Candidate Name: ______________________

Class	Adm No

H

millennia institute

2018 Preliminary Exams
Pre-University 3

MATHEMATICS **9758/01**

Paper 1 **11 Sept 2018**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, arrange your answers in NUMERICAL ORDER and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

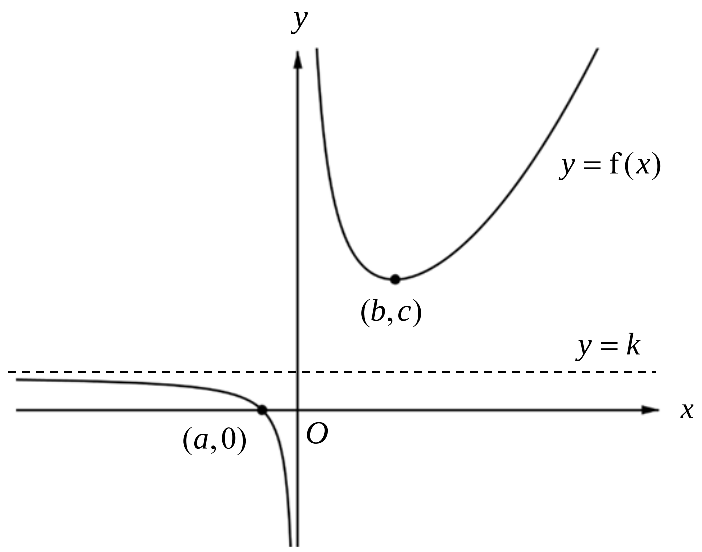
1 By using the series expansion of e^x from the List of Formulae (MF26), find $\sum_{r=2}^{\infty} \frac{3^r}{r!}$. [3]

2 Without using a calculator, solve the inequality

$$\frac{x-7}{x^2-x-6} \leq 1.$$ [4]

Hence solve the inequality $\frac{\ln x - 7}{(\ln x)^2 - \ln x - 6} - 1 \leq 0$. [2]

3 The diagram shows a sketch of the graph of $y = \mathrm{f}(x)$. The curve intersects the x-axis at the point $(a, 0)$ and has a turning point at the point (b, c), where a, b and c are constants. The y-axis and the line $y = k$, where k is a constant, are the asymptotes of the curve.



Sketch, on separate diagrams, the graphs of

(i) $y = \frac{1}{\mathrm{f}(x)}$, [3]

(ii) $y = \mathrm{f}'(x)$, [3]

stating clearly, where possible, the equations of any asymptotes and coordinates of any turning points and axial intercepts.

[Turn over

4 The curve C has equation $y=\dfrac{2x^2+13x+23}{x+\lambda}$, where λ is a positive constant.

(i) Sketch the graph of C, stating clearly the coordinates of any points of intersection with the axes and equations of any asymptotes. [3]

(ii) For $\lambda=3$, find algebraically the set of values that y can take. [3]

5 **(i)** Given that $\ln y=\sin 2x$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2y\cos 2x$. Hence find the values of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x=0$. [3]

(ii) Write down the first three non-zero terms in the Maclaurin series for y. [1]

(iii) It is given that the second and third non-zero terms in part **(ii)** are equal to the first and second non-zero terms in the series expansion of $\mathrm{e}^{px}\ln(1+qx)$ respectively. Using appropriate expansions from the List of Formulae (MF26), find the values of the constants p and q. [3]

Hence state the range of values of x for which the series expansion of $\mathrm{e}^{px}\ln(1+qx)$ is valid. [1]

6 A curve C has parametric equations

$$x=1-\mathrm{e}^{-t}, \qquad y=1+t^2.$$

(i) Find the exact gradient of C when $t=1$. [2]

(ii) By first finding an expression for $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of the parameter t, determine if C is concave upward for all real values of t. [2]

(iii) Find the cartesian equation of C. [2]

(iv) Hence find the volume of revolution when the region bounded by C, the x-axis and the lines $x=-\dfrac{1}{2}$ and $x=\dfrac{3}{5}$ is rotated completely about the x-axis, giving your answer correct to 3 decimal places. [2]

7 **(a)** **(i)** Find $\int \cos 2x \sin^5 2x \, \mathrm{d}x$. [3]

(ii) Use the substitution $u = \mathrm{e}^x$ to find $\displaystyle\int \frac{\mathrm{e}^x}{\sqrt{1-2\mathrm{e}^{2x}}} \, \mathrm{d}x$. [3]

(b) **(i)** Show that $\displaystyle\int x^2 \sin nx \, \mathrm{d}x = \frac{2}{n^3}\cos nx + \frac{2x}{n^2}\sin nx - \frac{x^2}{n}\cos nx + c$, where n is a positive integer and c is an arbitrary constant. [3]

(ii) Hence find $\displaystyle\int_{\pi}^{2\pi} x^2 \sin nx \, \mathrm{d}x$, giving your answer in the form $\dfrac{p}{n^3} - \dfrac{q\pi^2}{n}$, where the possible values of p and q are to be determined. [3]

8 The function f is defined by

$$\mathrm{f} : x \mapsto \sqrt{3}\cos x - \sin x, \quad x \in \mathbb{R}, \ -\pi \le x \le \pi.$$

(i) Express $\mathrm{f}(x)$ as $R\cos(x+\alpha)$, where R and α are constants to be found. [1]

(ii) Sketch the graph of $y = \mathrm{f}(x)$, stating clearly the exact coordinates of the end-points and turning points. Hence explain why f^{-1} does not exist. [2]

(iii) If the domain of f is further restricted to $-\dfrac{\pi}{6} \le x \le k$, state the maximum value of k for which f^{-1} exists. [1]

(iv) Using the value of k found in part **(iii)**, find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

Hence sketch, on the same diagram, the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$. [3]

The function g is defined by

$$\mathrm{g}(x) = \frac{1}{1-x}, \quad x \in \mathbb{R}, \ x \ne 0, \ x \ne 1.$$

Given that $\mathrm{g}^2(x) = \dfrac{x-1}{x}$ and $\mathrm{g}^3(x) = x$ for $x \in \mathbb{R}$, $x \ne 0$, $x \ne 1$, determine the value of $\mathrm{g}^{2018}(2)$. [2]

9 A toy all-terrain vehicle, Terra is designed to handle multiple terrains. In a test run, the vehicle moves on a path that consists of different sections of increasing difficulty. Terra completes the first section, second section, third section and so on until all sections have been completed. The total time (in seconds) taken by Terra to complete n sections is given by $\frac{3}{2}n^2+\frac{13}{2}n$.

(i) Find the time taken by Terra to complete the nth section. [2]

Terra was made to travel up a man-made hill of height 35 m, from the foot of the hill to the peak. Terra is designed to cover a vertical distance of 5 m in the first ten minutes. For each subsequent ten-minute period, Terra covers 80% of the vertical distance covered in the previous ten-minute period.

(ii) The vertical distances covered by Terra in the ten-minute periods form a geometric sequence. Explain if this sequence is convergent. [1]

(iii) Explain if Terra is able to reach the peak of the hill. [2]

After the completion of the fourth ten-minute period, Terra's vertical distance covered is changed such that in each subsequent ten-minute period, Terra covers 90% of the vertical distance covered in the previous ten-minute period.

(iv) Find the minimum number of ten-minute periods required for Terra to reach the peak of the hill. [3]

After installing the brake system on Terra, the design team wants to test its effectiveness. In this test, Terra moves with increasing speed from a stationary position for 4 seconds before the brake system is engaged. Terra then has to stop moving in 2 seconds. The test is done many times and is repeated immediately after the previous one.

Terra's motion is modelled by the function given by

$$\mathrm{f}(t)=\begin{cases} \frac{3}{2}t & , \quad 0\le t\le 4, \\ \frac{3}{2}t^2-18t+54 & , \quad 4<t\le 6, \end{cases}$$

where t denotes the time (in seconds) after the start of the first test and $\mathrm{f}(t)$ denotes Terra's speed (in metres per second) at time t. The repetition of the test can be represented by the relationship given by $\mathrm{f}(t)=\mathrm{f}(t+6)$ for $t\ge 0$.

(v) Sketch the graph of $y=\mathrm{f}(t)$ for $0\le t\le 14$. [2]

(vi) The distance covered by Terra is given by the area under the graph of $y=\mathrm{f}(t)$. Find the distance travelled by Terra from $t=4$ to $t=11$. [2]

10 Relative to the origin O, the position vectors of the points A and B are $\begin{pmatrix} 0 \\ 5 \\ 15 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 7 \\ 12 \end{pmatrix}$ respectively. The line l_1 has equation $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$. The line l_2 is parallel to the vector $\begin{pmatrix} -1 \\ 6 \\ 12 \end{pmatrix}$ and passes through A.

(i) Given that l_1 and l_2 intersect at the point C, find $\overrightarrow{OC}$. [2]

The point B lies on l_1 and is the foot of the perpendicular from A to l_1.

(ii) Find the cartesian equation of the line of reflection of l_2 in the line l_1. [2]

The equations of two planes π_1 and π_2 are as follows:

$$\pi_1: \ x + y - 2z = -1,$$
$$\pi_2: \ ax + y + 3z = 5,$$

where a is a constant.

(iii) Find the coordinates of the foot of the perpendicular from A to π_1 and hence, find the perpendicular distance from A to π_1. [4]

(iv) Given that the angle between l_1 and π_2 is 30°, find the value(s) of a. [2]

(v) Given instead that the line of intersection of π_1 and π_2 has the equation $\mathbf{r} = \mathbf{v} + \beta \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$, $\beta \in \mathbb{R}$, where $\mathbf{v}$ is the position vector of the point V on the line of intersection, determine the value of a and a possible position vector of V. [3]

11 A game designer created a computer game that involves an alien race inhabiting and developing a planet.

(i) The population of the aliens (in thousands) on the planet at time n years is denoted by x_n. The population of the aliens changes according to the relationship that the game designer devised as follows:

$$x_n = x_{n-1} + \frac{13}{2} - n \text{ for } n \geq 1.$$

The initial population of the aliens on the planet is 8000. By considering $\sum_{r=1}^{n}(x_r - x_{r-1})$, find an expression for x_n in terms of n. [4]

(ii) The game designer believes that the relationship devised in part **(i)** is overly simplistic and decides to use another model. The population of the aliens (in thousands) at time t years is denoted by x. It is set that the growth rate of x is proportional to $50-2t-x$.

It is given that the initial population of the alien on the planet is 8000 and the population grows at a rate of 14000 per year initially.

Show that the growth rate of x at time t years can be modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{3}(50-2t-x).$$ [2]

Using the substitution $u = 2t + x$, find x in terms of t. [5]

(iii) To add more variation into the game play, the game designer decides to use an alternative model that is given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{100}{(t+1)^3}, \ t \geq 0,$$

where x is the population of the aliens (in thousands) at time t years.

It is given that the initial population of the aliens on the planet is 8000 and the population grows at a rate of $41\frac{2}{3}$ thousand per year initially.

Find x in terms of t. [3]

End of Paper

Class	Adm No

Candidate Name: ______________________________

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millennia institute

2018 Preliminary Exams
Pre-University 3

MATHEMATICS **9758/02**

Paper 2 **13 Sept 2018**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

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Write in dark blue or black pen on both sides of the paper.
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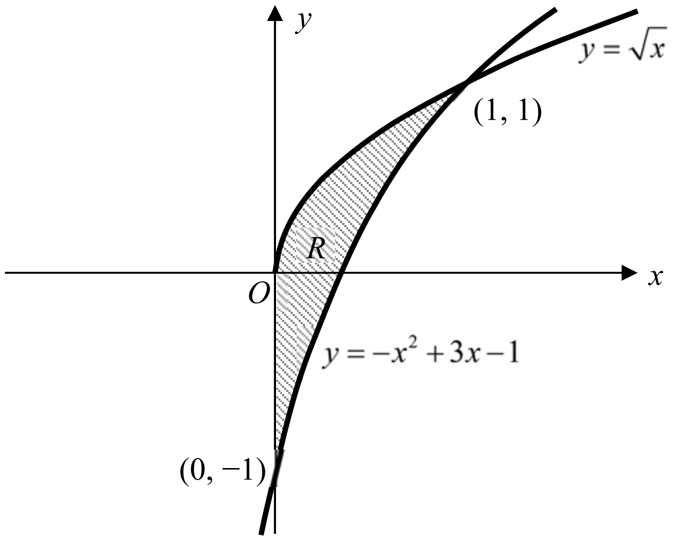
This question paper consists of 8 printed pages.

[Turn over

Section A: Pure Mathematics [40 marks]

1 In the diagram, the region R is bounded by the curves $y=-x^2+3x-1$, $y=\sqrt{x}$ and the y-axis.



Without using a graphing calculator, find the volume of the solid generated when R is rotated through 2π radians about the y-axis. [4]

2 **(a)** A curve C has parametric equations

$$x=k\cos t, \quad y=k\sin 2t, \quad \text{for } 0\le t\le \frac{\pi}{4},$$

where k is a positive constant.

(i) Find the equation of the normal to C at the point $(k\cos p, k\sin 2p)$, where p is a constant such that $0\le p<\dfrac{\pi}{4}$. [3]

(ii) Deduce the equation of the tangent to C at the point where $t=0$. [2]

(b) A rectangle of length $2|x|$ and breadth $2|y|$ is inscribed in the ellipse $\dfrac{x^2}{36}+\dfrac{y^2}{9}=1$ where (x, y) is any point on the ellipse.

(i) Show that the area of the rectangle, A, is given by $A=8\sqrt{9y^2-y^4}$. [2]

(ii) Find the maximum area of the rectangle. [4]

3 **(a)** Given that $a-\mathrm{i}$ is a root of the equation $z^3+4(1+\mathrm{i})z^2+(-2+9\mathrm{i})z-5+\mathrm{i}=0$, where a is a real constant, show that $(a^3+4a^2+3a)+(a^2+a)\mathrm{i}=0$ and find all the roots of the equation. [4]

Deduce the roots of the equation $z^3+8(1+\mathrm{i})z^2+4(-2+9\mathrm{i})z+8(-5+\mathrm{i})=0$. [2]

(b) It is given that $w=\left(1-\mathrm{i}\sqrt{3}\right)^4$.

(i) Without using a graphing calculator, find the modulus and argument of w. [2]

(ii) Hence find the three smallest positive whole number values of n for which $\dfrac{w^n}{w^*}$ is a real number. [4]

4 Relative to the origin O, the position vectors of points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. It is given that $\mathbf{a}$ and $\mathbf{b}-\mathbf{a}$ are perpendicular and C lies on AB produced such that $AC:AB=4:3$.

(i) If $\mathbf{a}$ is a unit vector, show that $|\mathbf{b}|>1$. [3]

Given further that $|\mathbf{b}|=2$, find the angle between $\mathbf{a}$ and $\mathbf{b}$. [1]

(ii) The direction cosines of $\mathbf{b}$ are 0.6, λ, μ and $\mathbf{b}$ is perpendicular to the y-axis. Find

(a) the angle $\mathbf{b}$ makes with the x-axis, [1]

(b) λ and μ. [2]

(iii) By expressing $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, show that $|\mathbf{c}\cdot\mathbf{a}| = \mathbf{a}\cdot\mathbf{a}$. [3]

Hence state the length of projection of $\mathbf{c}$ onto $\mathbf{a}$ in terms of $\mathbf{a}$. [1]

(iv) Give a geometrical interpretation of $|\mathbf{c}\times\mathbf{a}|$ and hence evaluate $\dfrac{|\mathbf{c}\times\mathbf{a}|}{|\mathbf{b}\times\mathbf{a}|}$. [2]

Section B: Probability and Statistics [60 marks]

5 A biased die is such that the probability of getting a score of 1, 2, 3, 4, 5 and 6 is $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, p$ and q respectively.

(i) If the mean score is 3.5, find p and q. [2]

A game is played by throwing the biased die and a fair coin together. If the coin shows a head, the player wins \$1 more than the score on the die. If the coin shows a tail, the player wins \$1 less than the score on the die. The winnings from one game is denoted by W.

(ii) Find $\mathrm{P}(W = w)$ for all possible values of w. [2]

(iii) Without any further calculations, state whether you would play this game if you need to pay \$4 at the start of the game, justifying your answer. [1]

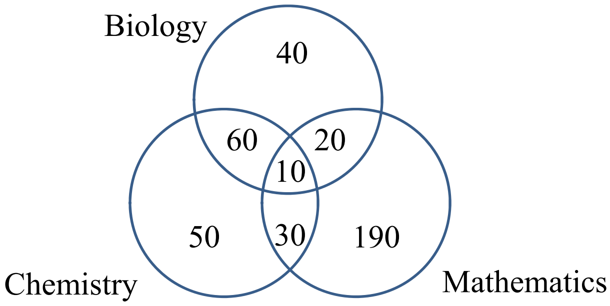
6 In the 2018 FIFA World Cup, the Morocco football team started the opening match against Iran with 1 goalkeeper, 3 defenders, 4 midfielders and 3 forwards.

(i) Before the match, the 11 selected players, together with their coach stand in a line for a photo shoot. In how many ways can this be done if the coach and the goalkeeper must stand at either ends and one particular defender and one particular forward must not stand together? [2]

(ii) During half-time, the coach gathered the 11 players at a round table with 12 numbered seats to discuss strategies for the second half of the match. In how many ways can this be done if the 3 defenders must be seated together, the 4 midfielders must be seated together and the 3 forwards must be seated together? [3]

(iii) After the match, the coach decides to construct four-lettered code-words from the 7 letters of the word MOROCCO. How many such code-words are there? [3]

7 The Venn diagram below shows the number of students studying Biology, Chemistry and Mathematics in a junior college.



One of the students is chosen at random.

B is the event that the student studies Biology
C is the event that the student studies Chemistry
M is the event that the student studies Mathematics

(i) Determine if B and C are independent, justifying your answer. [1]

(ii) Find

(a) $\mathrm{P}(B \cup C')$ and [1]

(b) $\mathrm{P}(C \mid M \cap B)$. [1]

The junior college also offers Further Mathematics as a subject. A student studying Further Mathematics must also study Mathematics. Given that the event that a randomly chosen student studying Further Mathematics is independent of C, find the largest possible number of students studying Further Mathematics. [3]

8 The random variable X has a normal distribution with mean μ and standard deviation 2.

(i) Given that $\mathrm{P}(X \le 1) = 0.1587$, find the value of μ, giving your answer to the nearest integer. [2]

(ii) Given further that $\mathrm{P}(X > 3k) = \mathrm{P}(X < 9k+4)$ and using the answer obtained in part **(i)**, find the value of k. [1]

The random variable Y is related to X by the formula $Y = 10 - X$.

(iii) Find $\mathrm{P}(\bar{Y} > 6)$, where $\bar{Y}$ is the mean of two independent observations of Y. [3]

[Turn over

9 A sample of 10 students are selected from a mixed school to participate in a survey on the school sports facilities. The number of male students in a sample of 10 is denoted by the random variable X and the proportion of male students in the mixed school is p.

(i) State, in the context of this question, two assumptions required for X to be well modelled by a binomial distribution. [2]

Assume now that these assumptions do in fact hold.

(ii) Given that $\mathrm{P}(X \le 1) = 0.05,$ write down an equation for the value of p, and find this value numerically. [2]

(iii) 8 such samples of 10 students are selected to participate in the household income survey. Find the probability that exactly 7 of the samples have at least 2 male students. [2]

For the rest of the question, take $p = 0.4$.

(iv) Find the most probable number of **female** students selected in a sample of 10 students. [2]

(v) 60 samples of 10 students each are selected to participate in the school climate survey. By using a suitable approximation, estimate the probability that the total number of male students selected exceeds 230. [3]

10 The distributor claims that the mean mass of cereal in a randomly chosen packet is 600 grams. A retailer suspects that the mean mass of the cereal is being overstated. He takes a random sample of 50 packets of cereal and weighs the content, x grams, in each packet. The results are summarised as follows:

$$\sum(x-600)=-8 \text{ and } \sum(x-600)^2=11.3$$

(i) Find unbiased estimates of the population mean and variance of the mass of cereal in a packet. [2]

(ii) Test, at the 1% level of significance, whether the distributor's claim has been overstated. [4]

(iii) State whether there is a need to assume a normal population in conducting the test in part **(ii)**, justifying your answer. [1]

(iv) Explain, in this context, the meaning of the p-value of the test obtained in part **(ii)**. [1]

The packaging process has been changed so that the mass of cereal in a randomly chosen packet follows a normal distribution with a standard deviation of 0.5 grams. The distributor now claims that the mean mass of cereal in a randomly chosen packet is μ_0 grams. The retailer selects a new random sample of 25 packets of cereal and the sample mean mass is found to be 600.6 grams. Find the range of possible values of μ_0 so that the retailer's suspicion that the mean mass differs from μ_0 grams is valid at the 5% level of significance. Give your answer correct to 2 decimal places. [4]

[Turn over

11 During the National Step Challenge period, participants are able to redeem rewards if they obtained 750 healthpoints in the first tier. In any day during the period, participants will earn 10 healthpoints when they clocked 5000 steps, and 25 healthpoints when they clocked 7500 steps and a maximum of 40 healthpoints when they clocked 10000 steps. Kenny recorded his steps count using his step tracker in the first two weeks as follows:

Day x	1	3	5	7	9	11	13
Steps y (in thousands)	3.8	5.9	6.7	7.5	6.9	8.2	8.5

(i) Draw a scatter diagram showing the above data. [1]

(ii) Suggest a possible reason why one of the step counts does not seem to follow the trend. [1]

Assume that the reason given in part **(ii)** is valid, the outlier is removed.

(iii) Using the remaining six points, find both equations of the least squares regression line of y on x and that of x on y. [2]

(iv) Interpret, in the context of this question, the gradient of the least squares regression line of y on x obtained in part **(iii)**. [1]

(v) Use a suitable regression line found in part **(iii)** to estimate the number of days taken by Kenny to clock 10000 steps. Justify the choice of the regression line used. [2]

It is decided to fit a model of the form $\ln(L-y)=a+bx$, where L is a suitable constant. The product moment correlation coefficient between x and $\ln(L-y)$ is denoted by r. The following table gives values of r for some possible values of L.

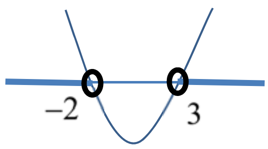
L	10.1	10.2	10.3
r		−0.98424	−0.98318

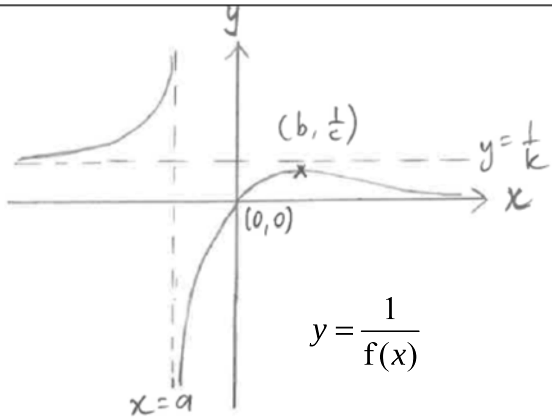
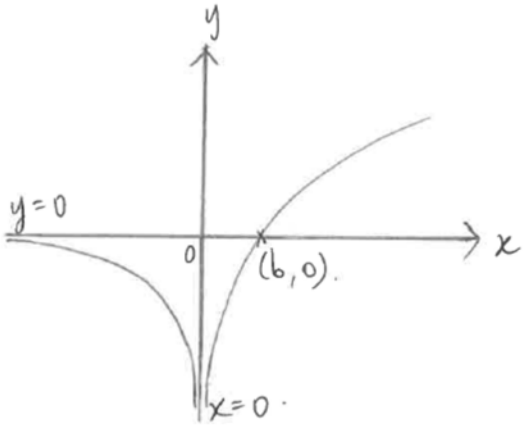
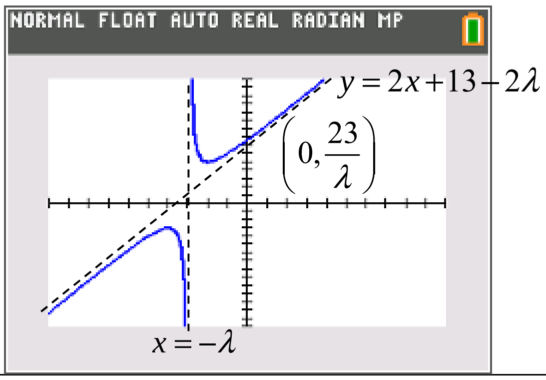
(vi) Calculate the value of r for $L=10.1$, giving your answer correct to 5 decimal places. Hence, with the help of the table, suggest with a reason which of 10.1, 10.2 or 10.3 is the most appropriate value for L. [2]

(vii) Using this value of L, calculate the values of a and b, and use them to predict the number of steps clocked by Kenny on Day 10. Comment on the reliability of this estimate. [3]

END OF PAPER

PU3 MATHEMATICS

Qn.	Suggested Solution
1	From MF26, $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots$ $= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots$ $= \sum_{r=0}^{\infty} \frac{x^r}{r!}$ $\sum_{r=2}^{\infty} \frac{3^r}{r!} = \sum_{r=0}^{\infty} \frac{3^r}{r!} - \sum_{r=0}^{1} \frac{3^r}{r!}$ $= \sum_{r=0}^{\infty} \frac{3^r}{r!} - \frac{3^0}{0!} - \frac{3^1}{1!}$ $= \mathrm{e}^3 - 4$
2	$\frac{x-7}{x^2-x-6} \le 1$ $\frac{x-7-(x^2-x-6)}{x^2-x-6} \le 0$ $\frac{-x^2+2x-1}{x^2-x-6} \le 0$ $\frac{x^2-2x+1}{x^2-x-6} \ge 0$ $\frac{(x-1)^2}{(x+2)(x-3)} \ge 0$ For $(x-1)^2 > 0$, $\frac{1}{(x+2)(x-3)} > 0$ $(x+2)(x-3) > 0$ $x < -2 \quad \text{or} \quad x > 3$  For $(x-1)^2 = 0$, $x = 1$ $\therefore x < -2 \quad \text{or} \quad x > 3 \quad \text{or} \quad x = 1$ $\frac{\ln x - 7}{(\ln x)^2 - \ln x - 6} \le 1$ Replacing x with $\ln x$: $\ln x < -2 \quad \text{or} \quad \ln x > 3 \quad \text{or} \quad \ln x = 1$ $0 < x < \mathrm{e}^{-2} \quad \text{or} \quad x > \mathrm{e}^3 \quad \text{or} \quad x = \mathrm{e}$

3(i)	
3(ii)	$y=\mathrm{f}'(x)$ 
4(i)	$y=\dfrac{2x^2+13x+23}{x+\lambda}$ $=2x+13-2\lambda+\dfrac{23-13\lambda+2\lambda^2}{x+\lambda}$ The asymptotes are $y=2x+13-2\lambda$ and $x=-\lambda$ When $x=0,\ y=\dfrac{23}{\lambda}$ When $y=0,\ \dfrac{2x^2+13x+23}{x+\lambda}=0$ $2x^2+13x+23=0$ Using GC, there are no real roots $\Rightarrow$ no $x-$intercepts 
4(ii)	**<u>Method 1: Discriminant</u>**

$$y=\frac{2x^2+13x+23}{x+3}$$

$$y(x+3)=2x^2+13x+23$$

$$2x^2+(13-y)x+23-3y=0$$

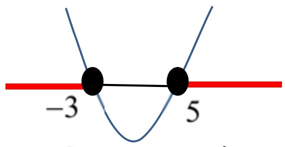
To determine the range of values of that y can take:

For real roots, $b^2-4ac\geq 0$

$$(13-y)^2-4(2)(23-3y)\geq 0$$

$$y^2-2y-15\geq 0$$

$$(y+3)(y-5)\geq 0$$



$y\leq -3$ or $y\geq 5 \Rightarrow \{y\in\mathbb{R}: y\leq -3 \text{ or } y\geq 5\}$

<u>Method 2: Differentiation</u>

$$y=\frac{2x^2+13x+23}{x+3}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(x+3)(4x+13)-(2x^2+13x+23)}{(x+3)^2}$$

$$=\frac{4x^2+25x+39-2x^2-13x-23}{(x+3)^2}$$

$$=\frac{2x^2+12x+16}{(x+3)^2}$$

To find turning points,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\frac{2x^2+12x+16}{(x+3)^2}=0$$

$$2x^2+12x+16=0$$

$$x^2+6x+8=0$$

$$(x+4)(x+2)=0$$

$\therefore x=-4$ or -2

When $x=-4, y=-3$,

When $x=-2, y=5$

$\therefore y\leq -3$ or $y\geq 5 \Rightarrow \{y\in\mathbb{R}: y\leq -3 \text{ or } y\geq 5\}$

5(i)

<u>Method 1: Implicit differentiation</u>

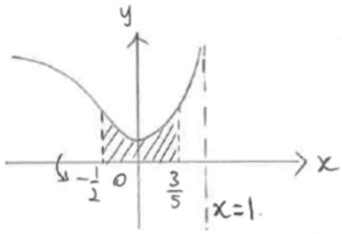
$$\ln y=\sin 2x$$

Differentiating both sides with respect to x:

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=2\cos 2x$$

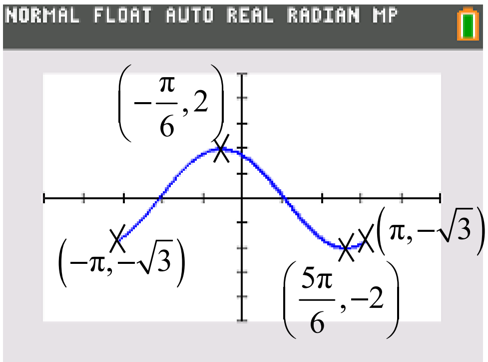
$$\frac{\mathrm{d}y}{\mathrm{d}x}=2y\cos 2x \text{ (shown)}$$

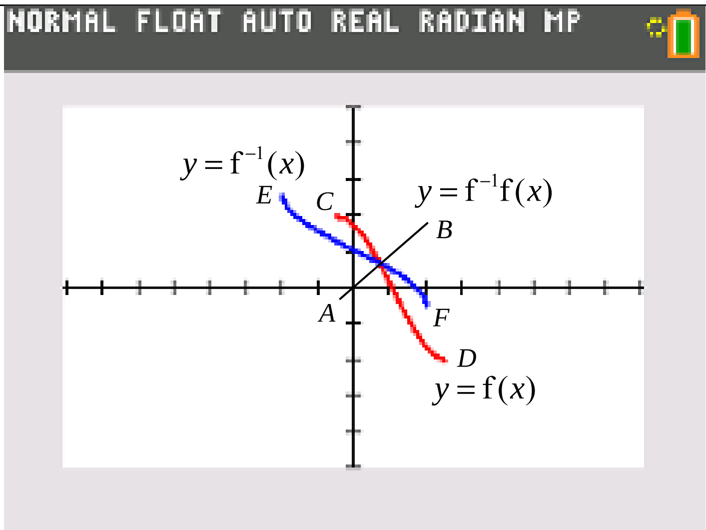
	<u>Method 2: Explicit differentiation</u> $\ln y = \sin 2x$ $y = \mathrm{e}^{\sin 2x}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cos 2x\ \mathrm{e}^{\sin 2x}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2y\cos 2x$ (shown) Differentiating both sides with respect to x: $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2\cos 2x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 4y\sin 2x$ When $x = 0,\ y = 1,\ \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2,\ \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 4$
5(ii)	$y = 1 + 2x + 4\left(\dfrac{x^2}{2!}\right) + \ldots = 1 + 2x + 2x^2 + \ldots$
5(iii)	$\mathrm{e}^{px}\ln(1+qx)$ $= \left(1 + px + \dfrac{(px)^2}{2!} + \ldots\right)\left(qx - \dfrac{(qx)^2}{2} + \ldots\right)$ $= \left(1 + px + \dfrac{p^2x^2}{2} + \ldots\right)\left(qx - \dfrac{q^2x^2}{2} + \ldots\right)$ $= qx - \dfrac{q^2x^2}{2} + pqx^2 + \ldots$ $= qx + \left(pq - \dfrac{q^2}{2}\right)x^2 + \ldots$ By comparison, $q = 2,\ pq - \dfrac{q^2}{2} = 2 \Rightarrow p = 2$ The expansion is valid for $-1 < 2x \le 1 \Rightarrow -\dfrac{1}{2} < x \le \dfrac{1}{2}$

6(i)	$x=1-\mathrm{e}^{-t}$, $\quad y=1+t^2$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\mathrm{e}^{-t}$, $\quad \frac{\mathrm{d}y}{\mathrm{d}t}=2t$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=2t\cdot\frac{1}{\mathrm{e}^{-t}}=2t\mathrm{e}^{t}$ When $t=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e}$ The required gradient is $2\mathrm{e}$
6(ii)	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=\left(2t\mathrm{e}^{t}+2\mathrm{e}^{t}\right)\cdot\frac{1}{\mathrm{e}^{-t}}=2\mathrm{e}^{2t}(t+1)$ Since $2\mathrm{e}^{2t}>0$ for $t\in\mathbb{R}$ but $t+1>0$ only for $t>-1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\mathrm{e}^{2t}(t+1)>0$ only for $t>-1$. $\therefore$ The curve is not concave upwards for all real values of t. **Note:** $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}\right)=\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\times\frac{\mathrm{d}t}{\mathrm{d}x}$
6(iii)	$x=1-\mathrm{e}^{-t}\Rightarrow \mathrm{e}^{-t}=1-x\Rightarrow t=-\ln(1-x)$ Substituting $t=-\ln(1-x)$ into $y=1+t^2$, $y=1+\left[\ln(1-x)\right]^2$
6(iv)	The required volume $=\pi\int_{-\frac{1}{2}}^{\frac{3}{5}}\left(1+\left[\ln(1-x)\right]^2\right)^2\ \mathrm{d}x$ $=4.6699=4.670$ units^3 (3 d.p) 

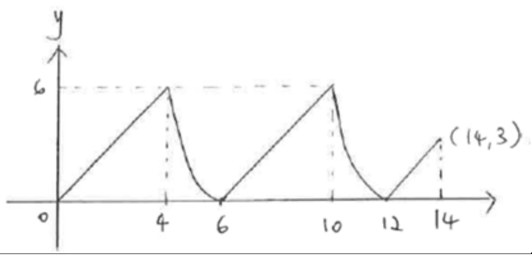
7(a)(i)	$\int \cos 2x \sin^5 2x \, \mathrm{d}x = \frac{1}{2}\int 2\cos 2x \sin^5 2x \, \mathrm{d}x$ $= \frac{1}{2} \cdot \frac{\sin^{5+1} 2x}{5+1} + c$ $= \frac{1}{12}\sin^6 2x + c$
7(a)(ii)	$\int \frac{\mathrm{e}^x}{\sqrt{1-2\mathrm{e}^{2x}}} \, \mathrm{d}x$ $u = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = u$ <u>Method 1</u> $\int \frac{\mathrm{e}^x}{\sqrt{1-2\mathrm{e}^{2x}}} \, \mathrm{d}x = \int \frac{u}{\sqrt{1-2u^2}} \cdot \frac{1}{u} \mathrm{d}u$ $= \int \frac{1}{\sqrt{1-2u^2}} \, \mathrm{d}u$ $= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{1}{2}-u^2}} \, \mathrm{d}u$ $= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - u^2}} \, \mathrm{d}u$ $= \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{2}u\right) + c$ $= \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{2}\mathrm{e}^x\right) + c$ <u>Method 2</u> $\int \frac{\mathrm{e}^x}{\sqrt{1-2\mathrm{e}^{2x}}} \, \mathrm{d}x = \int \frac{u}{\sqrt{1-2u^2}} \cdot \frac{1}{u} \mathrm{d}u$ $= \int \frac{1}{\sqrt{1-2u^2}} \, \mathrm{d}u$ $= \frac{1}{\sqrt{2}} \int \frac{\sqrt{2}}{\sqrt{1-\left(\sqrt{2}u\right)^2}} \, \mathrm{d}u$ $= \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{2}u\right) + c$ $= \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{2}\mathrm{e}^x\right) + c$

7(b)(i)	$\int x^2 \sin nx \, dx$ $= x^2\left(-\frac{\cos nx}{n}\right) + \frac{2}{n}\int x\cos nx \, dx$ $= -\frac{x^2}{n}\cos nx + \frac{2}{n}\left[x\left(\frac{\sin nx}{n}\right) - \frac{1}{n}\int \sin nx \, dx\right]$ $= -\frac{x^2}{n}\cos nx + \frac{2}{n}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right] + c$ $= \frac{2}{n^3}\cos nx + \frac{2x}{n^2}\sin nx - \frac{x^2}{n}\cos nx + c$
7(b)(ii)	$\int_{\pi}^{2\pi} x^2 \sin nx \, dx$ $= \left[\frac{2}{n^3}\cos nx - \frac{x^2}{n}\cos nx + \frac{2x}{n^3}\sin nx\right]_{\pi}^{2\pi}$ $= \frac{2}{n^3}\cos 2\pi n - \frac{(2\pi)^2}{n}\cos 2\pi \mathrm{n} + \frac{2}{n^3}(2\pi)\sin 2\pi n - \left[\frac{2}{n^3}\cos \pi n - \frac{\pi^2}{n}\cos \pi n + \frac{2}{n^3}(\pi)\sin \pi n\right]$ $= \frac{2}{n^3}(1) - \frac{4\pi^2}{n}(1) + 0 - \left[\frac{2}{n^3}\cos \pi n - \frac{\pi^2}{n}\cos \pi n + 0\right]$ $= \frac{2}{n^3} - \frac{4\pi^2}{n} - \left(\frac{2}{n^3} - \frac{\pi^2}{n}\right)\cos \pi n$ $\cos \pi n = \begin{cases} -1 & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even} \end{cases}$ For n odd, $\frac{2}{n^3} - \frac{4\pi^2}{n} - \left(\frac{2}{n^3} - \frac{\pi^2}{n}\right)\cos \pi n$ $= \frac{2}{n^3} - \frac{4\pi^2}{n} - \left(\frac{2}{n^3} - \frac{\pi^2}{n}\right)(-1)$ $= \frac{4}{n^3} - \frac{5\pi^2}{n} \Rightarrow p = 4, q = 5$ For n even, $\frac{2}{n^3} - \frac{4\pi^2}{n} - \left(\frac{2}{n^3} - \frac{\pi^2}{n}\right)\cos \pi n$ $= \frac{2}{n^3} - \frac{4\pi^2}{n} - \left(\frac{2}{n^3} - \frac{\pi^2}{n}\right)(1)$ $= -\frac{3\pi^2}{n} \Rightarrow p = 0, q = 3$

	$\therefore \int_{\pi}^{2\pi} x^2 \sin nx \, dx = \begin{cases} \dfrac{4}{n^3} - \dfrac{5\pi^2}{n} & \text{for } n \text{ odd} \\ -\dfrac{3\pi^2}{n} & \text{for } n \text{ even} \end{cases}$
8(i)	$R\cos\alpha = \sqrt{3}$; $R\sin\alpha = 1$ $R^2 = \sqrt{3}^2 + 1^2 \Rightarrow R = 2 \ (R > 0)$ $\tan\alpha = \dfrac{1}{\sqrt{3}} \Rightarrow \alpha = \dfrac{\pi}{6}$ (α is acute) $\mathrm{f}(x) = \sqrt{3}\cos x - \sin x = 2\cos\left(x + \dfrac{\pi}{6}\right)$
8(ii)	 A horizontal line $y = k$, $k \in (-2, 2)$ intersects the graph of $y = \mathrm{f}(x)$ at more than one point. f is not a one-one function. f^{-1} does not exist.
8(iii)	From the graph, the maximum value of k is $\dfrac{5\pi}{6}$
8(iv)	Let $y = \mathrm{f}(x)$ $y = 2\cos\left(x + \dfrac{\pi}{6}\right)$ $x = \cos^{-1}\left(\dfrac{y}{2}\right) - \dfrac{\pi}{6}$ $\mathrm{f}^{-1}(y) = \cos^{-1}\left(\dfrac{y}{2}\right) - \dfrac{\pi}{6}$ $\mathrm{f}^{-1}(x) = \cos^{-1}\left(\dfrac{x}{2}\right) - \dfrac{\pi}{6}$ Domain of f^{-1} = Range of $\mathrm{f} = [-2, 2]$
8	

	 Endpoints for $y=\mathrm{f}^{-1}\mathrm{f}(x): A\left(-\frac{\pi}{6},-\frac{\pi}{6}\right), B\left(\frac{5\pi}{6},\frac{5\pi}{6}\right)$ Endpoints for $y=\mathrm{f}(x): C\left(-\frac{\pi}{6},2\right), D\left(\frac{5\pi}{6},-2\right)$ Endpoints for $y=\mathrm{f}^{-1}(x): E\left(-2,\frac{5\pi}{6}\right), F\left(2,-\frac{\pi}{6}\right)$
8(v)	$\mathrm{g}^{2018}(2)=\mathrm{g}^3\mathrm{g}^{2015}(2)$ $=\mathrm{g}^{2015}(2)$ $=\mathrm{g}^3\mathrm{g}^{2012}(2)$ $=\mathrm{g}^{2012}(2)$ $=\ldots$ $=\mathrm{g}^3\mathrm{g}^2(2)$ $=\mathrm{g}^2(2)$ $=\frac{2-1}{2}=\frac{1}{2}$
9(i)	$T_n=S_n-S_{n-1}$ $=\frac{3}{2}n^2+\frac{13}{2}n-\left[\frac{3}{2}(n-1)^2+\frac{13}{2}(n-1)\right]$ $=\frac{3}{2}n^2+\frac{13}{2}n-\left[\frac{3}{2}n^2-3n+\frac{3}{2}+\frac{13}{2}n-\frac{13}{2}\right]$ $=3n+5$
9(ii)	The common ratio, $r=0.8$ Since $-1<r<1$, the sequence is convergent
9(iii)	$S_\infty=\frac{5}{1-0.8}=25\ <35$ $\therefore$ Terra will not reach the peak of the hill

9(iv)	<u>Method 1</u> $T_1,\ T_2,\quad T_3,\quad T_4,\qquad T_5$ $5, 5(0.8), 5(0.8^2), 5(0.8^3), 5(0.8^3)(0.9)$ $U_1 = 5(0.8^3)(0.9), U_2, ..., U_n$ $T_5 = U_1$ i.e. T_5 is the first term of the GP after r is changed from 0.8 to 0.9. $\left(T_1+T_2+T_3+T_4\right)+\left(U_1+U_2+U_3+....+U_n\right)=35$ $\frac{5(1-0.8^4)}{1-0.8}+\frac{5(0.8^3)(0.9)(1-0.9^n)}{1-0.9}=35$ $\frac{5(0.8^3)(0.9)(1-0.9^n)}{1-0.9}=20.24$ $1-0.9^n=0.87847$ $0.9^n=0.12153$ $n=\frac{\ln 0.12153}{\ln 0.9}=20.004$ The minmimum no. of 10-minute periods required $=4+21=25$ <u>Method 2</u> $T_1,\ T_2,\quad T_3,\quad T_4$ $5, 5(0.8), 5(0.8^2), 5(0.8^3)$ $U_1 = 5(0.8^3), U_2 = 5(0.8^3)(0.9), ..., U_n$ $T_4 = U_1$ i.e. T_4 is the first term of the GP. $\left(T_1+T_2+T_3\right)+\left(U_1+U_2+U_3+....+U_n\right)=35$ $\frac{5(1-0.8^3)}{1-0.8}+\frac{5(0.8^3)(1-0.9^n)}{1-0.9}=35$ $1-0.9^n=0.890625$ $0.9^n=0.109375$ $n=\frac{\ln 0.109375}{\ln 0.9}=21.004$ The minimum number of 10-minute periods required =3+22 =25

9(v)	
9(vi)	<u>Method 1</u> The distance covered $=\int_4^6 \frac{3}{2}t^2-18t+54\ \mathrm{d}t+\frac{1}{2}(4)(6)+\int_4^5 \frac{3}{2}t^2-18t+54\ \mathrm{d}t$ $=19.5$ m <u>Method 2</u> The distance covered $=\int_4^6 \frac{3}{2}t^2-18t+54\ \mathrm{d}t+\frac{1}{2}(4)(6)+\int_4^5 \frac{3}{2}t^2-18t+54\ \mathrm{d}t$ $=\left(\frac{1}{2}t^3-9t^2+54t\right)_4^6+12+\left(\frac{1}{2}t^3-9t^2+54t\right)_4^5$ $=(108-104)+12+\left(\frac{215}{2}-104\right)=19\frac{1}{2}$ m
10(i)	$l_1:\mathbf{r}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix},\lambda\in\mathbb{R}$ $l_2:\mathbf{r}=\begin{pmatrix}0\\5\\15\end{pmatrix}+\alpha\begin{pmatrix}-1\\6\\12\end{pmatrix},\alpha\in\mathbb{R}$

Equating the equations of both lines,

$$\begin{pmatrix}-2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}0\\5\\15\end{pmatrix}+\alpha\begin{pmatrix}-1\\6\\12\end{pmatrix}$$

$$\begin{pmatrix}-2-\lambda\\1+2\lambda\\3+3\lambda\end{pmatrix}=\begin{pmatrix}-\alpha\\5+6\alpha\\15+12\alpha\end{pmatrix}$$

$$2+\lambda=\alpha\Rightarrow\alpha-\lambda=2$$
$$1+2\lambda=5+6\alpha\Rightarrow 3\alpha-\lambda=-2$$

Solving both equations simultaneously,

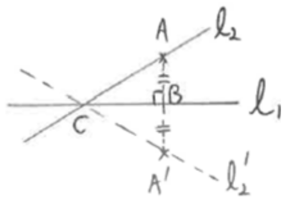
$\alpha=-2,\lambda=-4$

$$\overrightarrow{OC}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+(-4)\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}2\\-7\\-9\end{pmatrix}$$

10(ii)

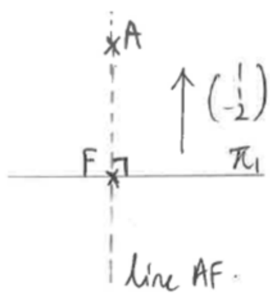
By Ratio Theroem, $\overrightarrow{OB}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$

$$\begin{pmatrix}-5\\7\\12\end{pmatrix}=\frac{1}{2}\left[\begin{pmatrix}0\\5\\15\end{pmatrix}+\overrightarrow{OA'}\right]$$



$$\overrightarrow{OA'}=\begin{pmatrix}-10\\9\\9\end{pmatrix}$$

$$\overrightarrow{CA'}=\begin{pmatrix}-10\\9\\9\end{pmatrix}-\begin{pmatrix}2\\-7\\-9\end{pmatrix}=\begin{pmatrix}-12\\16\\18\end{pmatrix}=2\begin{pmatrix}-6\\8\\9\end{pmatrix}$$

	<u>Method 1</u> A vector equation of the line of intersection is $\mathbf{r}=\begin{pmatrix}-10\\9\\9\end{pmatrix}+s\begin{pmatrix}-6\\8\\9\end{pmatrix},\ s\in\mathbb{R}$ Cartesian equation of the line of intersection is $\frac{-x-10}{6}=\frac{y-9}{8}=\frac{z-9}{9}$ <u>Method 2</u> A vector equation of the line of intersection is $\mathbf{r}=\begin{pmatrix}2\\-7\\-9\end{pmatrix}+t\begin{pmatrix}-6\\8\\9\end{pmatrix},\ t\in\mathbb{R}$ Cartesian equation of the line of intersection is $\frac{2-x}{6}=\frac{y+7}{8}=\frac{z+9}{9}$
10(iii)	$x+y-2z=-1\Leftrightarrow \mathbf{r}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=-1$ Let the point F be the foot of the perpendicular from A to π_1 Line AF : $\mathbf{r}=\begin{pmatrix}0\\5\\15\end{pmatrix}+\mu\begin{pmatrix}1\\1\\-2\end{pmatrix},\mu\in\mathbb{R}$ $\pi_1: x+y-2z=-1\Leftrightarrow \mathbf{r}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=-1$ By substitution, $\left[\begin{pmatrix}0\\5\\15\end{pmatrix}+\mu\begin{pmatrix}1\\1\\-2\end{pmatrix}\right]\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=-1$ $\begin{pmatrix}\mu\\5+\mu\\15-2\mu\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=-1$ $6\mu-25=-1\Rightarrow\mu=4$ 

$$\overrightarrow{OF}=\begin{pmatrix}0\\5\\15\end{pmatrix}+4\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}4\\9\\7\end{pmatrix}\Rightarrow F\text{ is }(4,9,7)$$

$$\overrightarrow{AF}=\begin{pmatrix}4\\9\\7\end{pmatrix}-\begin{pmatrix}0\\5\\15\end{pmatrix}=\begin{pmatrix}4\\4\\-8\end{pmatrix}$$

The required perpendicular distance

$$=\left|\overrightarrow{AF}\right|=\sqrt{4^2+4^2+(-8)^2}=\sqrt{96}=4\sqrt{6}\text{ units}$$

10(iv)

$$\sin 30^\circ=\frac{\begin{pmatrix}-1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}a\\1\\3\end{pmatrix}}{\sqrt{14}\sqrt{a^2+10}}$$

$$\left(\frac{1}{2}\right)\sqrt{14}\sqrt{a^2+10}=-a+11$$

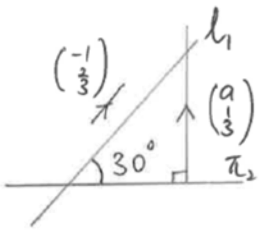
$$\sqrt{14(a^2+10)}=2(11-a)$$

$$14(a^2+10)=4(11-a)^2$$

$$10a^2+88a-344=0$$

$$a=2.9321\quad\text{or}\quad -11.732$$

$$a=2.93\text{ or }-11.7\text{ (3 s.f)}$$



10(v)

$$\pi_2: ax+y+3z=5\Leftrightarrow \mathbf{r}\cdot\begin{pmatrix}a\\1\\3\end{pmatrix}=5$$

A direction vector parallel to the line of intersection

$$=\begin{pmatrix}1\\1\\-2\end{pmatrix}\times\begin{pmatrix}a\\1\\3\end{pmatrix}=\begin{pmatrix}5\\-2a-3\\1-a\end{pmatrix}$$

By comparison to $\begin{pmatrix}5\\-1\\2\end{pmatrix}$, $-2a-3=-1\Rightarrow a=-1$

(verified by $1-a=2\Rightarrow a=-1$)

	$\pi_1: \; x+y-2z=-1$ $\pi_2: \; -x+y+3z=5$ Let $z=0$, $\quad x+y=-1$, $\qquad\qquad -x+y=5$ Solving both equations simultaneously, $y=2,\; x=-3$. $(-3,2,0)$ lies on the line of intersection. A possible position vector of V is $\begin{pmatrix}-3\\2\\0\end{pmatrix}$.
11(i)	$x_{n+1}-x_n=\frac{13}{2}-n$ $\sum_{r=1}^{n}\left(x_r-x_{r-1}\right)=\sum_{r=1}^{n}\left(\frac{13}{2}-r\right)$ $\begin{pmatrix} x_1-x_0 \\ +x_2-x_1 \\ +x_3-x_2 \\ \ldots \\ +x_{n-1}-x_{n-2} \\ +x_n-x_{n-1}\end{pmatrix}=\sum_{r=1}^{n}\frac{13}{2}-\sum_{r=1}^{n}r$ $x_n-x_0=\frac{13}{2}n-\frac{n}{2}(n+1)$ $x_n-8=6n-\frac{1}{2}n^2$ $x_n=-\frac{1}{2}n^2+6n+8$
11(ii)	$\frac{\mathrm{d}x}{\mathrm{d}t}=k\left(50-2t-x\right)$ Given that $\frac{\mathrm{d}x}{\mathrm{d}t}=14$, $x=8$ when $t=0$, $14=k\left(50-0-8\right)$ $k=\frac{1}{3}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{3}\left(50-2t-x\right)$ (shown)

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{3}(50-2t-x)$$

$$u=2t+x \Rightarrow x=u-2t$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}u}{\mathrm{d}t}-2$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}-2=\frac{1}{3}[50-2t-(u-2t)]$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{1}{3}(50-u)+2$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{1}{3}(56-u)$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{1}{3}(56-u)$$

$$\frac{1}{56-u}\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{1}{3}$$

$$\int \frac{1}{56-u}\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}t=\int \frac{1}{3}\,\mathrm{d}t$$

$$-\ln|56-u|=\frac{1}{3}t+C_1$$

$$\ln|56-u|=-\frac{1}{3}t+C_2 \text{ where } C_2=-C_1$$

$$|56-u|=\mathrm{e}^{-\frac{1}{3}t+C_2}$$

$$56-u=\pm\mathrm{e}^{C_2}\mathrm{e}^{-\frac{1}{3}t}$$

$$56-u=A\mathrm{e}^{-\frac{1}{3}t} \text{ where } A=\pm\mathrm{e}^{C_2}$$

$$u=56-A\mathrm{e}^{-\frac{1}{3}t}$$

$$2t+x=56-A\mathrm{e}^{-\frac{1}{3}t}$$

$$x=56-A\mathrm{e}^{-\frac{1}{3}t}-2t$$

Given that $x=8$ when $t=0$,

$$8=56-A\mathrm{e}^{-\frac{1}{3}(0)}-2(0) \Rightarrow A=48$$

$$x=56-48\mathrm{e}^{-\frac{1}{3}t}-2t$$

11(iii)	$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}=-\frac{100}{(t+1)^3}$ $\int \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\,\mathrm{d}t=-100\int (t+1)^{-3}\mathrm{d}t$ $\frac{\mathrm{d}x}{\mathrm{d}t}=50(t+1)^{-2}+C$ $\int \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t=50\int (t+1)^{-2}\mathrm{d}t$ $x=-50(t+1)^{-1}+Ct+D$ $x=-\frac{50}{t+1}+Ct+D$ Given that $\frac{\mathrm{d}x}{\mathrm{d}t}=41\frac{2}{3}=\frac{125}{3}$, $x=8$ when $t=0$, $\frac{125}{3}=50(0+1)^{-2}+C \Rightarrow C=-\frac{25}{3}$ $8=-\frac{50}{0+1}+0+D \Rightarrow D=58$ $x=-\frac{50}{t+1}-\frac{25}{3}t+58$

Section A

Qn	Suggested Solution
1	$y=-x^2+3x-1 \Rightarrow x^2-3x+1+y=0$ $\Rightarrow x=\dfrac{-(-3)\pm\sqrt{(-3)^2-4(1)(1+y)}}{2(1)}$ $\Rightarrow x=\dfrac{3\pm\sqrt{5-4y}}{2}$ Since $x<\dfrac{3}{2}$, $x=\dfrac{3-\sqrt{5-4y}}{2}$ Required volume = $\pi\left[\int_{-1}^{1}\left(\dfrac{3-\sqrt{5-4y}}{2}\right)^2 \mathrm{d}y-\int_0^1\left(y^2\right)^2 \mathrm{d}y\right]$ $=\pi\left[\dfrac{1}{4}\int_{-1}^{1}9-6\sqrt{5-4y}+(5-4y)\,\mathrm{d}y-\int_0^1 y^4\,\mathrm{d}y\right]$ $=\pi\left[\dfrac{1}{2}\int_{-1}^{1}7-2y-3\left(5-4y\right)^{\frac{1}{2}}\mathrm{d}y-\int_0^1 y^4\,\mathrm{d}y\right]$ $=\dfrac{\pi}{2}\left[7y-y^2+\dfrac{1}{2}\left(5-4y\right)^{\frac{3}{2}}\right]_{-1}^{1}-\left[\dfrac{\pi}{5}y^5\right]_0^1$ $=\dfrac{\pi}{2}\left[\left(7-1+\dfrac{1}{2}\right)-\left(-7-1+\dfrac{1}{2}9^{\frac{3}{2}}\right)\right]-\dfrac{\pi}{5}$ $=\dfrac{\pi}{2}\left(\dfrac{13}{2}-\dfrac{11}{2}\right)-\dfrac{\pi}{5}=\dfrac{\pi}{2}-\dfrac{\pi}{5}=0.3\pi$ units^3.

2(a) (i)	$x=k\cos t \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t}=-k\sin t$ $y=k\sin 2t \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}t}=2k\cos 2t$ $\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}t}\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{2k\cos 2t}{-k\sin t}=-\dfrac{2\cos 2t}{\sin t}$. At $(k\cos p, k\sin 2p)$, $t=p \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{2\cos 2p}{\sin p}$. Gradient of normal = $\dfrac{\sin p}{2\cos 2p}$. Equation of normal: $y-k\sin 2p=\dfrac{\sin p}{2\cos 2p}(x-k\cos p)$
2(a) (ii)	From **(a)(i)**, when $t=p=0$, equation of normal at $(k\cos 0, k\sin 2(0))$, i.e. $(k,0)$ is $y-k\sin 2(0)=\dfrac{\sin 0}{2\cos 2(0)}(x-k\cos 0)$, i.e. $y=0$. Hence, equation of tangent at $(k,0)$ is $x=k$.

2(b) (i)	$A=\left(2\lvert x\rvert\right)\left(2\lvert y\rvert\right)=4\sqrt{x^2}\sqrt{y^2}=4\sqrt{x^2y^2}$ $=4\sqrt{36\left(1-\dfrac{y^2}{9}\right)y^2}$ $=4\sqrt{\left(36-4y^2\right)y^2}$ $=8\sqrt{9y^2-y^4}$ (shown)

2(b) (ii)

At stationary points,

$$\frac{\mathrm{d}A}{\mathrm{d}y}=\frac{8}{2}\left(9y^2-y^4\right)^{-\frac{1}{2}}\left(18y-4y^3\right)=\frac{8y\left(9-2y^2\right)}{\sqrt{9y^2-y^4}}=0$$

$$\Rightarrow 8y\left(9-2y^2\right)=0$$

$$\Rightarrow y=0 \text{ (rejected as } y\neq 0\text{) or } y=\pm\sqrt{\frac{9}{2}}=\pm\frac{3}{\sqrt{2}}=\pm\frac{3}{2}\sqrt{2}.$$

Method 1: First derivative test

y	$\left(\pm\frac{3}{2}\sqrt{2}\right)^-$	$\pm\frac{3}{2}\sqrt{2}$	$\left(\pm\frac{3}{2}\sqrt{2}\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}y}$	+ve	0	-ve
Slope			

Method 2: Second derivative test

$$\sqrt{9y^2-y^4}\,\frac{\mathrm{d}A}{\mathrm{d}y}=8(9y-2y^3)$$

$$\sqrt{9y^2-y^4}\,\frac{\mathrm{d}^2A}{\mathrm{d}y^2}+\frac{\mathrm{d}A}{\mathrm{d}y}\left[\frac{\mathrm{d}}{\mathrm{d}y}\left(\sqrt{9y^2-y^4}\right)\right]=8(9-6y^2)$$

When $y=\pm\frac{3}{2}\sqrt{2}$, $4.5\frac{\mathrm{d}^2A}{\mathrm{d}y^2}+0=8(-18)\Rightarrow\frac{\mathrm{d}^2A}{\mathrm{d}y^2}=-32<0.$

Hence, A is a maximum when $y=\pm\frac{3}{2}\sqrt{2}$.

Required area $=8\sqrt{9\left(\pm\frac{3}{2}\sqrt{2}\right)^2-\left(\pm\frac{3}{2}\sqrt{2}\right)^4}$

$=8\sqrt{9\left(\frac{9}{2}\right)-\frac{81}{4}}=8\sqrt{\frac{81}{4}}$

$=36$ units^2.

3(a)	$a-\mathrm{i}$ is a root of $z^3+4(1+\mathrm{i})z^2+(-2+9\mathrm{i})z-5+\mathrm{i}=0$ $\Rightarrow(a-\mathrm{i})^3+4(1+\mathrm{i})(a-\mathrm{i})^2+(-2+9\mathrm{i})(a-\mathrm{i})-5+\mathrm{i}=0$ $\Rightarrow(a^3-3a^2\mathrm{i}-3a+\mathrm{i})+4(1+\mathrm{i})(a^2-2a\mathrm{i}-1)\ -2a+4+(3+9a)\mathrm{i}=0$

	$\Rightarrow a^3-5a+4-3a^2\text{i}+4\text{i}+9a\text{i}+4(a^2-2a\text{i}-1+a^2\text{i}+2a-\text{i})=0$ $\Rightarrow (a^3+4a^2+3a)+(a^2+a)\text{i}=0$. (shown) Comparing real and imaginary parts, $a(a+1)(a+3)=0 \Rightarrow a=0$ or $a=-1$ or $a=-3$ (N.A) $a(a+1)=0 \Rightarrow a=0$ or $a=-1$. Let the third root be w. Then $z^3+4(1+\text{i})z^2+(-2+9\text{i})z-5+\text{i}$ $=[z-(-\text{i})][z-(-1-\text{i})][z-(w)]=(z+\text{i})(z+1+\text{i})(z-w)$ Comparing constant term, $\text{i}(1+\text{i})(-w)=-5+\text{i}$ $\Rightarrow w=\dfrac{-5+\text{i}}{-\text{i}(1+\text{i})}=\dfrac{-5+\text{i}}{1-\text{i}}\times\dfrac{1+\text{i}}{1+\text{i}}=\dfrac{-5+\text{i}-5\text{i}-1}{2}=-3-2\text{i}.$ Hence, all the roots are $-\text{i}$, $-1-\text{i}$ and $-3-2\text{i}$. $z^3+8(1+\text{i})z^2+4(-2+9\text{i})z+8(-5+\text{i})=0$ $\dfrac{1}{8}z^3+(1+\text{i})z^2+\dfrac{1}{2}(-2+9\text{i})z+(-5+\text{i})=0$ $\left(\dfrac{1}{2}z\right)^3+4(1+\text{i})\left(\dfrac{1}{2}z\right)^2+(-2+9\text{i})\left(\dfrac{1}{2}z\right)+(-5+\text{i})=0$ $\Rightarrow \dfrac{1}{2}z=-\text{i}$ or $-1-\text{i}$ or $-3-2\text{i}$ (from first part) $\Rightarrow z=-2\text{i}$ or $-2-2\text{i}$ or $-6-4\text{i}$
3(b) (i)	$\left\|\left(1-\sqrt{3}\text{i}\right)^4\right\|=\left\|1-\sqrt{3}\text{i}\right\|^4=\left(\sqrt{1^2+(-\sqrt{3})^2}\right)^4=2^4=16.$ $\arg\left(1-\sqrt{3}\text{i}\right)^4=4\arg\left(1-\sqrt{3}\text{i}\right)=4\left(-\dfrac{\pi}{3}\right)=-\dfrac{4}{3}\pi.$ $\therefore \arg\left(1-\sqrt{3}\text{i}\right)^4=-\dfrac{4\pi}{3}+2\pi=\dfrac{2}{3}\pi.$ (for principal argument)
3(b) (ii)	$\dfrac{w^n}{w^*}$ is real $\Rightarrow \arg\left(\dfrac{w^n}{w^*}\right)=k\pi$, where $k\in\mathbf{Z}$ $\Rightarrow \arg\left(w^n\right)-\arg\left(w^*\right)=k\pi$ $\Rightarrow n\arg\left(w\right)+\arg\left(w\right)=k\pi$ $\Rightarrow \dfrac{2\pi}{3}(n+1)=k\pi$ (from first part) OR $n=\dfrac{3}{2}k-1$ Hence, the required values of n are 2, 5 and 8. **Alternatively**, $\dfrac{w^n}{w^*}$ is real $\Rightarrow \text{Im}\left(\dfrac{w^n}{w^*}\right)=0 \Rightarrow \sin\left[\arg\left(\dfrac{w^n}{w^*}\right)\right]=0$ $\Rightarrow \sin\left[\dfrac{2\pi}{3}(n+1)\right]=0 \Rightarrow \dfrac{2\pi}{3}(n+1)=k\pi$, where $k\in\mathbf{Z}$. Hence, the required values of n are 2, 5 and 8.

4(i)	$\mathbf{a} \perp \mathbf{b} - \mathbf{a} \Rightarrow \mathbf{a}\cdot(\mathbf{b}-\mathbf{a})=0$ $\Rightarrow \mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{a} = \lvert\mathbf{a}\rvert^2 = 1$ (since $\lvert\mathbf{a}\rvert = 1$) $\Rightarrow \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta = 1$ where θ is the angle between **a** and **b** $\Rightarrow \cos\theta = \dfrac{1}{\lvert\mathbf{b}\rvert} < 1 \Rightarrow \lvert\mathbf{b}\rvert > 1$. (shown) $\lvert\mathbf{b}\rvert = 2 \Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = 60^{\text{o}}$.
4(ii) (a)	Let α be the required angle. $\cos\alpha = 0.6 \Rightarrow \alpha = \cos^{-1} 0.6 = 53.1^{\text{o}}$. (1 d.p.)
4(ii) (b)	$\lambda = \cos 90^{\text{o}} = 0$. $0.6^2 + 0^2 + \mu^2 = 1 \Rightarrow \mu^2 = 1 - 0.36 = 0.64 \Rightarrow \mu = \pm 0.8$.
4(iii)	By Ratio Theorem, $\mathbf{b} = \dfrac{\mathbf{a}+3\mathbf{c}}{4} \Rightarrow \mathbf{c} = \dfrac{4\mathbf{b}-\mathbf{a}}{3}$. $\lvert\mathbf{c}\cdot\mathbf{a}\rvert = \left\lvert\left(\dfrac{4\mathbf{b}-\mathbf{a}}{3}\right)\cdot\mathbf{a}\right\rvert = \left\lvert\dfrac{4}{3}(\mathbf{b}\cdot\mathbf{a}) - \dfrac{1}{3}(\mathbf{a}\cdot\mathbf{a})\right\rvert$ $= \left\lvert\dfrac{4}{3}(\mathbf{a}\cdot\mathbf{a}) - \dfrac{1}{3}(\mathbf{a}\cdot\mathbf{a})\right\rvert$ (from (**i**)) $= \lvert\mathbf{a}\cdot\mathbf{a}\rvert = \left\lvert\lvert\mathbf{a}\rvert^2\right\rvert = \lvert\mathbf{a}\rvert^2 = \mathbf{a}\cdot\mathbf{a}$. (shown) Required length = $\dfrac{\lvert\mathbf{c}\cdot\mathbf{a}\rvert}{\lvert\mathbf{a}\rvert} = \dfrac{\mathbf{a}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert} = \dfrac{\lvert\mathbf{a}\rvert^2}{\lvert\mathbf{a}\rvert} = \lvert\mathbf{a}\rvert$.
4(iv)	$\lvert\mathbf{c}\times\mathbf{a}\rvert$ = Twice the area of triangle OAC $\lvert\mathbf{b}\times\mathbf{a}\rvert$ = Twice the area of triangle OAB = $(AB)h$, where h is the height of triangle OAB with AB as the base. Since A, B and C are collinear, $\lvert\mathbf{c}\times\mathbf{a}\rvert = (AC)h$. Hence, $\dfrac{\lvert\mathbf{c}\times\mathbf{a}\rvert}{\lvert\mathbf{b}\times\mathbf{a}\rvert} = \dfrac{(AC)h}{(AB)h} = \dfrac{AC}{AB} = \dfrac{4}{3}$.

Section B

5(i)	Mean score = $\dfrac{1}{5}+\dfrac{2}{6}+\dfrac{3}{7}+\dfrac{4}{8}+5p+6q = 3.5 \Rightarrow 5p+6q = \dfrac{214}{105}$ … (1) $\dfrac{1}{5}+\dfrac{1}{6}+\dfrac{1}{7}+\dfrac{1}{8}+p+q = 1 \Rightarrow p+q = \dfrac{307}{840}$ … (2) From graphing calculator, $p = \dfrac{13}{84}$ and $q = \dfrac{59}{280}$.
5(ii)	Let X be the score of the die and Y be the outcome of coin. For w = 0, 1, $\text{P}(W = w) = \text{P}(X = w+1,\ Y = \text{tail}) = \dfrac{1}{2}\text{P}(X = w+1)$. For w = 2, 3, 4, 5,

	P($W = w$) = P($X = w+1$, Y = tail) + P($X = w-1$, Y = head) $= \frac{1}{2}\left[\mathrm{P}(X = w-1) + \mathrm{P}(X = w+1)\right]$ For w = 6, 7, P($W = w$) = P($X = w-1$, Y = head) = $\frac{1}{2}\mathrm{P}(X = w-1)$. The probability distribution of W is given by: w: 0, 1, 2, 3, 4, 5, 6, 7 P($W = w$): $\frac{1}{10}$, $\frac{1}{12}$, $\frac{6}{35}$, $\frac{7}{48}$, $\frac{25}{168}$, $\frac{47}{280}$, $\frac{13}{168}$, $\frac{59}{560}$
5(iii)	Since the mean score of the die is 3.5 and the coin is fair, the expected winnings is \$3.5, hence I would not play this game since amount pay = \$4 > \$3.5 = expected winnings.

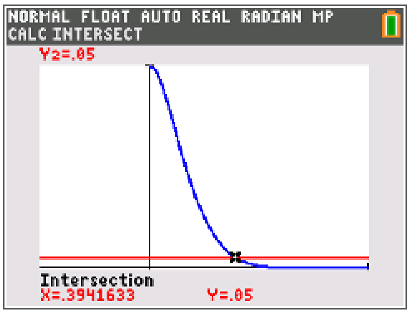
w	0	1	2	3	4	5	6	7
P($W = w$)	$\frac{1}{10}$	$\frac{1}{12}$	$\frac{6}{35}$	$\frac{7}{48}$	$\frac{25}{168}$	$\frac{47}{280}$	$\frac{13}{168}$	$\frac{59}{560}$

6(i)	Number of ways = $8! \times {}^{9}\mathrm{C}_2 \times 2! \times 2! = 5806080$. OR Number of ways = $10! \times 2! - 9! \times 2! \times 2 = 5806080$.
6(ii)	Number of ways = $(5-1)! \times 3! \times 4! \times 3! \times 12 = 248832$.
6(iii)	Case 1: All 4 letters are distinct Number of code-words = ${}^{4}\mathrm{C}_4 \times 4! = 24$. Case 2: Exactly 3 distinct letters Number of code-words = $\left(\frac{4!}{2!}\right) \times (6) = 12 \times 6 = 72$. OR $\left({}^{4}\mathrm{C}_2 \times 2!\right) \times \left({}^{2}\mathrm{C}_1 \times {}^{3}\mathrm{C}_2\right) = 12 \times 6 = 72$. OR $\left({}^{4}\mathrm{C}_2 \times 2! \times 2 \times 2\right) + \left({}^{4}\mathrm{C}_2 \times 2! \times 2\right) = 48 + 24 = 72$. Case 3: Exactly 2 distinct letters Number of code-words = $\left({}^{4}\mathrm{C}_1 \times 3\right) + \left({}^{4}\mathrm{C}_2\right) = 12 + 6 = 18$. Total number of code-words = 24 + 72 + 18 = 114.

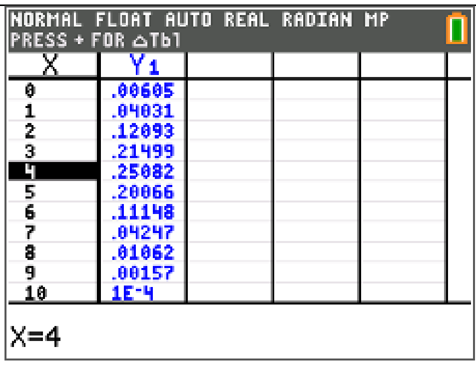
7(i)	P(B) = $\frac{130}{400} = \frac{13}{40}$ and P(C) = $\frac{150}{400} = \frac{3}{8}$. Since $\mathrm{P}(B \cap C) = \frac{70}{400} = \frac{7}{40} \neq \frac{39}{320} = \frac{13}{40} \times \frac{3}{8} = \mathrm{P}(B)\mathrm{P}(C)$, B and C are not independent.
7(ii) (a)	$\mathrm{P}(B \cup C') = \frac{320}{400} = \frac{4}{5}$.
7(ii) (b)	$\mathrm{P}(C \mid M \cap B) = \frac{10}{30} = \frac{1}{3}$.
7	Let x be the number of students studying Further Mathematics out of the 400 students and F be the event that the chosen student is studying Further Mathematics. $\mathrm{P}(F \cap C) = \mathrm{P}(F)\ \mathrm{P}(C)$ (since F and C are independent) $= \frac{3}{8} \times \frac{x}{400} = \frac{3x}{3200}$

	Since $F \subseteq M, F \cap C \subseteq M \cap C$. Hence, $\mathrm{P}(F) \le \mathrm{P}(M)$ and $\mathrm{P}(F \cap C) \le \mathrm{P}(M \cap C)$ $\Rightarrow \mathrm{P}(F) \le \frac{250}{400} = \frac{5}{8}$ and $\mathrm{P}(F \cap C) \le \frac{40}{400} = \frac{1}{10}$ $\Rightarrow \frac{x}{400} \le \frac{5}{8}$ and $\frac{3x}{3200} \le \frac{1}{10}$ $\Rightarrow x \le 250$ and $x \le \frac{320}{3} \Rightarrow x \le 106\frac{2}{3}$. Hence, required number of students is 106.

8(i)	Given: $X \sim \mathrm{N}\left(\mu, 2^2\right)$. $\mathrm{P}(X \le 1) = \mathrm{P}\left(Z \le \frac{1-\mu}{2}\right) = 0.1587$, where $Z \sim \mathrm{N}(0,1)$ $\Rightarrow \frac{1-\mu}{2} \approx -0.9998 \Rightarrow \mu = 3$. (nearest integer)
8(ii)	By symmetry, $\frac{(3k)+(9k+4)}{2} = \mu = 3 \Rightarrow k = \frac{1}{6}$.
8(iii)	$\mathrm{E}(\bar{Y}) = \mathrm{E}(Y) = 10 - \mathrm{E}(X) = 10 - 3 = 7$. $\mathrm{Var}(\bar{Y}) = \frac{\mathrm{Var}(Y)}{2} = \frac{\mathrm{Var}(X)}{2} = \frac{4}{2} = 2$. $\mathrm{P}\left(\bar{Y} > 6\right) \approx 0.76025 = 0.760$. (3 s.f.)

9(i)	Two assumptions are: 1. The probability of <u>a student being male</u> is the same for **each** <u>student</u>. 2. Whether <u>a student is male</u> is independent of other <u>students</u>.
9(ii)	$\mathrm{P}(X \le 1) = 0.05 \Rightarrow (1-p)^{10} + 10p(1-p)^9 = 0.05$ $\Rightarrow (1-p)^9(1+9p) = 0.05$  From graphing calculator, $p = 0.394$. (3 s.f.)
9(iii)	Let Y be the number of samples of 10 students, out of 8, with at least 2 male students. Then $Y \sim \mathrm{B}(8, \mathrm{P}(X \ge 2))$, i.e. $Y \sim \mathrm{B}(8, 0.95)$. Required probability $= \mathrm{P}(Y = 7) \approx 0.27933 = 0.279$. (3 s.f.) OR ${}^{8}\mathrm{C}_7 (0.95)^7 (0.05) = 0.279$. (3 s.f.)

9(iv)



x	$P(X=x)$
3	0.21499
4	0.25082
5	0.20066

From graphing calculator, the highest probability occurs when $X = 4$, so most probable number of male students in a sample of 10 is 4, i.e. most probable number of female students = 10 − 4 = 6.

9(v)

Let S be the sum of 60 independent observations of X.
$E(S) = 60E(X) = 60(10)(0.4) = 240$.
$Var(S) = 60Var(X) = 60(10)(0.4)(0.6) = 144$.
Since $n = 60$ is large, by Central Limit Theorem,
$S \sim N(240, 144)$ approximately.
$P(S > 230) \approx 0.79767 = 0.798$. (3 s.f.)

10(i)

Unbiased estimate of population mean mass,

$$\bar{x} = \frac{\sum(x-600)}{50} + 600 = \frac{-8}{50} + 600 = 599.84.$$

Unbiased estimate of population variance,

$$s^2 = \frac{1}{50-1}\left[\sum(x-600)^2 - \frac{1}{50}\left(\sum(x-600)\right)^2\right]$$

$$= \frac{1}{49}\left[11.3 - \frac{1}{50}(-8)^2\right] = \frac{10.02}{49} = \frac{501}{2450} = 0.204. \text{ (3 s.f.)}$$

10(ii)

Let X be the mass, in grams, of a randomly chosen packet of cereal.
$H_0 : \mu = 600$ (distributor's claim)
$H_1 : \mu < 600$ (claim is overstated)
Under H_0, since $n = 50$ is large, by Central Limit Theorem,

$$\bar{X} \sim N\left(600, \frac{1}{50} \times \frac{501}{2450}\right), \text{i.e. } N\left(600, \frac{5.01}{1225}\right) \text{ approximately.}$$

Use a z-test at $\alpha = 0.01$.
From graphing calculator, p-value = 0.00618 (3 s.f.)
Since p-value $= 0.00618 < 0.01 = \alpha$, we reject H_0.
There is sufficient evidence at 1% level of significance to conclude that the distributor has overstated the claim.

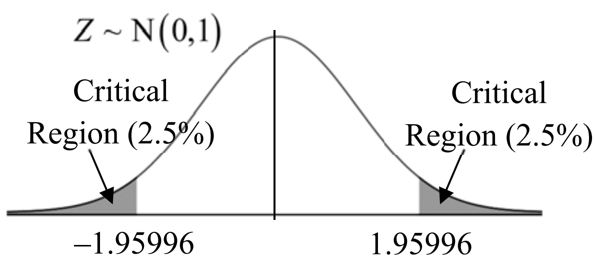
10(iii)

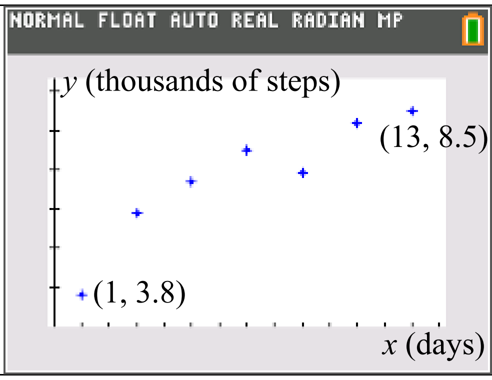
No. Since sample size <u>$n = 50$ is large</u>, <u>by Central Limit Theorem</u>, the sample mean mass of a packet of cereal will be <u>approximately normal</u>.

10(iv)

A p-value of <u>0.00618</u> means that there is a probability of <u>0.00618</u> of obtaining <u>a sample mean mass of 599.84 grams **or less**</u>, when the <u>population mean mass of cereals per packet is assumed to be 600 grams</u>.

Let Y be the mass, in grams, of a randomly chosen packet of cereal, after the change in packaging process.

	$H_0 : \mu = \mu_0$ (distributor's claim) $H_1 : \mu \neq \mu_0$ (retailer's suspicion) Under H_0, $\overline{Y} \sim N(\mu_0, \frac{0.5^2}{25})$, i.e. $N(\mu_0, 0.01)$. Standardising the test statistic, $Z = \frac{\overline{Y}-\mu_0}{\sqrt{\frac{0.5^2}{25}}} \sim N(0,1)$. Corresponding test statistic value: $z = \frac{600.6-\mu_0}{\sqrt{\frac{0.5^2}{25}}} = \frac{600.6-\mu_0}{\sqrt{0.01}}$ Critical values: –1.9599 and 1.9599 Critical region: $z \leq -1.95996$ or $z \geq 1.95996$  Since the retailer's suspicion is valid at 5% significance level, we **reject** H_0, $z = \frac{600.6-\mu_0}{\sqrt{\frac{0.5^2}{25}}}$ **falls in** the critical region. $\Rightarrow \frac{600.6-\mu_0}{\sqrt{0.01}} \leq -1.95996$ or $\frac{600.6-\mu_0}{\sqrt{0.01}} \geq 1.95996$ $\Rightarrow 600.7960 \leq \mu_0$ or $600.4040 \geq \mu_0$ $\Rightarrow \mu_0 \leq 600.40$ or $\mu_0 \geq 600.80$ (2 d.p.)

11(i)	
11(ii)	A possible reason is that the step tracker is not working properly on Day 9.
11(iii)	y on x line: $y = 0.34969x + 4.4354$ (5 s.f.), i.e. $y = 0.350x + 4.44$ (3 s.f.). x on y line: $x = 2.4769y - 10.094$ (5 s.f.), i.e. $x = 2.48y - 10.1$ (3 s.f.).
11(iv)	350 is the **expected** increase in the number of steps when the time increases by 1 day. OR Kenny is expected to increase 350 steps every day.

11(v)	Since x is the independent variable and y is the dependent variable, we should use the y on x regression line. When $y = 10$, $10 = 0.34969x + 4.4354 \Rightarrow x = \dfrac{10-4.4354}{0.34969} \approx 15.913 = 16.$ (nearest whole number) The estimated number of days = 16.
11(vi)	For $L = 10.1$, $r = -0.98533$ (5 d.p.) The most appropriate value for L is 10.1 since its corresponding value of $\lvert r \rvert$ is closest to 1 as compared to that of 10.2 and 10.3.
11 (vii)	The required equation is $\ln(10.1 - y) = 1.8184 - 0.10859x$ (5 s.f.) Hence, $a = 1.82$. (3 s.f.) and $b = -0.109$. (3 s.f.). When $x = 10$, $\ln(10.1 - y) = 1.8184 - 0.10859(10)$ $\Rightarrow y = 10.1 - e^{1.8184-1.0859} \approx 8.01973 = 8020.$ (3 s.f.) Kenny is expected to clock 8020 steps on Day 10. This estimate is reliable since $\lvert r \rvert = 0.98533$ is close to 1 and $x = 10$ is within the range of the data set.