NJC Paper 1

1 Given that $\mathbf{p} = 2\mathbf{i} + \alpha\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = \alpha\mathbf{i} + \mathbf{j} + 6\mathbf{k}$, where α is a real constant and $\mathbf{w}$ is the position vector of a variable point W relative to the origin such that $\mathbf{w} \times \mathbf{p} = \mathbf{q}$.

(i) Find the value of α . [2]

(ii) Find the set of vectors $\mathbf{w}$ in the form $\{\mathbf{w} : \mathbf{w} = \mathbf{a} + \lambda\mathbf{b}, \lambda \in \mathbb{R}\}$. [3]

2 **(a)** The sum, S_n , of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by $S_n = 3 + 7^{-2n}\left(n^2\right)$.

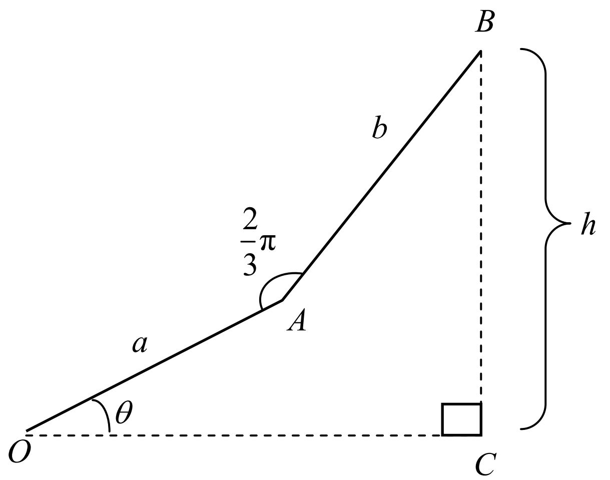
(i) Write down the value of $\sum_{r=1}^{\infty} u_r$. [1]

(ii) Find a formula for u_n for $n \geq 2$ and leave it in the form $7^{-2n}\ \mathrm{g}(n)$, where $\mathrm{g}(n)$ is an expression in terms of n. [2]

(b) Show that $\sum_{r=1}^{n}\left(\int_0^r \mathrm{e}^x - \mathrm{e}^{x-1}\ \mathrm{d}x\right) = \mathrm{e}^n + n\mathrm{e}^{-1} - (n+1)$.

Deduce the exact value of $\sum_{r=10}^{20}\left(\int_0^r \mathrm{e}^{x+2} - \mathrm{e}^{x+1}\ \mathrm{d}x\right)$. [5]

3 The diagram below shows two adjoining lines OA and AB where $OA = a$ m, $AB = b$ m and obtuse angle OAB is $\frac{2}{3}\pi$. C is a point such that OC and CB are perpendicular to each other, $BC = h$ m, and angle AOC is θ where $0 < \theta < \frac{\pi}{6}$.



(i) Show that

$h = \sqrt{a^2 + ab + b^2}\sin(\theta + \alpha)$, where α is a constant to be determined in terms of a and b. [4]

It is given that $a = 1$ and $b = 2$.

(ii) Find the rate of change of θ when $\theta = \frac{\pi}{12}$ and h is decreasing at a rate of 0.5 m per minute. [3]

(iii) When θ is a sufficiently small angle, show that $h \approx p\theta^2 + q\theta + \sqrt{3}$, where constants p and q are to be determined exactly. [3]

4 A researcher is investigating the elasticity of a new material. In the experiment, he stretched an extensible string of length 30 cm using a machine.

Each stretch is followed by a contraction. The initial stretch leads to an elongation of 10 cm and is followed by a contraction of 0.1 cm. The elongation resulting from each subsequent stretch is $\frac{10}{11}$ of the elongation caused by the previous stretch. Each subsequent contraction is 0.001 cm less than the previous contraction.

(i) Show that the length of the string after two stretches is 48.892 cm correct to 3 decimal places. [2]

(ii) Find the length of the string after it has been stretched n times, in terms of n. [3]

(iii) The string loses its elasticity completely when contraction exceeds elongation in a stretch. Find the minimum number of stretches for the string to lose its elasticity. [2]

(iv) The researcher coats a new string of the same initial length with another material. Now the string does not contract after every stretch while its elongation properties remain unchanged. Justify why it is impossible for the string to be elongated beyond 140 cm. [1]

5 **Do not use a calculator in answering this question.**

(a) Showing your working clearly, find the complex numbers z and w which satisfy the simultaneous equations

$$\mathrm{i}z + w = 2 \quad \text{and}$$

$$zw^* = 2 + 4\mathrm{i},$$

where w^* is the complex conjugate of w. [5]

(b) The complex number p is given by $a + \mathrm{i}b$, where $a > 0$, $b < 0$, $a^2 + b^2 > 1$ and $\tan^{-1}\left(\frac{b}{a}\right) = -\frac{2\pi}{9}$.

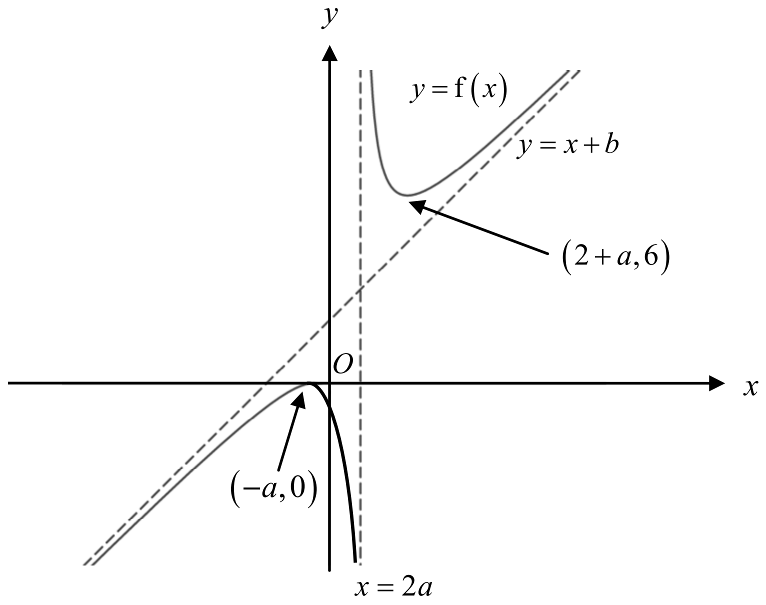
(i) Express the complex number $\frac{1}{p^2}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r is in terms of a and b, and $-\pi < \theta \le \pi$. [2]

(ii) On a single Argand diagram, illustrate the points P and Q representing the complex numbers p and $\frac{1}{p^2}$ respectively, labelling clearly their modulus and argument. [2]

(iii) It is given that $\angle OPQ = \alpha$. Using sine rule, show that $|p|^3 \approx \frac{\sqrt{3}}{2\alpha} - \frac{1}{2} - \frac{\alpha}{2\sqrt{3}}$ where α is small. [4]

6 The diagram shows the graph of the function $y = \mathrm{f}(x)$ where, $a,\ b \in \mathbb{R},\ b \geq 2$ and $0 < a < 1$.

The coordinates of the minimum point and maximum point on the curve are $(-a,\ 0)$ and $(2+a,\ 6)$ respectively. The equations of the asymptotes are $y = x + b$ and $x = 2a$.



On separate diagrams, sketch the graphs of the following functions, labelling the coordinates of any points of intersection with the x-axis, the coordinates of any turning points and the equations of any asymptotes.

(i) $y = \mathrm{f}(2x-1)+1$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

The two asymptotes of $y = \mathrm{f}(x)$ intersect at point *P*. Show that *P* lies on the line $y = mx + (b + 2a - 2am)$ for all real values of *m*. Hence, state the range of values of *m* for which the line $y = mx + (b + 2a - 2am)$ does not cut the curve $y = \mathrm{f}(x)$. [3]

7 **(a)** Find $\int \mathrm{e}^{x} \cos(2x)\,\mathrm{d}x$. [3]

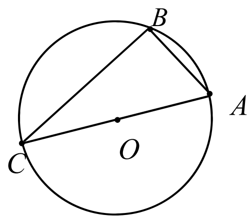
(b) The curve *C* has parametric equations

$$x = t - \mathrm{e}^{t}, \quad y = 3\cos^{2} t - 1, \quad \text{for } 0 < t < \pi.$$

(i) Use differentiation to find the exact x-coordinate of any turning point and determine the nature of the turning point. [3]

(ii) Find the exact area of the region bounded by the curve *C* and the line $y = 2$, expressing your answer in the form $a\pi + b + c\mathrm{e}^{\pi}$, where *a*, *b* and *c* are rational numbers to be determined. [5]

8



The diagram above shows the cross-section of a sphere containing the centre O of the sphere. The points A, B and C are on the circumference of the cross-section with the line segment AC passing through O. It is given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) Find $\overrightarrow{BC}$ in terms of **a** and **b**. [1]

(ii) D is a point on BC such that ΔOCD is similar to ΔACB. Find $\overrightarrow{OD}$ in terms of **a** and **b**. [2]

Point B lies on the x-z plane and has a positive z-component. It is also given that $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\angle OCB = \dfrac{\pi}{6}$.

(iii) Show that $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}$. [4]

(iv) Hence, determine whether the line passing through O and B and the line $\dfrac{x-2}{3} = \dfrac{y}{3} = z - 1$ are skew. [3]

9 The parametric equations of the curve C are

$$x = 2\sec t \text{ and } y = 3\tan t \text{, where } -\pi < t \le \pi,\ t \ne \pm\frac{\pi}{2}.$$

(i) Write down the Cartesian equation of C. [1]

(ii) Sketch the curve C, stating the equations of the asymptotes and the coordinates of the points where C crosses the axes, if any. [2]

(iii) The line $y = \sqrt{3}x + k$, where $k < 0$, is a tangent to C. Show that $k = -\sqrt{3}$. [3]

The region bounded by this tangent, the curve C and the x-axis is rotated completely about the x-axis. Calculate the volume obtained. [4]

10

(a) By using the substitution $u = \dfrac{y}{x}$, show that the differential equation

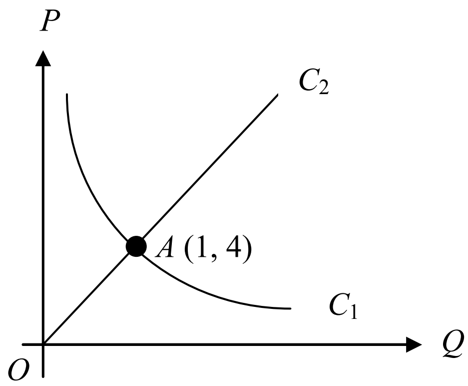
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 + xy + x^2}{x^2}, \text{ where } x > 0,$$

can be reduced to $\dfrac{1}{u^2+1}\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}$. Hence, find y in terms of x. [5]

(b) In the diagram below, the curve C_1 and the line C_2 illustrate the relationship between price (P dollars per kg) and quantity (Q tonnes) for consumers and producers respectively.

The curve C_1 shows the quantity of rice that consumers will buy at each price level while the line C_2 shows the quantity of rice that producers will produce at each price level. C_1 and C_2 intersect at point A, which has the coordinates (1, 4).

The quantity of rice that consumers will buy is inversely proportional to the price of the rice. The quantity of rice that producers will produce is directly proportional to the price.



(i) Interpret the coordinates of A in the context of the question. [1]

(ii) Solve for the equations of C_1 and C_2, expressing Q in terms of P. [2]

Shortage occurs when the quantity of rice consumers will buy exceeds the quantity of rice producers will produce. It is known that the rate of increase of P after time t months is directly proportional to the quantity of rice in shortage.

(iii) Given that the initial price is \$3 and that after 1 month, the price is \$3.65, find P in terms of t and sketch this solution curve, showing the long-term behaviour of P.

[7]

Suggest a reason why producers might use $P = aQ + b$, where $a, b \in \mathbb{R}^+$, instead of C_2 to model the relationship between price and quantity of rice produced. [1]

ANNEX B

NJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Vectors	$\alpha = -2$; $\left\{ \mathbf{w} : \mathbf{w} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \right\}$
2	Sigma Notation and Method of Difference	3; $7^{-2n}(8n-7)(7-6n)$; $e^{22} - e^{11} - 11e^2 + 11e$
3	Differentiation & Applications	-0.337 radians per minute; $h = -\dfrac{\sqrt{3}\theta^2}{2} + 2\theta + \sqrt{3}$
4	AP and GP	$30 + 110\left(1 - \left(\dfrac{10}{11}\right)^n\right) - \dfrac{n}{2000}(201 - n)$; 59
5	Complex numbers	$w = 3 - \mathrm{i},\ z = 1 + \mathrm{i}$ $w = -1 - \mathrm{i},\ z = 1 - 3\mathrm{i}$; $\dfrac{1}{p^2} = \dfrac{1}{a^2 + b^2} e^{\mathrm{i}\left(\frac{4\pi}{9}\right)}$; $\dfrac{\sqrt{3}}{2x} - \dfrac{1}{2} - \dfrac{1}{2\sqrt{3}} x$
6	Graphs and Transformation	$m \le 1$
7	Application of Integration	$\dfrac{e^x(\cos 2x + 2\sin 2x)}{5} + c$; x-coordinate of the minimum point at is $\dfrac{\pi}{2} - e^{\frac{\pi}{2}}$; $-\dfrac{3}{2}\pi - \dfrac{6}{5} + \dfrac{6}{5}e^{\pi}$
8	Vectors	$-(\mathbf{a} + \mathbf{b})$; $\dfrac{\mathbf{b} - \mathbf{a}}{2}$; lines are skew
9	Application of Integration	$\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$;

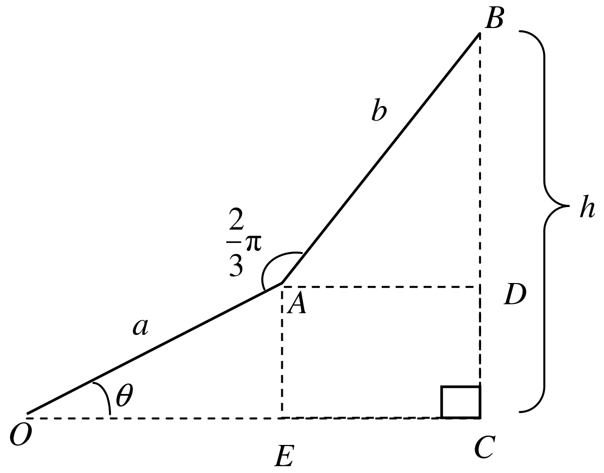
		$9.42\,(3\text{ sf})$
10	Differential Equations	$y = x\tan(\ln x + c)$; $C_1 : Q = \frac{4}{P}; C_2 : Q = \frac{p}{4}$; $P = \sqrt{16 - 7\mathrm{e}^{-0.961t}}$
11	Q11 Topic	
12	Q12 Topic	
13	Q13 Topic	

H2 Mathematics 2017 Prelim Exam Paper Solution

1	**(i)** Method 1 **p.q = 0** $\mathbf{p}.\mathbf{q} = 0$ $\begin{pmatrix}2\\ \alpha\\ 1\end{pmatrix}.\begin{pmatrix}\alpha\\ 1\\ 6\end{pmatrix}=0$ $2\alpha+\alpha+6=0$ $\alpha=-2$ Method 2 (for marking reference) Let $\mathbf{w}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$. $\mathbf{w}\times\mathbf{p}=\mathbf{q}$ $\begin{pmatrix}x\\ y\\ z\end{pmatrix}\times\begin{pmatrix}2\\ \alpha\\ 1\end{pmatrix}=\begin{pmatrix}\alpha\\ 1\\ 6\end{pmatrix}$ $\begin{pmatrix}y-\alpha z\\ 2z-x\\ \alpha x-2y\end{pmatrix}=\begin{pmatrix}\alpha\\ 1\\ 6\end{pmatrix}$ Thus, $y-\alpha z=\alpha$ ------(1) $2z-x=1$ ------(2) $\alpha x-2y=6$ ------(3) $(2)\times\alpha+(3)$: $2\alpha z-2y=\alpha+6$ $\Rightarrow 2(\alpha z-y)=\alpha+6$ $\Rightarrow 2(-\alpha)=\alpha+6$ (from (1)) $\Rightarrow \alpha=-2$ **(ii)**

	Let $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. $\mathbf{w} \times \mathbf{p} = \mathbf{q}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$ $\begin{pmatrix} y+2z \\ 2z-x \\ -2x-2y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$ $y+2z=-2$ ------(1) $2z-x=1$ ------(2) $-2x-2y=6$ ------(3) Let $z=\lambda, \lambda \in \mathbb{R}$. From (2): $x=-1+2\lambda$ From (1): $y+2\lambda=-2 \Rightarrow y=-2-2\lambda$ Thus, $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1+2\lambda \\ -2-2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$, which is the vector equation of the straight line. The set of vectors is $\left\{ \mathbf{w} : \mathbf{w} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \right\}$
2	**(a) (i)** By GC, sum to infinity is 3. **(a) (ii)**

	$u_n = S_n - S_{n-1}$ $= 3 + 7^{-2n}\left(n^2\right) - \left[3 + 7^{-2(n-1)}\left(n-1\right)^2\right]$ $= 3 - 3 + 7^{-2n}\left(n^2\right) - 7^{-2n+2}\left(n^2 - 2n + 1\right)$ $= 7^{-2n}\left(n^2 - 49n^2 + 98n - 49\right)$ $= 7^{-2n}\left(-48n^2 + 98n - 49\right)$ $= 7^{-2n}\left(8n-7\right)\left(7-6n\right)$ where $g(n) = -48n^2 + 98n - 49$ **(b)** $\sum_{r=1}^{n}\left(\int_0^r e^x - e^{x-1} dx\right)$ $= \sum_{r=1}^{n}\left[e^x - e^{x-1}\right]_0^r$ $= \sum_{r=1}^{n}\left(e^r - e^{r-1} - e^0 + e^{-1}\right)$ $= e^1 - e^0 - e^0 + e^{-1}$ $+ e^2 - e^1 - e^0 + e^{-1}$ $+ e^3 - e^2 - e^0 + e^{-1}$ ... $+ e^n - e^{n-1} - e^0 + e^{-1}$ $= e^n - 1 - n(1) + ne^{-1}$ $= e^n + ne^{-1} - \left(n+1\right)$ $\sum_{r=10}^{20}\left(\int_0^r e^{x+2} - e^{x+1}\ dx\right)$ $= e^2\sum_{r=1}^{20}\left(\int_0^r e^x - e^{x-1}\ dx\right) - e^2\sum_{r=1}^{9}\left(\int_0^r e^x - e^{x-1}\ dx\right)$ $= e^2\left[e^{20} + 20e^{-1} - \left(20+1\right) - \left(e^9 + 9e^{-1} - 10\right)\right]$ $= e^{22} - e^{11} - 11e^2 + 11e$
3	<u>Method 1</u>



$h = BD + DC$, $DC = a\sin\theta$, $BD = b\sin\angle BAD$

$\angle BAD + \angle DAE + \angle OAE = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$

$\Rightarrow \angle BAD + \frac{\pi}{2} + \left(\frac{\pi}{2} - \theta\right) = \frac{4\pi}{3}$

$\Rightarrow \angle BAD = \theta + \frac{\pi}{3}$

$\Rightarrow BD = b\sin\left(\theta + \frac{\pi}{3}\right)$

$\therefore h = a\sin\theta + b\sin\left(\theta + \frac{\pi}{3}\right)$

Using $\sin(A+B) = \sin A\cos B + \cos A\sin B$, we have

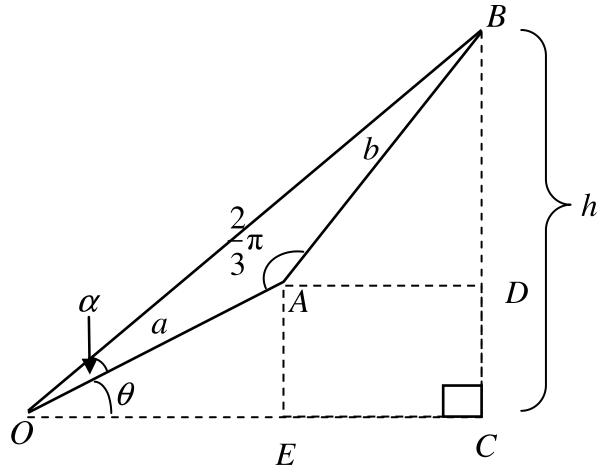
$$\begin{aligned} h &= a\sin\theta + b\sin\left(\theta + \frac{\pi}{3}\right) \\ &= a\sin\theta + b\sin\theta\cos\left(\frac{\pi}{3}\right) + b\cos\theta\sin\left(\frac{\pi}{3}\right) \\ &= \left(a + \frac{b}{2}\right)\sin\theta + \frac{b\sqrt{3}}{2}\cos\theta \\ &= R\sin(\theta + \alpha), \end{aligned}$$

where $R=\sqrt{\left(a+\frac{b}{2}\right)^2+\left(\frac{\sqrt{3}}{2}b\right)^2}=\sqrt{a^2+ab+\frac{b^2}{4}+\frac{3b^2}{4}}=\sqrt{a^2+ab+b^2}$

$$\tan\alpha=\frac{\frac{\sqrt{3}}{2}b}{a+\frac{b}{2}}=\frac{\sqrt{3}b}{2a+b}\Rightarrow\alpha=\tan^{-1}\left(\frac{\sqrt{3}b}{2a+b}\right)$$

$$\therefore h=\sqrt{a^2+ab+b^2}\sin\left[\theta+\tan^{-1}\left(\frac{\sqrt{3}b}{2a+b}\right)\right]$$

<u>Method 2</u>



$$\sin(\theta+\alpha)=\frac{h}{OB}$$

$$OB^2=a^2+b^2-2ab\cos\frac{2\pi}{3}$$

$$OB=\sqrt{a^2+b^2+ab}$$

$$\frac{\sin\alpha}{b}=\frac{\sin\frac{2\pi}{3}}{\sqrt{a^2+b^2+ab}}$$

$$\sin\alpha=\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}$$

$$\alpha=\sin^{-1}\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}$$

$$h=\sqrt{a^2+ab+b^2}\,\sin\left(\theta+\sin^{-1}\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}\right)$$

(ii)

Since $a=1, b=2, \alpha=\tan^{-1}\left(\frac{2\sqrt{3}}{2+2}\right)=\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$$h=\sqrt{7}\sin\left[\theta+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right], \frac{\mathrm{d}h}{\mathrm{d}\theta}=\sqrt{7}\cos\left[\theta+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

At $\theta=\frac{\pi}{12}$,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{\sqrt{7}\cos\left[\frac{\pi}{12}+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]}\times(-0.5)$$

$=-0.337$ radians per minute

(iii)

$$\alpha=\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\Rightarrow\sin\alpha=\sqrt{\frac{3}{7}}, \cos\alpha=\frac{2}{\sqrt{7}}$$

$$h=\left(a+\frac{b}{2}\right)\sin\theta+\frac{b\sqrt{3}}{2}\cos\theta$$

If θ is small,

$$\begin{aligned}h&=\sqrt{7}\sin(\theta+\alpha)\\&=\sqrt{7}\sin\theta\cos\alpha+\sqrt{7}\cos\theta\sin\alpha\\&\approx\sqrt{7}\theta\left(\frac{2}{\sqrt{7}}\right)+\sqrt{7}\left(1-\frac{\theta^2}{2}\right)\left(\sqrt{\frac{3}{7}}\right)\\&=2\theta+\sqrt{3}-\frac{\sqrt{3}\theta^2}{2}\\&=-\frac{\sqrt{3}\theta^2}{2}+2\theta+\sqrt{3}\end{aligned}$$

4 **(i)**

Stretch count, n	Length of string before stretch	Elongation after stretch, u_n	Contraction after stretch, t_n	Final length of string
1	30	10	0.1	$30+10-0.1=39.9$

2	39.9	$10\left(\frac{10}{11}\right)$	$0.1-0.001$ $=0.099$	$39.9+10\left(\frac{10}{11}\right)-0.099$ $=48.8919$

$30+10-0.1+10\left(\frac{10}{11}\right)-0.099$

$=48.8919$

$=48.892\ (3\text{ dp})$

(ii)

Total length of string

$=30+u_1+u_2+...+u_n-(t_1+t_2+...+t_n)$

$=30+10+10\left(\frac{10}{11}\right)+...+10\left(\frac{10}{11}\right)^{n-1}$

$-\left(0.1+\left(0.1-0.001(1)\right)+...+\left(0.1-0.001(n)\right)\right)$

$$\sum_{i=1}^{n}u_i=\frac{10\left[1-\left(\frac{10}{11}\right)^n\right]}{1-\frac{10}{11}}=110\left(1-\left(\frac{10}{11}\right)^n\right)$$

$$\sum_{i=1}^{n}t_i=\frac{n}{2}\left[2(0.1)+(n-1)(-0.001)\right]=\frac{n}{2000}(201-n)$$

Length of string after n stretches

$=30+110\left(1-\left(\frac{10}{11}\right)^n\right)-\frac{n}{2000}(201-n)$

(iii)

$t_n>u_n$

$0.1+(n-1)(-0.001)>(10)\left(\frac{10}{11}\right)^{n-1}$

$0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}>0$

Using GC,

when $n=58$, $0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}=-7.1364\times10^{-4}$

when $n=59$, $0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}=0.00226$

Therefore, the minimum number of stretches is 59.

(iv)

$$S_\infty = 30 + \frac{10}{1-\frac{10}{11}} = 140 \quad (\text{since } 0 < r < 1)$$

Since the sum to infinity, S_∞ is 140, it is impossible for the string to be stretched beyond 140cm.

OR

The theoretical maximum is 140 cm so it is impossible for the strong to be stretched beyond 140 cm.

5 **(a)**

$= 2$ ------(1)

$2+4\text{i}$ ------(2)

From (1),

$$z = \frac{2-w}{\text{i}} = -\text{i}(2-w) \text{ ------(3)}$$

Substitute (3) into (2) and let $w = x + \text{i}y$:

$$-\text{i}(2-w)w^* = 2+4\text{i}$$
$$-\text{i}\left(2w^* - ww^*\right) = 2+4\text{i}$$
$$-\text{i}\left[2(x-\text{i}y) - \left(x^2+y^2\right)\right] = 2+4\text{i}$$
$$-2y - \text{i}\left(2x - x^2 - y^2\right) = 2+4\text{i}$$

Comparing real and imaginary parts,

$$-2y = 2 \Rightarrow y = -1$$
$$-2x + x^2 + y^2 = 4$$
$$\Rightarrow -2x + x^2 + (-1)^2 = 4$$
$$\Rightarrow x^2 - 2x - 3 = 0$$
$$\Rightarrow (x-3)(x+1) = 0$$
$$\Rightarrow x = 3 \text{ or } x = -1$$

$\therefore w = 3 - \text{i}$ or $w = -1 - \text{i}$

If $w = 3 - \text{i}$, $z = -\text{i}\left(2 - (3-\text{i})\right) = 1 + \text{i}$.

If $w = -1 - \text{i}$, $z = -\text{i}\left(2 - (-1-\text{i})\right) = 1 - 3\text{i}$.

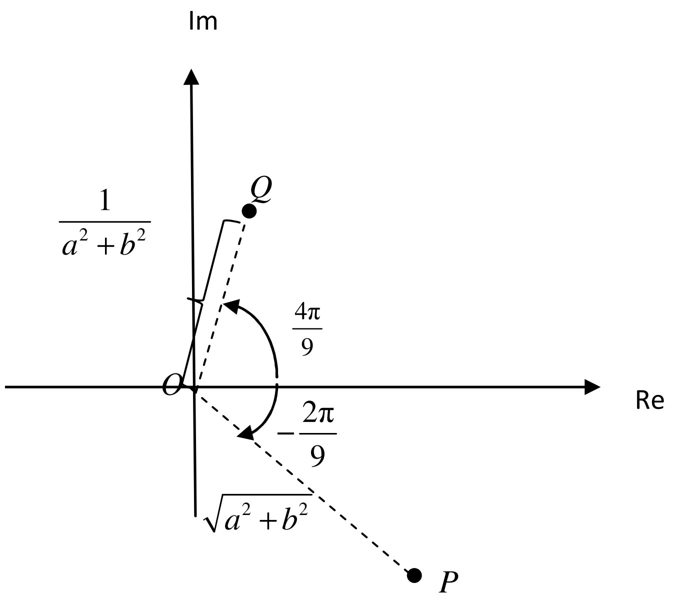
(b)(i)

$$\left|\frac{1}{p^2}\right| = \frac{1}{|p|^2} = \frac{1}{\left(\sqrt{a^2+b^2}\right)^2} = \frac{1}{a^2+b^2}$$

$$\arg\left(\frac{1}{p^2}\right) = -2\arg(p) = -2\left(\frac{-2\pi}{9}\right) = \frac{4\pi}{9}$$

$$\therefore \frac{1}{p^2} = \frac{1}{a^2+b^2}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{9}\right)}$$

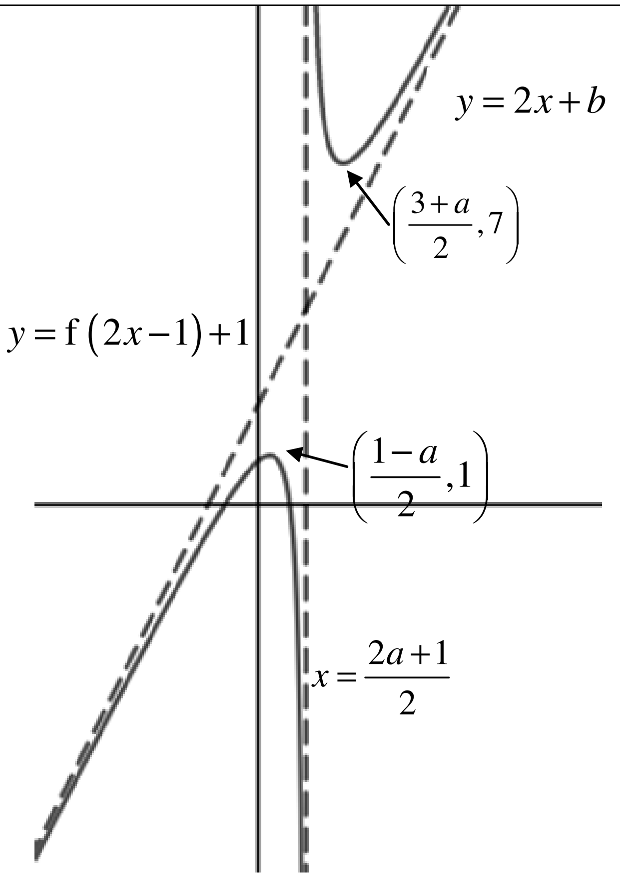
(b)(ii)



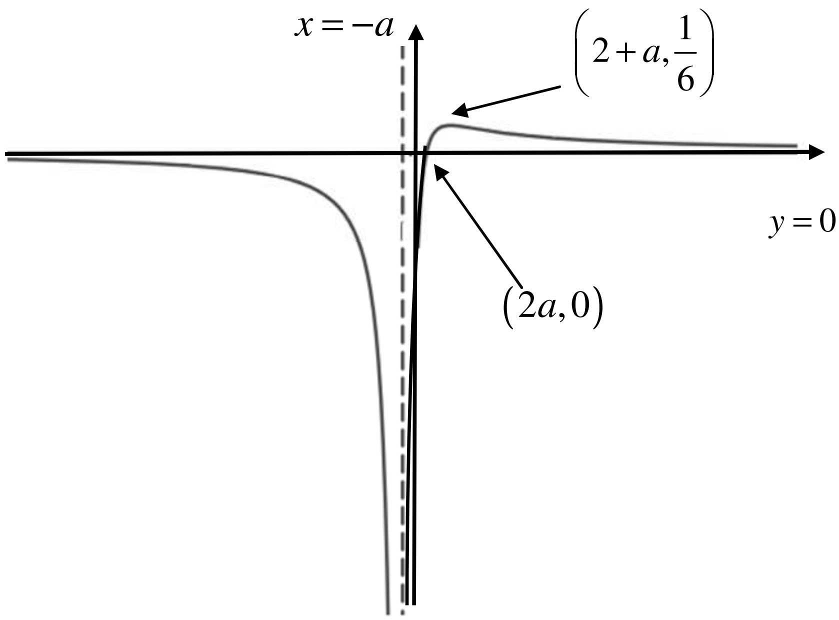
(b)(iii)

Given $\angle OPQ = \alpha$, $\dfrac{\sin\alpha}{1/\left(a^2+b^2\right)} = \dfrac{\sin\left(\frac{\pi}{3}-\alpha\right)}{\sqrt{a^2+b^2}}$

	$\left(a^2+b^2\right)^{\frac{3}{2}}=\dfrac{\sqrt{3}\cos\alpha-\sin\alpha}{2\sin\alpha}$ $\approx\dfrac{\sqrt{3}\left(1-\dfrac{x^2}{2}\right)-\left(x-\dfrac{x^3}{6}\right)}{2\left(x-\dfrac{x^3}{6}\right)}$ $=\dfrac{1}{2}\left[\dfrac{1}{x}\left(\sqrt{3}-x-\dfrac{\sqrt{3}x^2}{2}+\dfrac{x^3}{6}\right)\left(1-\dfrac{x^2}{6}\right)^{-1}\right]$ $\approx\dfrac{1}{2}\left[\dfrac{1}{x}\left(\sqrt{3}-x-\dfrac{\sqrt{3}x^2}{2}+\dfrac{x^3}{6}\right)\left(1+(-1)\left(-\dfrac{x^2}{6}\right)\right)\right]$ $=\dfrac{1}{2}\left[\left(\dfrac{\sqrt{3}}{x}-1-\dfrac{\sqrt{3}x}{2}+\dfrac{x^2}{6}\right)\left(1+\dfrac{x^2}{6}\right)\right]$ $=\dfrac{1}{2}\left[\dfrac{\sqrt{3}}{x}+\dfrac{\sqrt{3}}{6}x-1-\dfrac{x^2}{6}-\dfrac{\sqrt{3}x}{2}+\dfrac{x^2}{6}\right]$ $=\dfrac{1}{2}\left[\dfrac{\sqrt{3}}{x}-\dfrac{2\sqrt{3}}{6}x-1\right]$ $=\dfrac{\sqrt{3}}{2x}-\dfrac{1}{2}-\dfrac{1}{2\sqrt{3}}x$
6	**(i)**



(ii)



(iii)

Point P is $(2a, 2a+b)$

$$\frac{y-(2a+b)}{x-2a}=m \Rightarrow y=mx-2am+2a+b$$

Hence, P lies on the line $y=mx+(b+2a-2am)$ for $m\in\mathbb{R}$.

From the graph, $m\le 1$ for the line not to cut $y=\mathrm{f}(x)$.

---

7 **(a)**

$$\int \mathrm{e}^x\cos 2x\,\mathrm{d}x=\mathrm{e}^x\cos 2x+2\int \mathrm{e}^x\sin 2x\,\mathrm{d}x$$

$$=\mathrm{e}^x\cos 2x+2\left(\mathrm{e}^x\sin 2x-2\int \mathrm{e}^x\cos 2x\,\mathrm{d}x\right)$$

$$5\int \mathrm{e}^x\cos 2x\,\mathrm{d}x=\mathrm{e}^x\cos 2x+2\mathrm{e}^x\sin 2x$$

$$\therefore \int \mathrm{e}^x\cos 2x\,\mathrm{d}x=\frac{\mathrm{e}^x(\cos 2x+2\sin 2x)}{5}+c$$

(b)(i)

$$x=t-\mathrm{e}^t,\quad y=3\cos^2 t-1$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=1-\mathrm{e}^t,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=6\cos t(-\sin t)=-3\sin(2t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-3\sin(2t)}{1-\mathrm{e}^t}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow \sin(2t)=0$$

$$\Rightarrow 2t=0\text{ (N.A.) or }2t=\pi\text{ or }2t=2\pi\text{(N.A.)}$$

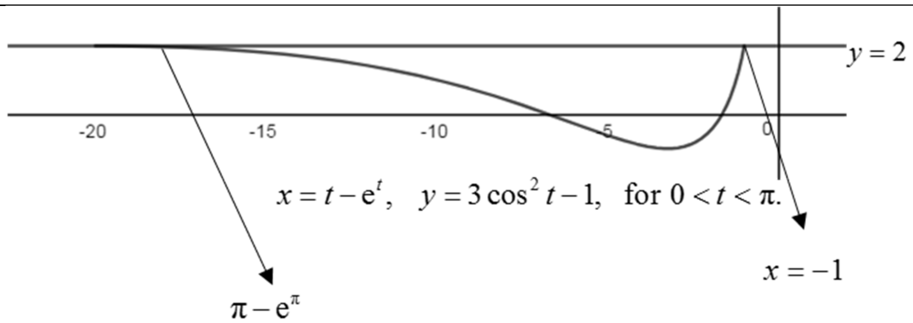
$$\Rightarrow t=\frac{\pi}{2}$$

t	1.6	$\frac{\pi}{2}$	1.5
x	-3.35303	-3.2396811	-2.981689
$\frac{\mathrm{d}y}{\mathrm{d}x}$	-0.0443	0	0.122

NB: t increases as x decreases.

Hence x-coordinate of the minimum point at is $\frac{\pi}{2}-\mathrm{e}^{\frac{\pi}{2}}$.

(b)(ii)



When $y=2$,

$2=3\cos^2 t-1$

$\Rightarrow \cos t=\pm 1$

$\Rightarrow t=0,\pi$

When $t=0, x=0-\mathrm{e}^0=-1$

When $t=\pi, x=\pi-\mathrm{e}^{\pi}=-19.9991$

Area required

$$=\int_{\pi-\mathrm{e}^{\pi}}^{-1}(2-y)\,\mathrm{d}x$$

$$=\int_{\pi}^{0}\left(2-\left(3\cos^2 t-1\right)\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$$

$$=\int_{\pi}^{0}\left(3-3\cos^2 t\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$$

$$=3\int_{\pi}^{0}\left(1-\cos^2 t\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$$

$$=3\int_{\pi}^{0}\left(1-\frac{\cos 2t+1}{2}\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$$

$$=3\int_{\pi}^{0}\left(\frac{1}{2}-\frac{\cos 2t}{2}\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$$

$$=\frac{3}{2}\int_{\pi}^{0}\left(1-\cos 2t\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$$

$$=\frac{3}{2}\int_{\pi}^{0}\left(1-\cos 2t-\mathrm{e}^t+\mathrm{e}^t\cos 2t\right)\mathrm{d}t$$

$$=\frac{3}{2}\left[t-\frac{\sin 2t}{2}-\mathrm{e}^t+\frac{\mathrm{e}^t\left(\cos 2t+2\sin 2t\right)}{5}\right]_{\pi}^{0}$$

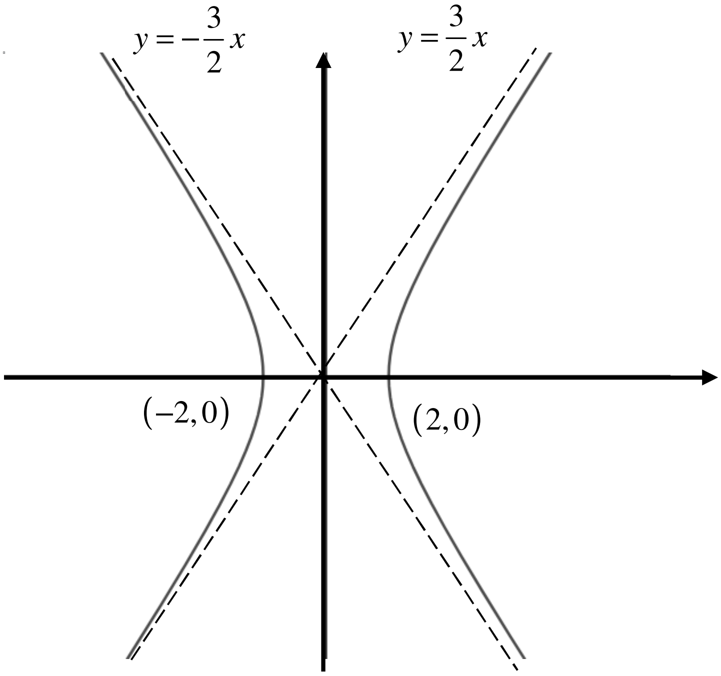
$$=\frac{3}{2}\left[-\frac{4}{5}-\pi+\frac{4e^{\pi}}{5}\right]$$

$$=-\frac{3}{2}\pi-\frac{6}{5}+\frac{6}{5}\mathrm{e}^{\pi},\ \text{where } a=-\frac{3}{2},\ b=-\frac{6}{5},\ c=\frac{6}{5}$$

8	**(i)** $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=-\mathbf{a}-\mathbf{b}=-(\mathbf{a}+\mathbf{b})$ **(ii)** Since ΔOCD is similar to ΔACB, OD parallel to AB. $\frac{OD}{AB}=\frac{CO}{CA}=\frac{1}{2}$. $\overrightarrow{OD}=\frac{1(-\mathbf{a})+1(\mathbf{b})}{2}=\frac{\mathbf{b}-\mathbf{a}}{2}$ **(iii)** Let $\overrightarrow{OB}=\begin{pmatrix}x\\0\\z\end{pmatrix}$. $\overrightarrow{CB}=\begin{pmatrix}x\\0\\z\end{pmatrix}-\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}x-2\\0\\z\end{pmatrix}$ $\overrightarrow{CB}.\overrightarrow{CO}=\begin{pmatrix}x-2\\0\\z\end{pmatrix}.\begin{pmatrix}-2\\0\\0\end{pmatrix}=\left\|\overrightarrow{CB}\right\|(2)\cos\frac{\pi}{6}---(1)$ $\left\|\overrightarrow{CB}\right\|=4\cos\frac{\pi}{6}=2\sqrt{3}$ $-2x+4=2\sqrt{3}(2)\frac{\sqrt{3}}{2}$ $x=-1$ $\left\|\overrightarrow{OB}\right\|=\left\|\begin{pmatrix}-1\\0\\z\end{pmatrix}\right\|=2$ $(-1)^2+z^2=2^2$ $z^2=3$ $z=\pm\sqrt{3}$ $\overrightarrow{OB}=\begin{pmatrix}-1\\0\\\sqrt{3}\end{pmatrix}$ or $\begin{pmatrix}-1\\0\\-\sqrt{3}\end{pmatrix}$ (rejected $\because$ z-component >0).

	(iv) Equation of line passing through *OB*: $\overrightarrow{OB} = \lambda\begin{pmatrix}-1\\0\\\sqrt{3}\end{pmatrix},\ \lambda \in \square$ $\frac{x-2}{3} = \mu \Rightarrow x = 2+3\mu$ $\frac{y}{3} = \mu \Rightarrow y = 3\mu$ $z-1 = \mu \Rightarrow z = \mu+1$ Equation of line: $\mathbf{r} = \begin{pmatrix}2\\0\\1\end{pmatrix}+\mu\begin{pmatrix}3\\3\\1\end{pmatrix}, \mu \in \square$ Direction vector of line is not parallel to direction vector of line passing through *O* and *B* since direction vectors of both lines are not scalar multiple of each other. Solving equations simultaneously: $\begin{pmatrix}2+3\mu\\3\mu\\\mu+1\end{pmatrix} = \lambda\begin{pmatrix}-1\\0\\\sqrt{3}\end{pmatrix}$ There is no value of λ and μ that satisfy the above equation. Since the lines are not parallel and non-intersecting, the lines are skew.
9	**(i)** $x = 2\sec t$ and $y = 3\tan t$ $1+\tan^2 t = \sec^2 t$ $\Rightarrow 1+\frac{y^2}{9} = \frac{x^2}{4}$ $\Rightarrow \frac{x^2}{4}-\frac{y^2}{9} = 1$

	(ii)



(iii)

Method 1

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=3\sec^2 t\cdot\frac{1}{2\sec t\tan t}=1.5\operatorname{cosec} t$$

$1.5\operatorname{cosec} t=\sqrt{3}$

$t=\frac{\pi}{3}$ or $\frac{2\pi}{3}$

When $t=\frac{\pi}{3}$,

$x=2\sec\frac{\pi}{3}=4$, $y=3\tan\frac{\pi}{3}=3\sqrt{3}.$

Equation of tangent:

$y-3\sqrt{3}=\sqrt{3}(x-4)$

$\Rightarrow y=\sqrt{3}x-4\sqrt{3}+3\sqrt{3}$

$\Rightarrow y=\sqrt{3}x-\sqrt{3}$

$\therefore k=-\sqrt{3}$ (Shown)

When $t = \dfrac{2\pi}{3}$,

$x = 2\sec\dfrac{2\pi}{3} = -4$, $y = 3\tan\dfrac{2\pi}{3} = -3\sqrt{3}$.

Equation of tangent:

$$y + 3\sqrt{3} = \sqrt{3}(x+4)$$
$$\Rightarrow y = \sqrt{3}x + 4\sqrt{3} - 3\sqrt{3}$$
$$\Rightarrow y = \sqrt{3}x + \sqrt{3}$$
$$\therefore k = \sqrt{3} \text{ (N.A. } \because k < 0)$$

Method 2

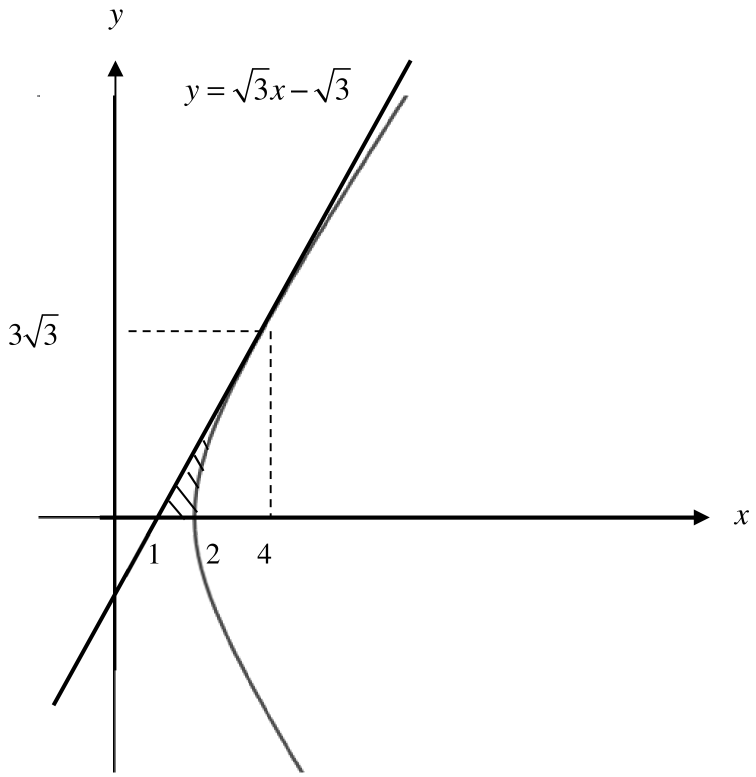
$$\frac{x^2}{4} - \frac{\left(\sqrt{3}x + k\right)^2}{9} = 1$$

$$\Rightarrow -3x^2 - \left(8\sqrt{3}k\right)x - \left(36 + 4k^2\right) = 0$$

Since the line $y = \sqrt{3}x + k$, where $k < 0$, is a tangent to C, there should be repeated roots.

Thus,

$$\left(8\sqrt{3}k\right)^2 - 4(-3)\left(-36 - 4k^2\right) = 0$$
$$\Rightarrow 192k^2 - 432 - 48k^2 = 0$$
$$\Rightarrow 144k^2 = 432$$
$$\Rightarrow k^2 = 3$$
$$\Rightarrow k = \sqrt{3} \text{ (N.A. } \because k < 0) \text{ or } k = -\sqrt{3} \text{ (Shown)}$$



For $k=-\sqrt{3}$,

$-3x^2-\left(8\sqrt{3}\left(-\sqrt{3}\right)\right)x-\left(36+4(3)\right)=0$

$3x^2-24x+48=0$

$x^2-8x+16=0$

$\left(x-4\right)^2=0$

$\Rightarrow x=4$

$y^2=9\left(\frac{x^2}{4}-1\right)$

When $x=4$, $y=\sqrt{27}=3\sqrt{3}\ (y>0)$

$=\frac{1}{3}\pi\left(3\sqrt{3}\right)^2(3)-\pi\int_2^4 y^2\ \mathrm{d}x$

$=27\pi-9\pi\int_2^4\left(\frac{x^2}{4}-1\right)\ \mathrm{d}x$

$=27\pi-9\pi\left(\frac{8}{3}\right)$

$=3\pi$

$=9.42\ (3\text{ sf})$

10

(a)

$$u=\frac{y}{x}\Rightarrow y=ux\ ,\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2+xy+x^2}{x^2}=\left(\frac{y}{x}\right)^2+\frac{y}{x}+1\text{ ---(1)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}x+u\text{ -- (2)}$$

Sub (2) into (1):

$$\frac{\mathrm{d}u}{\mathrm{d}x}x+u=u^2+u+1\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}x=u^2+1$$

$$\frac{1}{u^2+1}\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$$

$$\int\frac{1}{u^2+1}\mathrm{d}u=\int\frac{1}{x}\mathrm{d}x$$

$\tan^{-1}u=\ln|x|+c$, where c is an arbitrary constant.

$\tan^{-1}u=\ln x+c$ (since $x>0$)

$$u=\tan(\ln x+c)$$

$$y=x\tan(\ln x+c)$$

(b)(i)

Point A shows that at 4 dollars per kg, 1 tonne of rice is produced and all of it is bought by the consumers.

This is the equilibrium point where the price is 4 dollars per kg and the quantity produced/consumed is 1 tonne.

(b)(ii)

$C_1: Q=\dfrac{k_1}{P}$

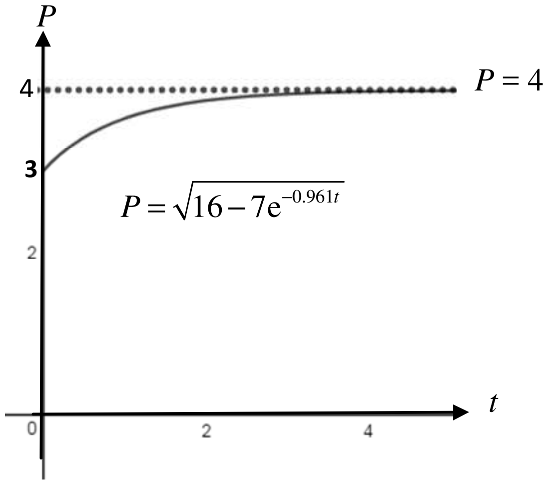
$C_2: Q=k_2P$

When $Q=1, P=4$,

$$k_1=4, k_2=\frac{1}{4}.$$

$$C_1: Q=\frac{4}{P}; C_2: Q=\frac{P}{4}$$

	Hence, $C_1 : Q = \dfrac{4}{P}; C_2 : Q = \dfrac{p}{4}$. **(b)(iii)** $\dfrac{\mathrm{d}P}{\mathrm{d}t} = k_3\left(\dfrac{4}{P} - \dfrac{P}{4}\right)$ $\dfrac{\mathrm{d}P}{\mathrm{d}t} = k_3\left(\dfrac{16-P^2}{4P}\right)$ $\int \dfrac{4P}{16-P^2}\,\mathrm{d}P = \int k_3 \mathrm{d}t$ $-2\int \dfrac{-2P}{16-P^2}\,\mathrm{d}P = \int k_3 \mathrm{d}t$ $-2\ln\left\lvert 16-P^2\right\rvert = k_3 t + c$ $\ln\left\lvert 16-P^2\right\rvert = \dfrac{-k_3}{2}t + \dfrac{-c}{2}$ $\left\lvert 16-P^2\right\rvert = \mathrm{e}^{\frac{-k_3}{2}t+\frac{-c}{2}}$ $16-P^2 = \left(\pm\mathrm{e}^{\frac{-c}{2}}\right)\mathrm{e}^{\frac{-k_3}{2}t}$ $16-P^2 = A\mathrm{e}^{Bt}, A = \pm\mathrm{e}^{\frac{-c}{2}}, B = \dfrac{-k_3}{2}$ $\sqrt{16-A\mathrm{e}^{Bt}} = P \ \left(P>0\right)$ When $t=0, P=3$: $\sqrt{16-A\mathrm{e}^{B(0)}} = 3$ $16-A = 3^2$ $A = 7$ When $t=1, P=3.65$: $\sqrt{16-7\mathrm{e}^{B}} = 3.65$ $B = \ln\dfrac{16-3.65^2}{7} = -0.96102663 = -0.961$ $\therefore P = \sqrt{16-7\mathrm{e}^{-0.961t}}$

	 Rice production will only occur if the <u>price is able to at least cover</u> the initial cost of investment.

NJC Paper 2

1 There are 3 bike-sharing companies in the current market. For each ride, α- bike charges a certain amount per 5 min block or part thereof, β- bike charges a certain amount per 10 min block or part thereof and μ- bike charges a certain amount per 15 min block or part thereof. Rebecca rode each of the bike-sharing companies' bikes once in each month. The table below shows the amount of time Rebecca clocked for each ride and her total spending for each month. In celebration of the company's first anniversary, the pricings in February and March 2017 of μ- bikes are a 5% discount off the immediate previous month's pricing.

	January 2017	**February 2017**	**March 2017**
α- bike	25 min	17 min	36 min
β- bike	30 min	10 min	39 min
μ- bike	15 min	44 min	33 min
Total spending	\$5.70	\$5.72	\$9.71

Determine which bike-sharing company offers the cheapest rate (without any discount) for a 40-min ride. Justify your answer clearly. [4]

2 A function f is said to self-inverse if $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ for all x in the domain of f.

The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{7-3x}{3-x}, \qquad x \in \mathbb{R}, x \neq 3,$$
$$\mathrm{g}: x \mapsto |(2-x)(1+x)|, \qquad x \in \mathbb{R}, x \in (-\infty,-1].$$

(i) Explain why f^{-1} exists and show that f is self-inverse. Hence, or otherwise, evaluate $\mathrm{f}^{2003}(5)$. [4]

(ii) Find an expression for $\mathrm{g}^{-1}(x)$. [3]

(iii) Sketch, on the same diagram, the graphs of $y=\mathrm{g}(x)$ and $y=\mathrm{g}^{-1}(x)$, illustrating clearly the relationship between the two graphs, and labelling the axial intercept(s), if any. Write down the set of values of x that satisfies the equation $\mathrm{g}\mathrm{g}^{-1}(x)=x$. [3]

(iv) Show that $\mathrm{f}\mathrm{g}^{-1}$ exists. Find the exact range of $\mathrm{f}\mathrm{g}^{-1}$. [3]

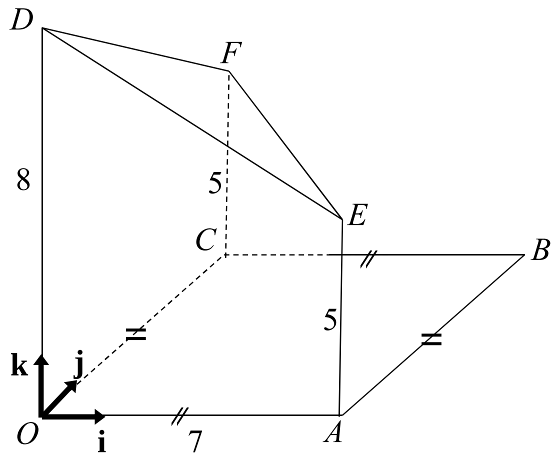
3 Using differentiation, find the Maclaurin's series of $\dfrac{\mathrm{e}^{2x}}{1+x^2}$, in ascending powers of x up to and including x^3. [6]

Let $\mathrm{h}(x)=\dfrac{\mathrm{e}^{2x}}{1+x^2}$ and the cubic polynomial obtained above be $\mathrm{f}(x)$.

Find, for $-2 \le x \le 2$, the set of values of x for which the value of $\mathrm{f}(x)$ is within ± 0.5 of the value of $\mathrm{h}(x)$. [3]

4 The diagram (not drawn to scale) shows the structure of a partially constructed building that is built on a horizontal ground. The building has a square base foundation of 7 m in length. Points O, A, B and C are the corners of the foundation of the building. The building currently consists of three vertical pillars OD, AE and CF of heights 8 m, 5 m and 5 m respectively.

A canvas is currently attached at D, E and F, forming a temporary shelter for the building. O is taken as the origin and vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, each of length 1 m, are taken along OA, OC and OD respectively.



(i) Find a Cartesian equation of the plane that represents the canvas DEF. [3]

(ii) Find the acute angle which the canvas DEF makes with the horizontal ground. [2]

(iii) Given that the canvas is to be extended along the plane DEF till it touches the horizontal ground, explain why point B will lie beneath the canvas. [2]

A cement roof is to be built to replace the extended canvas. A vertical partition wall is also to be built such that it is d m away from and parallel to the plane $ODFC$, where $0 < d < 7$.

(iv) Find the exact vector equation of the line where the roof meets the partition wall. Show your working clearly, leaving your answer in terms of d. [4]

(v) A lighting point, P, is to be placed on the roof such that it is closest to B. Find the position vector of P. [3]

5 A delegation of four students is to be selected from five badminton players, m floorball players, where $m > 3$, and six swimmers to attend the opening ceremony of the 2017 National Games. A pair of twins is among the floorball players. The delegation is to consist of at least one player from each sport.

(i) Show that the number of ways to select the delegation in which neither of the twins is selected is $k(m-2)(m+6)$, where k is an integer to be determined. [3]

(ii) Given that the number of ways to select a delegation in which neither of the twins is selected is more than twice the number of ways to select a delegation which includes exactly one of the twins, find the least value of m. [2]

The pair of twins, one badminton player, one swimmer and two teachers, have been selected to attend a welcome lunch at the opening ceremony. Find the number of ways in which the group can be seated at a round table with distinguishable seats if the pair of twins is to be seated together and the teachers are separated. [3]

6 In the fishery sciences, researchers often need to determine the length of a fish as a function of its age. The table below shows the average length, L inches, at age, t years, of a kind of fish called the North Sea Sole.

t	1	2	3	4	5	6	7	8
L	3.6	7.5	10.1	11.7	12.7	13.4	14.0	14.4

(i) Draw a scatter diagram of these data, and explain how you know from your diagram that the relationship between L and t should not be modelled by an equation of the form $L = at + b$. [3]

(ii) Which of the formulae $L = a\sqrt{t} + b$ and $L = c\ln t + d$, where a, b, c and d are constants, is the better model for the relationship between L and t? Explain fully how you decided, and find the constants for the better formula. [3]

(iii) Use the formula you chose from part **(ii)** to estimate the average length of a six-month old Sole. Explain whether your estimate is reliable. [2]

A popular approach to determine the average length of a fish as a function of its age is the von Bertalanffy model. The model shows the relationship between the average length that is yet to be grown, G inches, at age, t years. The maximum average length attained by the Sole is 14.8 inches.

(iv) The product moment correlation between L and t is given as r_1 while that between G and t is given as r_2. State the relationship between r_1 and r_2. [1]

7 There are three identically shaped balls, numbered from 1 to 3, in a bag. Balls are drawn one by one at random and with replacement. The random variable X is the number of draws needed for any ball to be drawn a second time. The two draws of the same ball do not need to be consecutive.

(i) Show that $\mathrm{P}(X = 4) = \dfrac{2}{9}$ and find the probability distribution of X. [3]

(ii) Show that $\mathrm{E}(X) = \dfrac{26}{9}$ and find the exact value of $\mathrm{Var}(X)$. [3]

(iii) The mean for forty-four independent observations of X is denoted by $\overline{X}$. Using a suitable approximation, find the probability that $\overline{X}$ exceeds 3. [3]

8 Heart rate, also known as pulse, is the number of times a person's heart beats per minute. The normal heart rate of teenagers has a mean of 75 at the resting state.

Obesity is a leading preventable cause of death worldwide. It is most commonly caused by a combination of excessive food intake, lack of physical activity and genetic susceptibility. To examine the effect of obesity on heart rate, 70 obese teenagers are randomly selected and their heart rates h are measured in a resting state. The results are summarised as follows.

$$n = 70 \qquad \sum h = 5411 \qquad \sum h^2 = 426433$$

The Health Promotion Board (HPB) wishes to test whether the mean heart rate for obese teenagers differs from the normal heart rate by carrying out a hypothesis test.

(i) Explain whether HPB should use a 1-tail test or a 2-tail test. [1]

(ii) Explain why HPB is able to carry out a hypothesis test without knowing anything about the distribution and variance of the heart rates. [2]

(iii) Find the unbiased estimates of the population mean and variance, and carry out the test at the 10% level of significance for the HPB. [6]

A researcher wishes to test whether obese teenagers have a **higher** mean heart rate. He finds that the mean heart rate for 80 randomly obese teenagers is 79.4, then carries out a hypothesis test at the 10% level of significance.

(iv) Explain, with justification, how the population variance of the heart rates will affect the conclusion made by the researcher. [3]

(v) Show that the probability of any normal variable lying within one standard deviation from its mean is approximately 0.683. [1]

By considering **(iv)** and **(v)**, explain why it is likely for the researcher to reject the null hypothesis in this test if it is assumed that heart rates follow a normal distribution at the resting state. [1]

9 The number of days of gestation for a Dutch Belted cow is normally distributed, with a mean of μ days and a standard deviation of σ days. 8.08% of this cattle breed has a gestation period shorter than 278 days whereas 21.2% has a gestation period longer than 289 days. Find the values of μ and σ, giving your answers correct to 3 significant figures. [3]

(i) Find the probability that the mean gestation period for thirty-two randomly chosen Dutch Belted cows is more than 287 days. State a necessary assumption for your calculation to be valid. [3]

For another cattle breed, the Jersey cow, the number of days of gestation is normally distributed with a mean of 278 days and a standard deviation of 2.5 days.

During gestation, a randomly chosen pregnant Dutch Belted cow eats 29 kg of feed daily while a randomly chosen pregnant Jersey cow eats 26 kg of feed daily.

(ii) Find the value of a such that during their respective gestation periods, there is a probability of 0.35 that the amount of feed consumed by a randomly chosen pregnant Jersey cow exceeds half of the amount consumed by a randomly chosen pregnant Dutch Belted cow by less than a kg. Express your answer to the nearest kg. [2]

(iii) Calculate the probability that during their respective gestation periods, the difference between the amount of feed consumed by three randomly chosen pregnant Dutch Belted cows and four randomly chosen pregnant Jersey cows is more than 4000 kg. State clearly the parameters of the distribution used in the calculation. [3]

10 Factory A manufactures a large batch of light bulbs. It is known that on average, 1 out of 200 light bulbs manufactured by Factory A, is defective. A random sample of 180 light bulbs is inspected. The batch is accepted if the sample contains less than r defective light bulbs.

(i) Explain why the context above may not be well-modelled by a binomial distribution. [1]

Assume now that the context above is well-modelled by a binomial distribution.

(ii) Determine the value of r such that the probability of accepting the batch is 0.998. [1]

In Factory B, a random sample of 30 light bulbs is taken from a large batch. If the sample contains no defective light bulbs, the batch is accepted. The batch is rejected if the sample contains more than two defective light bulbs. If the sample contains one or two defective light bulbs, a second random sample of 30 light bulbs is chosen and the batch is accepted only if this second sample contains no defectives. It is known that Factory B produces $(100p)$% defective light bulbs.

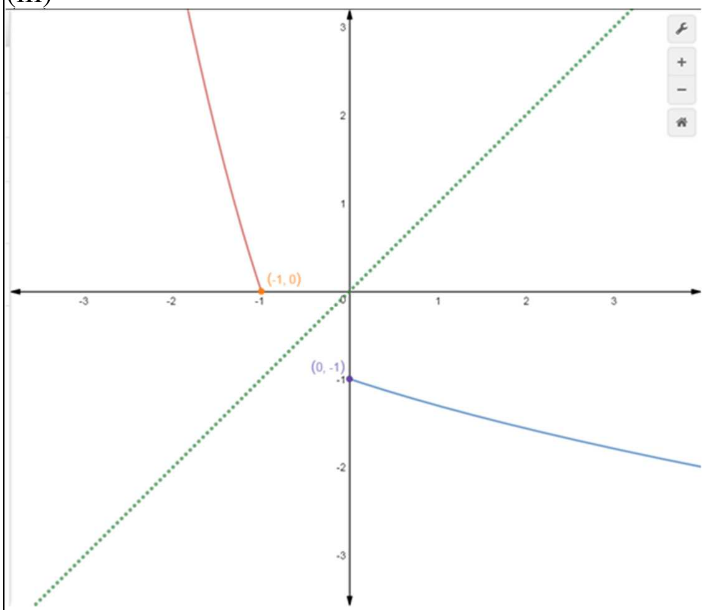
(iii) Find the probability that the batch is accepted. Leave your answer in terms of p. [3]

Forty random samples of 30 light bulbs are taken from each of the two factories A and B.

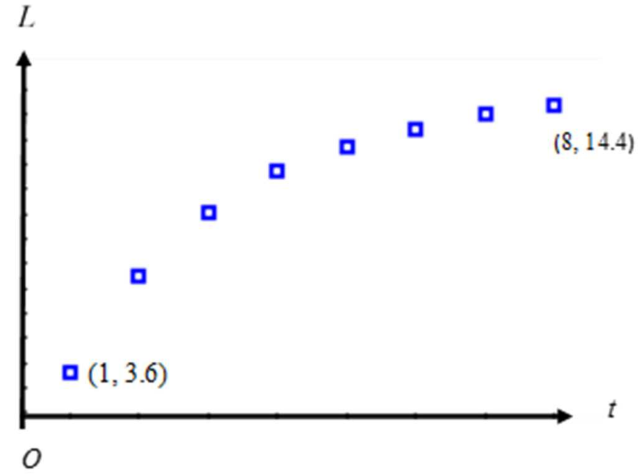
(iv) Given that $p = 0.007$ and there is exactly one defective bulb, find the probability that it is from Factory B. [4]

ANNEX B

NJC H2 Math JC2 Preliminary Examination Paper 2

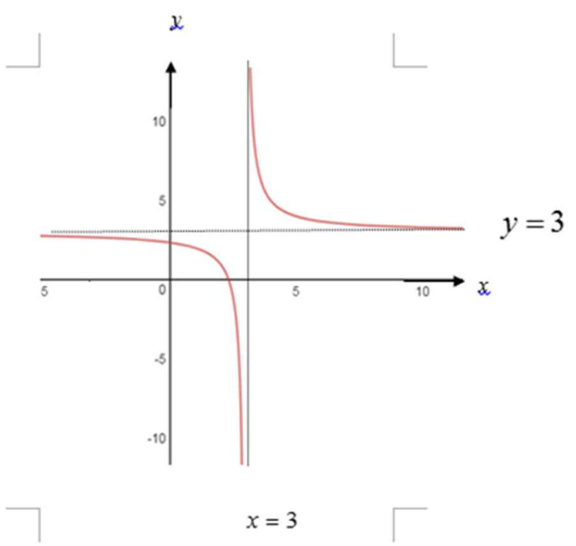
<table>
<tr><th>QN</th><th>Topic Set</th><th>Answers</th></tr>
<tr><td>1</td><td>Equations and Inequalities</td><td>$\alpha = \$0.41, \beta = \$0.84, \mu = \$1.14$.</td></tr>
<tr><td>2</td><td>Functions</td><td>(i)
$\mathrm{f}^{2003}(5) = 4.$
(ii)
$\mathrm{g}^{-1}(x) = \frac{1}{2} - \sqrt{x + \frac{9}{4}}$
(iii)


$x \geq 0$
(iv)
$R_{\mathrm{f\,g}^{-1}} = [2.5, 3)$</td></tr>
<tr><td>3</td><td>Maclaurin series</td><td>$y \approx 1 + 2x + x^2 - \frac{2x^3}{3}$
$-0.952 \leq x \leq 1.07$</td></tr>
<tr><td>4</td><td>Vectors</td><td>(i)
$3x + 3y + 7z = 56$
(ii)
$\theta \approx 31.2°$ (1 dec place) (iii)
(iv)</td></tr>
</table>

		$\mathbf{r} = \begin{pmatrix} d \\ \frac{56-3d}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}$, where $\lambda \in \mathbb{R}$. (v) $\overrightarrow{OP} = \frac{1}{67}\begin{pmatrix} 511 \\ 511 \\ 98 \end{pmatrix}$
5	P&C, Probability	(i) $k = 15$ (ii) 9 (iii) 144
6	Correlation & Linear Regression	(i)  (iii) $L = 0.261(3 \text{ sf})$ (iv) $r_2 = -r_1$
7	DRV	(i) $\mathrm{P}(X = 4) = \frac{2}{9}$ (ii)

		$\mathrm{E}(X)=\frac{26}{9}$ (iii) $P\left(\overline{X}>3\right)=0.159$
8	Hypothesis Testing	(iii) p-value = 0.0768 (iv) $\sigma^2 \le 943$
9	Normal Distribution	(i) $\mathrm{P}(\overline{D}>287)=0.0119$ (ii) $a \approx 3058$ (iii) $\mathrm{P}(\lvert C_1 - C_2\rvert > 4000)=0.660$
10	Binomial Distribution	(ii) $r=5$ (iii) $\left(1-p\right)^{30}+30p\left(1-p\right)^{59}+435p^2\left(1-p\right)^{58}$ (iv) 0.584
11	Q11 Topic	
12	Q12 Topic	
13	Q13 Topic	

H2 Mathematics 2017 Prelim Exam Paper Solution

1	Let α, β and μ be the original amount charged per 5 min, 10 min and 15 min block for each ride by α- bike, β- bike and μ- bike respectively. $5\alpha + 3\beta + \mu = 5.7$ ---------- (1) $4\alpha + \beta + 3(0.95\mu) = 5.72$ ---------- (2) $8\alpha + 4\beta + 3(0.95^2 \mu) = 9.71$ ---------- (3) Solving the above 3 equations simultaneously by GC, $\alpha = \$0.4079329609, \beta = \$0.8402234637, \mu = \$1.139664804$ $\alpha = \$0.41, \beta = \$0.84, \mu = \$1.14$. Original pricing per 40-min block: **Using calculator values** α- bike: $\$0.4079329609 \times 8 = \$3.26$ β- bike: $\$0.84 \times 4 = \$3.36$ μ- bike: $\$1.14 \times 3 = \$3.42$ Thus, α- bike offers the cheapest rate for a 40-min ride.
2	**(i)** 

Since any horizontal line $\underline{y = a, a \in \mathbb{R}}$, intersects the graph of $y = \mathrm{f}(x)$ <u>at most once</u>, the function f is one-one. It follows that f^{-1} exists.

OR

Since any horizontal line $\underline{y = a, a \in \mathrm{R}_f}$, intersects the graph of $y = \mathrm{f}(x)$ <u>exactly once</u>, the function f is one-one. It follows that f^{-1} exists.

Let $y = \dfrac{7-3x}{3-x}$

$y(3-x) = 7-x$

$x = \dfrac{7-3y}{3-y}$

Since $\mathrm{f}^{-1}(x) = \dfrac{7-3x}{3-x}$, $x \in \mathbb{R}, x \neq 3$,

$\therefore \mathrm{f}^{-1} = \mathrm{f}$. (shown)

$D_{\mathrm{f}^{-1}} = R_{\mathrm{f}} = (-\infty, 3) \cup (3, \infty) = D_{\mathrm{f}}$

Note that $\mathrm{f}^{-1}\mathrm{f}(x) = x$. Therefore, $\mathrm{f}^{2003}(5) = \underbrace{\mathrm{fff...f}}_{2003\,\text{times}}(5) = \mathrm{f}\left(\underbrace{\mathrm{f}^{-1}\mathrm{f......f}^{-1}\mathrm{f}}_{1000\text{ times of } \mathrm{f}^{-1}\mathrm{f}}(5)\right) = \mathrm{f}(5) = 4.$

(ii)

$$|(2-x)(1+x)| = \begin{cases} (2-x)(1+x), & -1 \le x \le 2, \\ -(2-x)(1+x), & x < -1 \text{ or } x > 2. \end{cases}$$

For $x \in (-\infty, -1]$, $y = -(2-x)(1+x)$

Method 1

$x^2 - x - 2 - y = 0$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2-y)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{9+4y}}{2}$$

Method 2

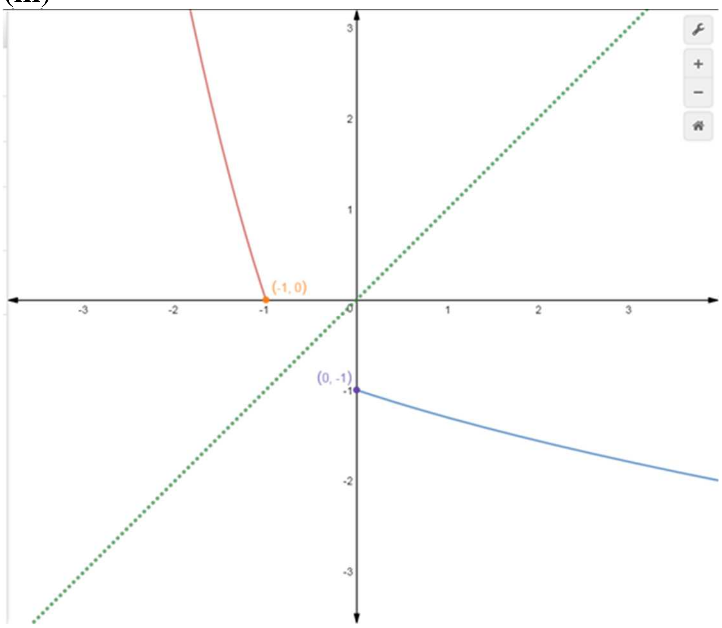
$y = x^2 - x - 2 = (x-0.5)^2 - 2.25$

$x = 0.5 \pm \sqrt{y + 2.25}$

$$x=\frac{1+\sqrt{9+4y}}{2} \text{ (rejectd } \because x\le -1) \text{ or } \frac{1-\sqrt{9+4y}}{2}$$

$$\therefore \mathrm{g}^{-1}(x)=\frac{1}{2}-\sqrt{x+\frac{9}{4}}$$

(iii)



For $\mathrm{g}\,\mathrm{g}^{-1}(x)=x$,

$D_{\mathrm{g}\mathrm{g}^{-1}}=D_{\mathrm{g}^{-1}}$.

$\therefore x\in[0,\infty)$ or $x\ge 0$

(iv)

Since $R_{\mathrm{g}^{-1}}=(-\infty,-1]$ and $\mathrm{D_f}=(-\infty,\infty)\setminus\{3\}$

$R_{g^{-1}}\subseteq D_f$.

$\therefore \mathrm{fg}^{-1}$ exists.

Using the graph of $y=\mathrm{g}^{-1}(x)$ in part **(ii)**, $R_{\mathrm{g}^{-1}}=(-\infty,-1]$.

From graph of $y=\mathrm{f}(x)$ in **(i)** in $(-\infty,-1]$.

$\therefore R_{\mathrm{f}\mathrm{g}^{-1}}=[2.5,3)$

3

$$y=\frac{e^{2x}}{1+x^2}$$

$$\left(1+x^2\right)y=e^{2x}$$

$$\left(1+x^2\right)y'+2xy=2e^{2x}$$

$$\left(1+x^2\right)y''+2xy'+2xy'+2y=4e^{2x}$$

$$\Rightarrow\left(1+x^2\right)y''+4xy'+2y=4e^{2x}$$

$$\left(1+x^2\right)y'''+2xy''+4y'+4xy''+2y'=8e^{2x}$$

$$\Rightarrow\left(1+x^2\right)y'''+6xy''+6y'=8e^{2x}$$

When $x=0,\ y=1,\ y'=2,\ y''=2,\ y'''=-4$

$$\begin{aligned} y&=\frac{e^{2x}}{1+x^2}\\ &=1+2x+2\left(\frac{x^2}{2!}\right)-4\left(\frac{x^3}{3!}\right)+\cdots\\ &\approx 1+2x+x^2-\frac{2x^3}{3}\end{aligned}$$

$$a=2,\quad b=-\frac{2}{3}$$

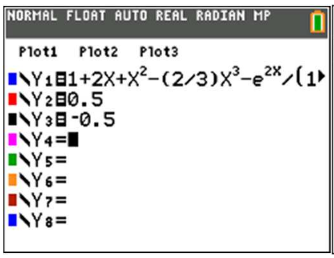
(a)

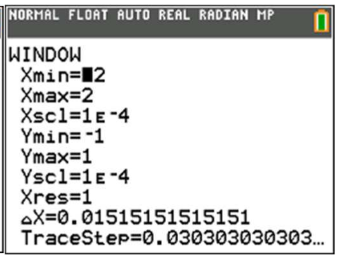
For $-2\le x\le 2$,

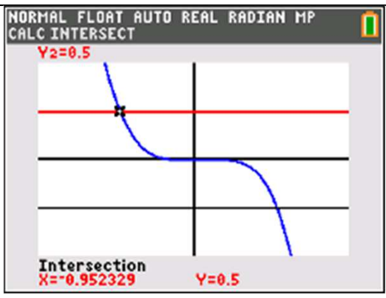
$$\left|f(x)-h(x)\right|\le 0.5$$

$$-0.5\le 1+2x+x^2-\frac{2x^3}{3}-\frac{e^{2x}}{1+x^2}\le 0.5$$

By GC,







From the diagram above,

$-0.95233 \le x \le 1.072619$

$\therefore -0.952 \le x \le 1.07$

4 **(i)**

$$\overrightarrow{OD} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}, \overrightarrow{OE} = \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix}, \overrightarrow{OF} = \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix}. \text{ Hence,}$$

$$\overrightarrow{DE} = \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}, \overrightarrow{DF} = \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix} \text{ and } \overrightarrow{EF} = \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \\ 0 \end{pmatrix}$$

A vector perpendicular to the plane is

$$= \overrightarrow{DE} \times \overrightarrow{DF}$$

$$= \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 21 \\ 21 \\ 49 \end{pmatrix} = 7\begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}$$

Cartesian equation of the plane is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} \Rightarrow \mathbf{r} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = 56$$

$\therefore 3x + 3y + 7z = 56$

(ii)

Let the required angle be θ

$$\cos\theta = \frac{\left|\begin{pmatrix}3\\3\\7\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{3^2+3^2+7^2}\sqrt{1}} = \frac{7}{\sqrt{67}}$$

$\theta \approx 31.2°$ (1 dec place)
(or 0.545 rad)

(iii)
Method 1

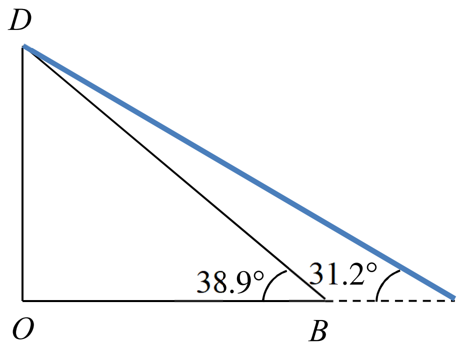
$\left|\overrightarrow{OB}\right| = \sqrt{7^2+7^2} = \sqrt{98}$

$\left|\overrightarrow{OD}\right| = 9$

Angle between DB and the ground
$= \angle OBD$

$= \tan^{-1}\left(\dfrac{8}{\sqrt{7^2+7^2}}\right)$

$\approx 38.9°$



From the diagram, the canvas will cover *B*.

Method 2

$$\overrightarrow{OB} = \begin{pmatrix}7\\7\\0\end{pmatrix}$$

Equation of perpendicular line passing through *B*, *l*:

$$\mathbf{r} = \begin{pmatrix}7\\7\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\0\\1\end{pmatrix}, \lambda \in \mathbb{R}$$

Using normal of plane to be $\begin{pmatrix}3\\3\\7\end{pmatrix}$ i.e. all entries are positive:

solve the equation of plane *DEF* and *l*:

$$\left[\begin{pmatrix}7\\7\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\cdot\begin{pmatrix}3\\3\\7\end{pmatrix}=56$$

$\lambda=2$

Since $\lambda=2>0$, l and plane *DEF* intersect above the horizontal ground. So the canvas covers the point *B*.

Method 3

$$\begin{pmatrix}7\\7\\0\end{pmatrix}\cdot\begin{pmatrix}3\\3\\7\end{pmatrix}=42<56$$

Distance from *O* to plane parallel to DEF and passing through *B* is smaller than the distance between *O* and plane *DEF*. Hence *B* is beneath the canvas.

(iv)

Normal vector of the vertical wall is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $(d,0,0)$ lies on the vertical wall.

$$\begin{pmatrix}d\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=d\Rightarrow \mathbf{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=d$$

Hence the equation of the vertical wall is $\mathbf{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=d$.

Direction vector of the line of intersection is

$$\begin{pmatrix}3\\3\\7\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\7\\-3\end{pmatrix}$$

Let $(x,\mathrm{y},0)$ be the common point on lying on the two planes.

$$\begin{pmatrix}x\\y\\0\end{pmatrix}\cdot\begin{pmatrix}3\\3\\7\end{pmatrix}=56\Rightarrow 3x+3y=56$$

	$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = d \Rightarrow x = d$ Solving the above equations simultaneously $3d + 3y = 56 \Rightarrow y = \dfrac{56-3d}{3}$ $\therefore \mathbf{r} = \begin{pmatrix} d \\ \dfrac{56-3d}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}$, where $\lambda \in \mathbb{R}$. **(v)** For P to shine the brightest at point B, P must be as near as possible to B. Thus P is the foot of perpendicular from B to the roof. Equation of the line passes through B and P: $\mathbf{r} = \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}, \alpha \in \mathbb{R}$ Thus $\overrightarrow{OP} = \begin{pmatrix} 7+3\alpha \\ 7+3\alpha \\ 7\alpha \end{pmatrix}$ for some α. Since P lies on the roof, $\begin{pmatrix} 7+3\alpha \\ 7+3\alpha \\ 7\alpha \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = 56 \Rightarrow 42 + 67\alpha = 56$ $\therefore \alpha = \dfrac{14}{67}$ Substitute $\alpha = \dfrac{14}{67}$ into $\overrightarrow{OP} = \begin{pmatrix} 7+3\alpha \\ 7+3\alpha \\ 7\alpha \end{pmatrix}$.

	$\overrightarrow{OP}=\begin{pmatrix}7+3\left(\frac{14}{67}\right)\\7+3\left(\frac{14}{67}\right)\\7\left(\frac{14}{67}\right)\end{pmatrix}=\begin{pmatrix}\frac{511}{67}\\\frac{511}{67}\\\frac{98}{67}\end{pmatrix}$ **Alternatively, use projection vector:** $\overrightarrow{BD}=\begin{pmatrix}0\\0\\8\end{pmatrix}-\begin{pmatrix}7\\7\\0\end{pmatrix}=\begin{pmatrix}-7\\-7\\8\end{pmatrix}$ To check for the direction of normal vector of *DEF* **n=** $\begin{pmatrix}3\\3\\7\end{pmatrix}$ $\begin{pmatrix}3\\3\\7\end{pmatrix}.\overrightarrow{BD}=\begin{pmatrix}3\\3\\7\end{pmatrix}.\begin{pmatrix}-7\\-7\\8\end{pmatrix}=-21-21+56>0$ Hence, angle between $\overrightarrow{BD}$ and **n** is acute. $\overrightarrow{BP}=\left(\overrightarrow{BD}.\hat{\mathbf{n}}\right)\hat{\mathbf{n}}$ $=\left(\begin{pmatrix}-7\\-7\\8\end{pmatrix}.\frac{\begin{pmatrix}3\\3\\7\end{pmatrix}}{\sqrt{9+9+49}}\right)\frac{\begin{pmatrix}3\\3\\7\end{pmatrix}}{\sqrt{9+9+49}}$ $=\frac{14}{67}\begin{pmatrix}3\\3\\7\end{pmatrix}$ $\overrightarrow{OP}=\overrightarrow{OB}+\overrightarrow{BP}$ $=\begin{pmatrix}7\\7\\0\end{pmatrix}+\frac{14}{67}\begin{pmatrix}3\\3\\7\end{pmatrix}$ $=\frac{1}{67}\begin{pmatrix}511\\511\\98\end{pmatrix}$
5	**(i)**

	5	$(m-2)$	6

	Badminton Players	Floorball Players	Swimmers Players
Case 1	1	1	2
Case 2	1	2	1
Case 3	2	1	1

Case 1: Number of selections is $\binom{5}{1}\binom{m-2}{1}\binom{6}{2}$

Case 2: Number of selections is $\binom{5}{1}\binom{m-2}{2}\binom{6}{1}$

Case 3: Number of selections is $\binom{5}{2}\binom{m-2}{1}\binom{6}{1}$

Total number of selections

$$=\binom{5}{1}\binom{m-2}{1}\binom{6}{2}+\binom{5}{1}\binom{m-2}{2}\binom{6}{1}+\binom{5}{2}\binom{m-2}{1}\binom{6}{1}$$

$$=75(m-2)+30\frac{(m-2)(m-3)}{2!}+60(m-2)$$
$$=135(m-2)+15(m-2)(m-3)$$
$$=15(m-2)(9+m-3)$$
$$=15(m-2)(m+6)$$

$\therefore k=15$

<u>Alternative method:</u>

$$\binom{5}{1}\binom{m-2}{1}\binom{6}{1}\binom{m-2+5+6-3}{1}/2!$$

(ii)
Number of ways to select exactly one of the twins

$$=\binom{5}{1}\binom{2}{1}\binom{6}{2}+\binom{5}{1}\binom{m-2}{1}\binom{2}{1}\binom{6}{1}+\binom{5}{2}\binom{2}{1}\binom{6}{1}$$

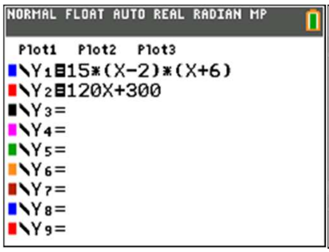
$=150+60(m-2)+120$

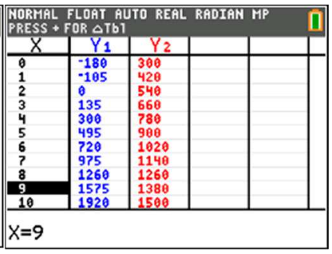
$=60m+150$

Number of ways that the twins are not selected > 2 times the number of ways that exactly one of the twins is selected.

$15(m-2)(m+6)>2(60m+150)$

By GC,





least value of m is 9.

Last part

Step 1: Arrange 3 units at the round table = 3!/3

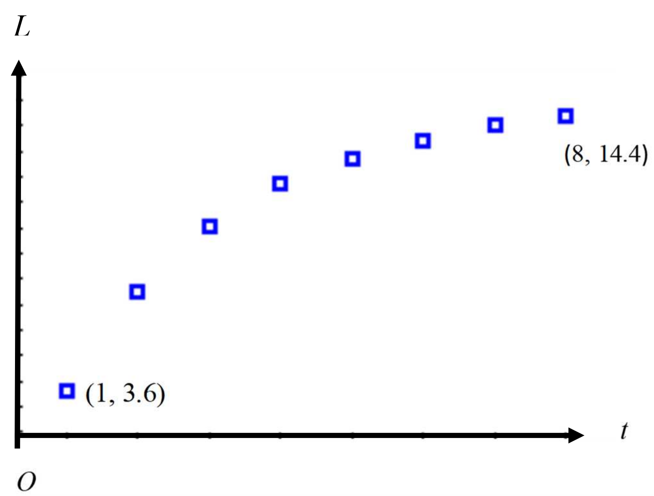
Step 2: Arrange the twins among themselves = 2!

Step 3: Slot in the teachers = $\binom{3}{2}\times 2!$

Number of ways for the twins to be seated together and teachers are separated

$=\frac{3!}{3}\times 2!\times\binom{3}{2}\times 2!\times 6=144$

6



	As L <u>increases at a decreasing rate/concave downwards</u> with respect to t, the linear model $L = at + b$ should not be used. **(ii)** The r value for $L = a\sqrt{t} + b$ is 0.972. The r value for $L = c\ln t + d$ is 0.996. Since the value of |r| for $L = c\ln t + d$, is closer to 1, $L = c\ln t + d$ is a better model. $\therefore c = 5.28248 \approx 5.28$ $\therefore d = 3.92267 \approx 3.92$ **(iii)** $L = 5.28248\ln(0.5) + 3.92267$ $= 0.2611$ $= 0.261(3 \text{ sf})$ This estimate is not reliable as as the age of the Sole is <u>out of the range of the data.</u> **(iv)** Since $G = 14.8 - L$ r_1 is positive but r_2 is negative. $\therefore r_2 = -r_1$
7	**(i)** $\mathrm{P}(X = 4)$ $= \frac{3}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{3}{3}$ $= \frac{2}{9}$ x | 2 | 3 | 4 $\mathrm{P}(X = x)$ | $\frac{1}{3}$ | $\frac{4}{9}$ | $\frac{2}{9}$ **(ii)**

	$\mathrm{E}(X)$ $=\frac{1}{3}\times2+\frac{4}{9}\times3+\frac{2}{9}\times4$ $=\frac{26}{9}$ $\mathrm{E}(X^2)$ $=\frac{1}{3}\times2^2+\frac{4}{9}\times3^2+\frac{2}{9}\times4^2$ $=\frac{80}{9}$ Var(X) $=\frac{80}{9}-\left(\frac{26}{9}\right)^2$ $=\frac{44}{81}$ **(iii)** Since n =44 <u>is large</u>, by Central Limit Theorem, $\overline{X} \sim N\left(\frac{26}{9},\frac{44}{81}\div44\right)$ approx. $P\left(\overline{X}>3\right)$ $=0.159$ (By GC)

8	**(i)** 2-tail as HPB is looking for a change in either way. **(ii)** Central Limit Theorem states that the sample <u>mean</u> heart rate will follow a normal distribution approximately when the sample is large (in this case, 70 > 20). An unbiased estimate for the unknown population variance can be found obtained from the sample. **(iii)** Unbiased estimate of population mean $\bar{h}=\frac{5411}{70}=77.3$. Unbiased estimate of population variance, $s^2=\frac{1}{69}\left(426433-\frac{5411^2}{70}\right)=118.3$. Let μ denote the mean heart rate of the teenagers in the obesity group. To test at 10% significance level: H_0: $\mu=75$ H_1: $\mu\neq 75$ Under H_0, since n is large, by CLT, $\overline{H}\sim \mathrm{N}\left(75,\frac{118.3}{70}\right)$ approximately, (AND/OR $\frac{\overline{H}-75}{\sqrt{\frac{118.3}{70}}}\sim \mathrm{N}(0,1)$) By GC, p-value = 0.0768 < 0.10. (Alternatively, CR: $\lvert z\rvert>1.645$, $z=1.769$ is in CR) Hence we <u>reject H_0</u> at the <u>10% level of significance</u> and conclude there is <u>sufficient evidence</u> that obesity will cause change in the mean heart rate. **(iv)** An one-tail test is used instead: H_0: $\mu=75$ H_1: $\mu>75$ CR: $z>z_{0.9}=1.28155$ To reject H_0,

	$$\frac{79.4-75}{\sqrt{\frac{\sigma^2}{80}}} \geq 1.28155$$ $$\sigma^2 \leq \left(\frac{4.4\sqrt{80}}{1.28155}\right)^2 = 943 \text{ (3 sf)}$$ The researcher should conclude that obese teenagers evidentially has a higher mean heart rate if and only if the variance is not more (less) than 943. **(v)** $\mathrm{P}(\mu-\sigma < X < \mu+\sigma) = \mathrm{P}(-1 < Z < 1) = 0.68268 \approx 0.683$. Since heart rates follow a normal distribution, $\mathrm{P}(\mu-\sigma < H < \mu+\sigma) \approx 0.683$ We know that from (iv), null hypothesis will be rejected whenever $\sigma \leq 30.7$. Taking $\sigma = 30.7$, under H_0, $\mathrm{P}(75-30.7 < H < 75+30.7) \approx 0.683$ $\Rightarrow \mathrm{P}(44.3 < H < 105.7) \approx 0.683$ and null hypothesis will be rejected. We can say that for $\sigma \leq 30.7$ and when null hypothesis is rejected , $\mathrm{P}(44.3 < H < 105.7) \geq 0.683$ or $\mathrm{P}(H < 44.3) + \mathrm{P}(H > 105.7) < 0.317$ We know that the teenager's heart rate is rarely below 44.3 or above 105.7 in a resting state, so it is likely for the researcher to reject the null hypothesis. In reality, it is unlikely for sigma to be as large as 30.7 such that the probability for *H* to be within one standard deviation from mean to be 0.683.
9	Let *D* be the random variable denoting the number of days of gestation for a Dutch Belted cow. $\mathrm{P}(D < 278) = 0.0808$ $\mathrm{P}(Z < \frac{278-\mu}{\sigma}) = 0.0808$ $\frac{278-\mu}{\sigma} = -1.39971 - - - (1)$ $\mathrm{P}(D > 289) = 0.212$ $\mathrm{P}(Z < \frac{289-\mu}{\sigma}) = 0.788$ $\frac{289-\mu}{\sigma} = 0.799501 - - - (2)$ Solving (1)&(2), $\mu = 285.001064$ and $\sigma = 5.0017961$

	$\mu = 285$ (3 s.f.) and $\sigma = 5.00$ (3 s.f.) . **(i)** $\overline{D} \sim \mathrm{N}(285.001064, \frac{5.0017961^2}{32})$. $\mathrm{P}(\overline{D} > 287) = 0.0118629 = 0.0119$ (3 s.f.) The <u>number of days of gestation for a Dutch Belted cow</u> is <u>independent</u> of <u>the number of days of gestation of another Dutch Belted cow.</u> **(ii)** $J \sim \mathrm{N}(278, 2.50^2)$ $D \sim \mathrm{N}(285.001064, 5.0017961^2)$ Let $X = 26J - \frac{1}{2}29D$ $X \sim \mathrm{N}(3095.48457\ ,\ 9485.02698)$ $\mathrm{P}(0 < X < a) = 0.35$ $\therefore a = 3057.95778 \approx 3058$ **(iii)** $D \sim \mathrm{N}(285.001064, 5.0017961^2)$ $J \sim \mathrm{N}(278, 2.50^2)$ Let C_1 denote the random variable of the amount of feed cosumed by 3 pregnant Dutch belted cows. Let C_2 denote the random variable of the amount of feed cosumed by 4 pregnant Jersey cows. $C_1 = 29(D_1 + D_2 + D_3) \sim \mathrm{N}(24795.09257, 63120.32374)$ $C_2 = 26(\mathrm{J}_1 + J_2 + J_3 + J_4) \sim \mathrm{N}(28912, 16900)$ $C_1 - C_2 \sim \mathrm{N}(-4117, 80020.3237)$ $\mathrm{P}(\lvert C_1 - C_2 \rvert > 4000)$ $= \mathrm{P}(C_1 - C_2 < -4000) + \mathrm{P}(C_1 - C_2 > 4000)$ $= 0.6604182314$ $= 0.660$ (3 s.f.) Or

$D \sim \mathrm{N}(285.00, 5.0018^2)$

$J \sim \mathrm{N}(278, 2.50^2)$

Let C_1 denote the random variable of the amount of feed cosumed by 3 pregnant Dutch belted cows.

Let C_2 denote the random variable of the amount of feed cosumed by 4 pregnant Jersey cows.

$C_1 = 29(D_1 + D_2 + D_3) \sim \mathrm{N}(24795, 63120.42217)$

$C_2 = 26(J_1 + J_2 + J_3 + J_4) \sim \mathrm{N}(28912, 16900)$

$C_1 - C_2 \sim \mathrm{N}(-4117, 80020.422)$

$$\begin{aligned}
\mathrm{P}(|C_1 - C_2| > 4000) \\
&= \mathrm{P}(C_1 - C_2 < -4000) + \mathrm{P}(C_1 - C_2 > 4000) \\
&= 0.66041814 \\
&= 0.660 \text{ (3 s.f.)}
\end{aligned}$$

10

(i)

The event of a bulb being defective may not be independent of another bulb being defective.

(ii)

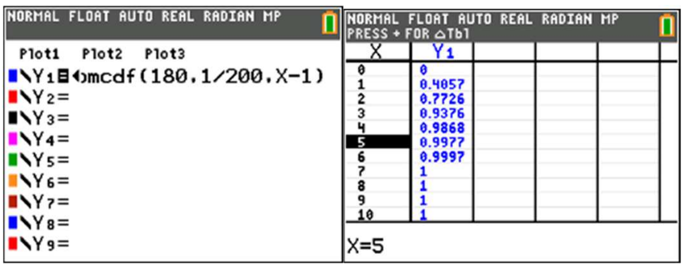
Let X be the random variable for the number of defective light bulbs produced by Factory A.

$$X \sim \mathrm{B}\left(180, \frac{1}{200}\right)$$

Given

$$\mathrm{P}(X < r) = 0.998$$
$$\Rightarrow \mathrm{P}(X \le r-1) = 0.998$$

By GC,



$\therefore r = 5$

(iii)

	Let Y be the random variable for the number of defective light bulbs produced by Factory B. $Y \sim \mathrm{B}(30, p)$ P(the batch is accepted) $= \mathrm{P}(Y_1 = 0) + \mathrm{P}(Y_1 = 1 \text{ or } 2)\mathrm{P}(Y_2 = 0)$ $= \binom{30}{0} p^0 (1-p)^{30}$ $+ \left[\binom{30}{1} p(1-p)^{29} + \binom{30}{2} p^2 (1-p)^{28}\right]\binom{30}{0} p^0 (1-p)^{30}$ $= (1-p)^{30} + \left[30p(1-p)^{29} + 435p^2(1-p)^{28}\right](1-p)^{30}$ $= (1-p)^{30} + 30p(1-p)^{59} + 435p^2(1-p)^{58}$ **(iv)** Let U be the random variable for the number of defective light bulbs produced by Factory A. Let V be the random variable for the number of defective light bulbs produced by Factory B. $U \sim \mathrm{B}\left(1200, \frac{1}{200}\right)$ $V \sim \mathrm{B}(1200, 0.007)$ P(1 bulb from B is defective| there is exactly one defective bulb)

	$= \dfrac{\mathrm{P}(U=0,V=1)}{\mathrm{P}(U=0,V=1)+\mathrm{P}(U=1,V=0)}$ ≈ 0.5838223 $= 0.584$ (3 s.f.) Reference for $\dfrac{\mathrm{P}(U=0,V=1)}{\mathrm{P}(U=0,V=1)+\mathrm{P}(U=1,V=0)}$: $\dfrac{\left[\binom{1200}{1}0.007^{1}(1-0.007)^{1199}\times\binom{1200}{0}\left(\frac{1}{200}\right)^{0}\left(1-\frac{1}{200}\right)^{1200}\right]}{\left[\begin{array}{l}\binom{1200}{1}0.007^{1}(1-0.007)^{1199}\times\binom{1200}{0}\left(\frac{1}{200}\right)^{0}\left(1-\frac{1}{200}\right)^{1200}\\ +\binom{1200}{0}0.007^{0}(1-0.007)^{1200}\times\binom{1200}{1}\left(\frac{1}{200}\right)^{1}\left(1-\frac{1}{200}\right)^{1199}\end{array}\right]}$