the scatter diagram.	
(iii)	(B) $t = ce^{-x} + d$ is a better model because as x increases, t decreases at a decreasing rate. As for the model $t = a(x-10)^2 + b$, t might decrease at a decreasing rate initially as x increases (for $x < 10$) but it will increase eventually (for $x > 10$). This does make sense for the context of the question since more practice on a longer period ought to improve his proficiency thus taking less time to complete the stage of the computer game.	
(iv)	For $t = ce^{-x} + d$: $r = 0.9852371391 \approx 0.985$ For $t = a(x-10)^2 + b$: $r = 0.954421565 \approx 0.954$	
(v)	A suitable regression line: $t = 89.60094125\ e^{-x} + 16.73059924$ i.e. $t = 89.6\ e^{-x} + 16.7$ When $x = 10$, $t = 89.60094125\ e^{-10} + 16.73059924$ $= 16.73466712 \approx 16.7$ mins The estimate is not reliable because: (1) $x = 10$ is outside of data range (extrapolation) (2) Andy has already obtained timings less than 16.7 minutes for both week 5 and week 6. It is more likely that the timing will be shorter than 16.7 min by week 8. If the wrong line is chosen: $t = 0.5024319662(x-10)^2 + 2.706822862$ When $x = 10$. $t = 0.5024319662(-2)^2 + 2.706822862$ $= 4.716550727 \approx 4.72$ mins The estimate is not reliable because $x = 10$ is outside of data range (extrapolation).	