ANDERSON SERANGOON JUNIOR COLLEGE

H2 MATHEMATICS

9758/01

Preliminary Examination Paper 1
(100 marks)

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS /

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
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4	
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10	
11	
12	
Total	

This document consists of 24 printed pages.

[Turn over

1 **Do not use a calculator in answering this question.**

It is given that $\mathrm{f}(z) = z^4 + 2\sqrt{2}z^3 + z^2 + 8\sqrt{2}z - 12$. One of the roots of the equation $\mathrm{f}(z) = 0$ is given by $2\mathrm{i}$. By factorising $\mathrm{f}(z)$ as a product of two quadratic factors, obtain the other roots of the equation. [4]

2 Amazoon TV, BingeWatch, Cinematic are three TV streaming services which offer customers a monthly subscription package. Promotions for TV streaming services subscriptions are available for PhoneHub members as well as for customers who pay by credit cards. This is shown in the following table.

TV Streaming Service	Promotion for PhoneHub members	Promotion for credit card
Amazoon TV	Free for first 3 months	10% discount on all monthly subscriptions
BingeWatch	Not Applicable	
Cinematic	Free for first month	

- Aaron is a PhoneHub member but does not own a credit card. He subscribed to all three services and his annual expenditure is \$360.00.
- Serene is not a PhoneHub member and does not own a credit card. She decided to subscribe to BingeWatch and Cinematic services. She pays \$300 for her annual subscriptions.
- Jaycee is a PhoneHub member who subscribed to all three services. She uses her credit card to pay for Amazoon TV and BingeWatch services and pays for Cinematic service by cash. Her yearly expenditure for the three services is \$337.20.

How much is the monthly subscription for each streaming service? [4]

3 Do not use a calculator in answering this question.

A geometric series has first term a and common ratio r, where $a > 0$ and $r \neq 0$. An arithmetic series has the same first term a and common difference d, where $d \neq 0$. It is given that the second and fourth terms of the geometric series are equal to the third and fifth terms of the arithmetic series respectively.

(i) Show that $r^3 - 2r + 1 = 0$ and explain why the common ratio cannot be 1 even though $r = 1$ is a root of the equation. [3]

(ii) Given that the geometric series is convergent, find the exact value of its common ratio. [2]

4 Curve C_1 has equation $y=\frac{1}{x}$ and curve C_2 has equation $y=\frac{1}{\sqrt{2-x^2}}$. Region Q is bounded by line $x=\frac{1}{\sqrt{2}}$ and both curves C_1 and C_2.

(i) Find the exact area of region Q. [3]

(ii) The region bounded by the line $y=5$ and the curves C_1 and C_2 in the first quadrant is rotated about the x–axis by 2π radians. Find the volume of the solid formed. [3]

5 (i) It is given that $y = \mathrm{e}^{\sqrt{1+x}}$, $x > -1$.

Show that $2\ln y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \dfrac{2}{y}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \dfrac{\mathrm{d}y}{\mathrm{d}x}$. [2]

(ii) By further differentiation of this result, or otherwise, find the Maclaurin's series of y up to and including the term in x^3. [3]

(iii) Deduce the series expansion of $y = \dfrac{e^{\sqrt{1+x}}}{\sqrt{1+x}}$ up to and including the term in x^2. [1]

6 (a) Evaluate $\int_{-a}^{a} \left| x^2 + 1 - e^x \right| dx$ where $a > 0$, giving your answer in terms of a. [3]

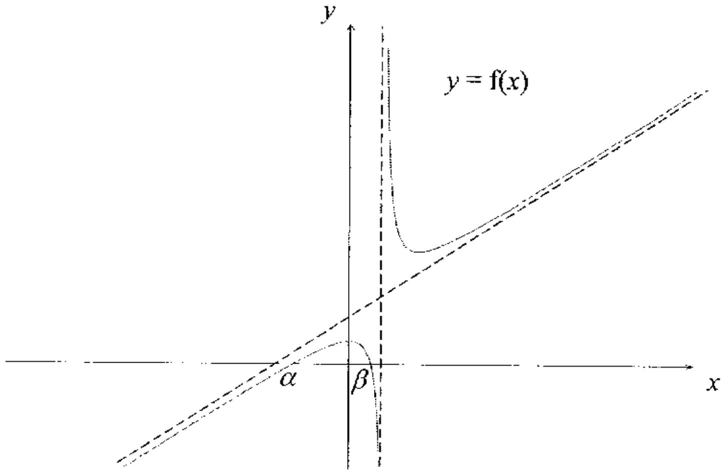
(b) By means of substituting $x = \tan\theta$, or otherwise, show that

$$\int \frac{x^2}{\left(1+x^2\right)^2}\,\mathrm{d}x = \frac{1}{2}\left(\tan^{-1} x + \mathrm{f}(x)\right) + C,$$

where C is an arbitrary unknown constant and $\mathrm{f}(x)$ is an expression to be determined in non-trigonometric form. [4]

7 **(i)** The curve C_1 with equation $y = \frac{(x+2)^2}{x+1}$ is transformed onto the curve C_2 with equation $y = \mathrm{f}(x)$. The curve C_1 has a minimum turning point (0, 4) which corresponds to the point with coordinates (a, b) on the curve C_2, where $a, b > 0$. Given that f(x) has the form $\frac{p^2x^2}{px-1} + q$, where p, q are positive constants, express p and q in terms of a and b. [4]

(ii) The curve of $y = \mathrm{f}(x)$ has a maximum point and a minimum point at $(0, q)$ and (a, b) respectively, and intersects the x-axis at α and β, as shown in the diagram below. The equation of the vertical asymptote is $x = \frac{1}{p}$.



Sketch the curve $y = \frac{1}{\mathrm{f}(x)}$. Your diagram should indicate clearly, in terms of a, b, α and β, the equations of any asymptote(s) as well as the coordinates of turning points and axial intercepts. [3]

8 (i) By using method of difference, show that $\sum_{r=1}^{n}\left((r+1)^3-r^3\right)=(n+1)^3-1$. [2]

(ii) By simplifying $(r+1)^3-r^3$ or otherwise, show that $\sum_{r=1}^{n} r^2=\frac{n}{6}(n+1)(2n+1)$. [4]

(iii) Hence find $\sum_{r=5}^{N}(r+2)^2$, leaving your answer in terms of N. [3]

9 A curve C has parametric equations

$$x = \tan 2\theta - 1, \qquad y = 1 - 2\sec 2\theta, \qquad \text{for } -\frac{\pi}{4} < \theta < \frac{\pi}{4}.$$

(i) Find the coordinates of the point where C cuts the y-axis, giving your answer in exact form. [2]

(ii) Show that $\frac{dy}{dx} = -2\sin 2\theta$. Find the coordinates of the point where $\theta = 0$ and determine the gradient of C at this point. What can be said about the values of x and y and the gradient to C as $\theta \to -\frac{\pi}{4}$ and $\theta \to \frac{\pi}{4}$? [5]

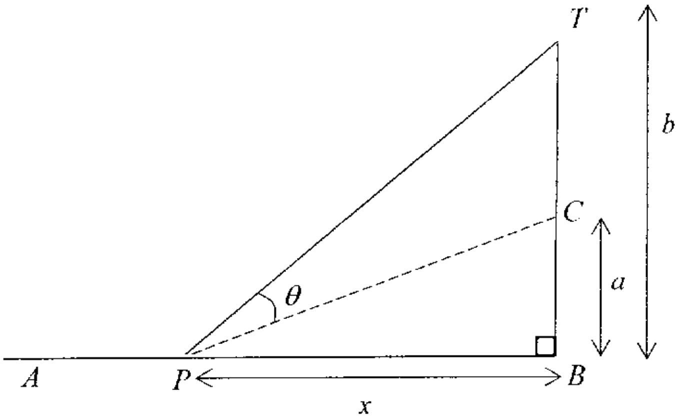
It is given that C has two oblique asymptotes which intersect each other at the point $(-1,1)$.

(iii) Find the equations of the asymptotes. Hence sketch C, showing clearly the features of the curve at the points where $\theta = 0$ and where $\theta \to -\frac{\pi}{4}$ and $\theta \to \frac{\pi}{4}$. [3]

(iv) Find the cartesian equation of C, expressing y in terms of x. [2]

[Turn over

10 In the diagram below, A and B are two fixed points on horizontal ground and an observer is standing at point P which is x metres away from the base of a building BC of height a m. A transmission tower CT is fixed at the top of the building and the total height of the building and the tower is given as b metres. You may assume that B, C and T lies on a straight line.



It is given that the angle TPC is θ radians.

(i) Show that $\tan\theta = \dfrac{(b-a)x}{x^2+ab}$. [2]

(ii) Find, in terms of a and b, the value of x such that θ is a maximum. Find also the corresponding value of $\tan\theta$. [4]

For the remaining parts of the question, take $a = 50$ and $b = 70$.

(iii) Find the range of values of $\tan\theta$ as x varies between 30 and 80. [2]

10 [Continued]

(iv) The observer at P starts to walk towards B with a steady speed of 3 ms^{-1}. Find the rate of change of θ when he is 10 m from B. [4]

11 Intercepting drones are programmed to travel in straight paths on the plane p with equation

$$6x-5y+2z+5=0.$$

For one particular intercepting drone, such a path is given by the line l with cartesian equation

$$\frac{x-10}{15}=\frac{y-15}{14}=\frac{z-5}{-10}.$$

(i) Show that the path l lies on plane p. [2]

This particular intercepting drone is always activated from a fixed point on the ground, which is represented by the xy plane.

(ii) Find the coordinates of the fixed point on the ground where this particular intercepting drone takes off. [2]

11 [Continued]

An unauthorised drone takes off from the ground traversing along the straight path m having an equation

$$\mathbf{r} = (30-10t)\mathbf{i} + (9t-10)\mathbf{j} + 15t\ \mathbf{k}$$

where t is the time taken from take-off measured in minutes.

(iii) Find when and where the unauthorised drone hits the plane p. [3]

(iv) Determine the shortest distance between the point where the unauthorised drone hits the plane p and the path l traversed by the particular intercepting drone. [3]

(v) Find the location of the point on path l that gives this shortest distance. [3]

12 Two students are trying to do mathematical modelling for a recent disease outbreak in a particular city with a population of 10 000. There are 5 people being infected by the disease initially.

Student A proposed modelling the scenario using the following differential equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = RI, \qquad \text{--- (I)}$$

where I is the number of people in the city being infected at time t (days) after the initial outbreak and R is a positive constant.

(i) Show that $I = 5e^{Rt}$. [3]

Another student B proposed using a geometric progression for his mathematical model. His model can be predicted by the equation

$$I_n = 5\left(2^n\right), \qquad \text{--- (II)}$$

where the n^{th} term of the geometric progression, I_n, gives the number of people infected with the disease at the beginning of the n^{th} day after the initial outbreak.

It is given that the geometric progression has common ratio 2.

(ii) Explain, in the context of the infectious outbreak, the meaning of the common ratio. [1]

It is given that both models give the same prediction of the number of people infected at the start of every single day after the initial outbreak.

(iii) Show that $R = \ln 2$ [1]

(iv) Explain whether model (I) or model (II) is a better model. [1]

Both students then went to do some research and discovered another model in the form of the following differential equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = aI(10000 - I), \qquad \text{--- (III)}$$

where a is a positive constant.

It is given that the greatest value of $\frac{\mathrm{d}I}{\mathrm{d}t}$ is 2500.

(v) Show that $a = 0.0001$. [2]

(vi) By solving the differential equation (III), show that $I = \dfrac{10000}{1+1999\mathrm{e}^{t}}$. [5]

(vii) Sketch the graph showing how the number of infected people varies with time in model (III). [2]

Question 1

Since all the coefficients of $f(z)$ are all real, when $(2i)$ is a root of $f(z)=0$, then $(-2i)$ is another root.

Thus the product of quadratic factors $= [z-(2i)][z-(-2i)]$

$$= [z-(2i)][z+(2i)]$$

$$= z^2+4$$

$$f(z) = z^4 + 2\sqrt{2}z^3 + z^2 + 8\sqrt{2}z - 12$$
$$= (z^2+4)(z^2+Az-3)$$

By comparing coefficients in the terms in z: $4A = 8\sqrt{2} \Rightarrow A = 2\sqrt{2}$

Thus $f(z) = (z^2+4)(z^2+2\sqrt{2}z-3)$

Consider, $z^2+2\sqrt{2}z-3=0$.

Thus, we have $z = \dfrac{-2\sqrt{2} \pm \sqrt{8+12}}{2} = \dfrac{-2\sqrt{2} \pm \sqrt{20}}{2} = -\sqrt{2} \pm \sqrt{5}$

Hence the other roots are: $-2i$, $-\sqrt{2}-\sqrt{5}$ and $-\sqrt{2}-\sqrt{5}$.

[Turn over

Question 2

Let a, b and c to be the monthly subscription cost for Amazoon TV, BingeWatch, and Cinematic.

Aaron : $9a+12b+11c=360$

Serene : $12b+12c=300$

Jaycee : $0.9(9a+12b)+11c=337.2$

Solving the SLE $\Rightarrow \begin{cases} 9a+12b+11c=360 \\ 12b+12c=300 \\ 8.1a+10.8b+11c=337.2 \end{cases} \Rightarrow \begin{cases} a=8 \\ b=13 \\ c=12 \end{cases}$

The monthly subscription for Amazoon TV, BingeWatch, and Cinematic are \$8, \$13 and \$12 respectively.

Question 3(i)

$$ar = a + (3-1)d \Rightarrow d = \frac{a(r-1)}{2} \qquad \text{--- (1)}$$

$$ar^3 = a + (5-1)d \Rightarrow d = \frac{a(r^3-1)}{4} \qquad \text{--- (2)}$$

Subst (1) into (2),

$$\frac{a(r^3-1)}{4} = \frac{a(r-1)}{2}$$

$$a(r^3-1) = 2a(r-1)$$

Since $a > 0$,

$$r^3 - 1 = 2(r-1)$$

$$r^3 - 2r + 1 = 0$$

If the common ratio were to be 1, from equation (1), the common difference will be 0. But the question mentioned $d \neq 0$. That is why the common ratio cannot be 1 even though $r = 1$ is a root of the equation.

Question 3(ii)

Let $r^3 - 2r + 1 = (r-1)(Ar^2 + Br + C)$.

Clearly, $A = 1$ and $C = -1$.

Compare coefficient of r : $\quad -2 = A - B$

$$B = A + 2 = 1$$

So, $r^3 - 2r + 1 = (r-1)(r^2 + r - 1) = 0$

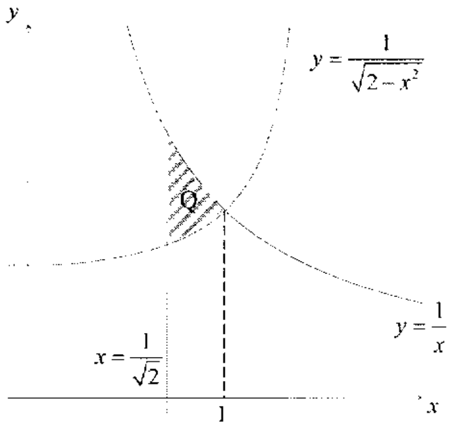
Consider $r^2 + r - 1 = 0$,

$$r = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

Since geometric series is convergent, $r \neq \frac{-1-\sqrt{5}}{2}$.

Therefore, the common ratio of the convergent geometric series is $\frac{-1+\sqrt{5}}{2}$.

Question 4(a)



Let $\frac{1}{x}=\frac{1}{\sqrt{2-x^2}}$

$\Rightarrow x^2=2-x^2$

$\Rightarrow x=1$ (since $x>0$)

$\therefore$ Area of Region Q

$=\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{x}\,dx-\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{\sqrt{2-x^2}}\,dx$

$=\left[\ln x-\sin^{-1}\frac{x}{\sqrt{2}}\right]_{\frac{1}{\sqrt{2}}}^{1}$

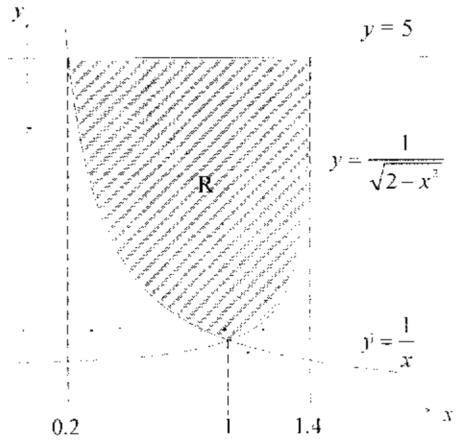
$=\left(\ln 1-\sin^{-1}\frac{1}{\sqrt{2}}\right)-\left(\ln\frac{1}{\sqrt{2}}-\sin^{-1}\frac{1}{2}\right)$

$=0-\frac{\pi}{4}-\ln\frac{1}{\sqrt{2}}+\frac{\pi}{6}$

$=\frac{1}{2}\ln 2-\frac{\pi}{12}$ units^2

*Accept $\ln\sqrt{2}-\frac{\pi}{12}$ and $-\frac{1}{2}\ln\frac{1}{2}-\frac{\pi}{12}$

Question 4(b)



Let $5=\frac{1}{x}$ and $5=\frac{1}{\sqrt{2-x^2}}$

From GC, $x=0.2$ and $x=1.4$ (since $x>0$)

Required volume

$=$ Vol of cylinder $-\pi\int_{0.2}^{1}\left(\frac{1}{x}\right)^2dx-\pi\int_{1}^{1.4}\left(\frac{1}{\sqrt{2-x^2}}\right)^2dx$

$=\pi(5)^2(1.4-0.2)-12.566-3.9158$

$=77.8$ units^3 (3 s.f.)

Question 5(i)

$y = e^{\sqrt{1+x}}$

$\ln y = \sqrt{1+x}$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$2\ln y\frac{dy}{dx} = y$

$2\ln y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) = \frac{dy}{dx}$

$2\ln y\frac{d^2y}{dx^2} + \frac{2}{y}\left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$

Question 5(ii)

$2\ln y\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2}\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) + \frac{4}{y}\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2\left(-\frac{1}{y^2}\right)\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

When $x = 0$,

$y = e$

$\frac{dy}{dx} = \frac{e}{2}$

$\frac{d^2y}{dx^2} = 0$

$\frac{d^3y}{dx^3} = \frac{e}{8}$

Maclaurin's series of y is $y = e + \left(\frac{e}{2}\right)(x) + (0)\frac{x^2}{2!} + \left(\frac{e}{8}\right)\frac{x^3}{3!} + \ldots$

$$y = e + \frac{e}{2}x + \frac{e}{48}x^3 + \ldots$$

Question 5(iii)

$e^{\sqrt{1+x}} = e + \frac{e}{2}x + \frac{e}{48}x^3 + \ldots$

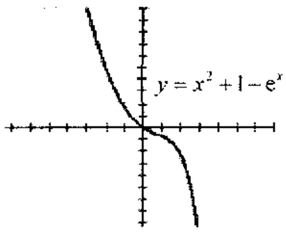
Differentiating both sides with respect to x,

$\frac{1}{2}\frac{e^{\sqrt{1+x}}}{\sqrt{1+x}} = \frac{e}{2} + \frac{e}{48}\left(3x^2\right) + \ldots$

$\frac{e^{\sqrt{1+x}}}{\sqrt{1+x}} = e + \frac{e}{8}x^2 + \ldots$

Question 6(a)

From graph, $x^2+1-e^x<0$ for $x>0$.



$$\int_{-a}^{a}\left|x^2+1-e^x\right|dx$$
$$=\int_{-a}^{0}x^2+1-e^x\,dx+\int_0^a -x^2-1+e^x\,dx$$
$$=\left[\frac{x^3}{3}+x-e^x\right]_{-a}^{0}+\left[-\frac{x^3}{3}-x+e^x\right]_0^a$$
$$=-2+e^{-a}+e^a$$

Question 6(b)

Differentiating $x=\tan\theta$ with respect to θ

$$\frac{dx}{d\theta}=\sec^2\theta$$

$$\int\frac{x^2}{\left(1+x^2\right)^2}dx=\int\frac{\tan^2\theta}{\left(1+\tan^2\theta\right)^2}\left(\sec^2\theta\right)d\theta$$
$$=\int\frac{\tan^2\theta}{\sec^2\theta}d\theta$$
$$=\int\sin^2\theta\,d\theta$$
$$=\frac{1}{2}\int\left(1-\cos 2\theta\right)d\theta$$
$$=\frac{1}{2}\left(\theta-\frac{\sin 2\theta}{2}\right)+c$$, where c is an arbitrary constant

Since $x=\tan\theta$, $\sin\theta=\frac{x}{\sqrt{1+x^2}}$ and $\cos\theta=\frac{1}{\sqrt{1+x^2}}$

$$\therefore\int\frac{x^2}{\left(1+x^2\right)^2}dx=\frac{1}{2}\left(\theta-\frac{\sin 2\theta}{2}\right)+c$$
$$=\frac{1}{2}\left(\theta-\frac{2\sin\theta\cos\theta}{2}\right)+c$$
$$=\frac{1}{2}\left(\tan^{-1}x-\frac{x}{1+x^2}\right)+c$$

Question 7(i)

$$y=\frac{(x+2)^2}{x+1} \xrightarrow{\text{Replace } x \text{ with } x-2} y=\frac{\left[(x-2)+2\right]^2}{(x-2)+1}=\frac{x^2}{x-1}$$ (Translation of 2 units in the positive x-direction)

$$\xrightarrow{\text{Replace } x \text{ with } px} y=\frac{(px)^2}{(px)-1}=\frac{p^2x^2}{px-1}$$ (Scaling of scale factor $\frac{1}{p}$ along the x-axis)

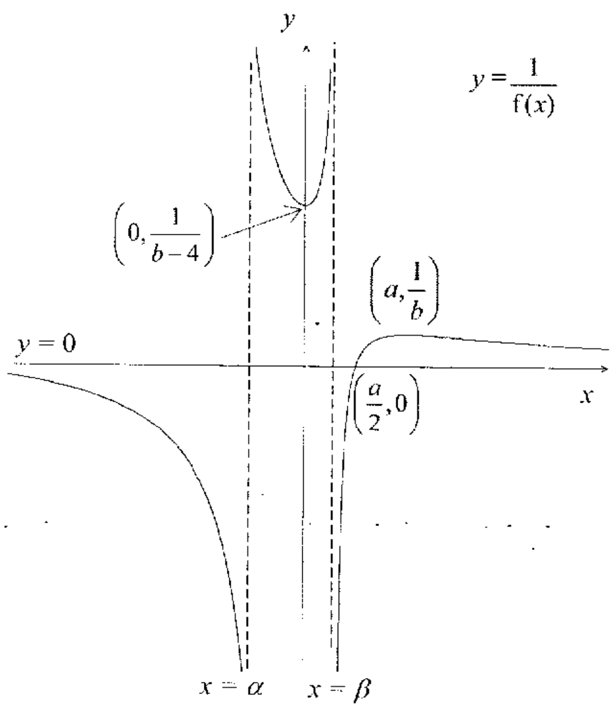
$$\xrightarrow{\text{Replace } y \text{ with } y-q} y=\frac{p^2x^2}{px-1}+q$$ (Translation of q units in the positive y-direction)

The minimum turning point on C_1, (0, 4) corresponds to (a, b) on C_2.

Hence $a=\frac{1}{p}(0+2) \Rightarrow p=\frac{2}{a}$

$b=4+q \Rightarrow q=b-4$

Question 7(ii)



Question 8(i)

$$\sum_{r=1}^{n}\left((r+1)^3-r^3\right)=2^3-1^3$$
$$+3^3-2^3$$
$$+4^3-3^3$$
$$\vdots$$
$$+\quad (n)^3-(n-1)^3$$
$$+(n+1)^3-(n)^3$$
$$=(n+1)^3-1$$

Question 8(ii)

$$(r+1)^3-r^3=r^3+3r^2+3r+1-r^3$$
$$=3r^2+3r+1$$

$$\sum_{r=1}^{n}\left((r+1)^3-r^3\right)=\sum_{r=1}^{n}\left(3r^2+3r+1\right)$$

$$(n+1)^3-1=3\sum_{r=1}^{n}r^2+3\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$$

$$n^3+3n^2+3n=3\sum_{r=1}^{n}r^2+3\left(\frac{n}{2}\right)(1+n)+n$$

$$3\sum_{r=1}^{n}r^2=n^3+3n^2+3n-\frac{3}{2}n-\frac{3}{2}n^2-n$$

$$\sum_{r=1}^{n}r^2=\frac{1}{6}\left(2n^3+3n^2+n\right)$$

$$=\frac{n}{6}(n+1)(2n+1)=\frac{n}{6}\left(2n^2+3n+1\right)$$

Question 8(iii)

$$\sum_{r=5}^{N}(r+2)^2=\sum_{r=7}^{N+2}r^2$$

$$=\sum_{r=1}^{N+2}r^2-\sum_{r=1}^{6}r^2$$

$$=\frac{N+2}{6}\big((N+2)+1\big)\big(2(N+2)+1\big)-\frac{6}{6}(6+1)\big(2(6)+1\big)$$

$$=\frac{1}{6}(N+2)(N+3)(2N+5)-91$$

Alternatively

$$\sum_{r=5}^{N}(r+2)^2=\sum_{r=5}^{N}(r^2+4r+4)$$

$$=\sum_{r=5}^{N}r^2+\sum_{r=5}^{N}(4r+4)$$

$$=\sum_{r=1}^{N}r^2-\sum_{r=1}^{4}r^2+\sum_{r=5}^{N}(4r+4)$$

$$=\frac{N}{6}(N+1)(2N+1)-\frac{4}{6}(5)(9)+\frac{N-4}{2}(24+4N+4)$$

$$=\frac{N}{6}(N+1)(2N+1)-30+(N-4)(14+2N)$$

$$=\frac{N}{6}(2N^2+3N+1)-30+14N+2N^2-56$$

$$=\frac{N^3}{3}+\frac{5}{3}N^2+\frac{37}{6}N-86$$

Question 9(i)

When $x=0$,

$$\tan 2\theta - 1 = 0 \Rightarrow 2\theta = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{8}$$

$$y = 1 - 2\sec 2\left(\frac{\pi}{8}\right) = 1 - 2\sec\frac{\pi}{4} = 1 - 2\sqrt{2}$$

The coordinates of the point where C cuts the y-axis is $(0, 1-2\sqrt{2})$

Question 9(ii)

$$\frac{dx}{d\theta} = 2\sec^2 2\theta\,;\ \frac{dy}{d\theta} = -4\sec 2\theta \tan 2\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-4\sec 2\theta \tan 2\theta}{2\sec^2 2\theta}$$

$$= -2\sin 2\theta$$

When $\theta = 0$,

$x=-1$ and $y=-1$ and $\frac{dy}{dx} = 0$.

The coordinates are $(-1,-1)$ and the gradient of C at this point is 0.

As $\theta \to -\frac{\pi}{4}$,

$x \to -\infty$ and $y \to -\infty$

$\frac{dy}{dx} \to 2$

As $\theta \to \frac{\pi}{4}$

$x \to +\infty$ and $y \to -\infty$

$\frac{dy}{dx} \to -2$

Question 9(iii)

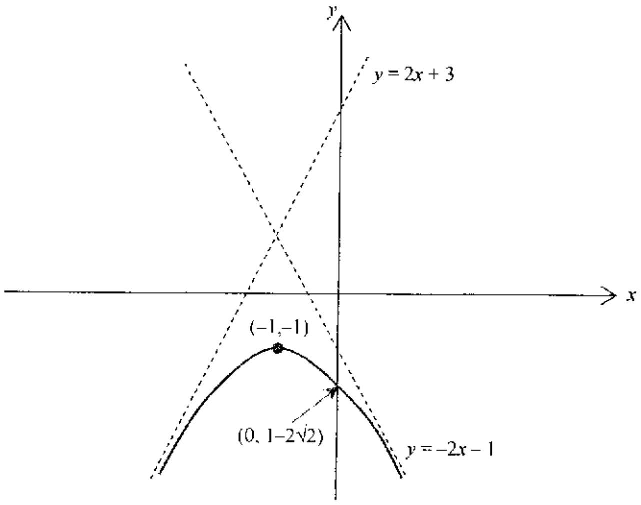
Let the equations of the asymptote be $y=2x+a$ and $y=-2x+b$.

Substitute $x=-1$ and $y=1$ into the equations,

$1=2(-1)+a \Rightarrow a=3$

$1=-2(-1)+b \Rightarrow b=-1$

Therefore the equations of the asymptotes are $y=2x+3$ and $y=-2x-1$.



Question 9(iv)

Let $\tan 2\theta = x+1$ and $\sec 2\theta = \dfrac{1-y}{2}$

Since $1+\tan^2 2\theta = \sec^2 2\theta$

$$1+(x+1)^2=\left(\frac{1-y}{2}\right)^2$$

$$(1-y)^2=4+4(x+1)^2$$

$$y-1=\pm\sqrt{4+4(x+1)^2}$$

$$y-1=-\sqrt{4+4(x+1)^2} \qquad \text{reject } y-1=\sqrt{4+4(x+1)^2}$$

$$y=1-\sqrt{4+4(x+1)^2}$$

Question 10(i)

Let $\alpha = \angle TPB$ and $\beta = \angle CPB$

$$\tan\theta = \tan(\alpha - \beta)$$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\tan\theta = \frac{\frac{b}{x} - \frac{a}{x}}{1 + \left(\frac{b}{x}\right)\left(\frac{a}{x}\right)}$$

$$\tan\theta = \frac{(b-a)x}{x^2 + ab}$$

Question 10(ii)

$$\frac{d}{dx}(\tan\theta) = \frac{d}{dx}\left(\frac{(b-a)x}{x^2+ab}\right)$$

$$\sec^2\theta\frac{d\theta}{dx} = \frac{(x^2+ab)(b-a) - (2x)(b-a)x}{(x^2+ab)^2}$$

$$\sec^2\theta\frac{d\theta}{dx} = \frac{(ab-x^2)(b-a)}{(x^2+ab)^2}$$

OR

$$\theta = \tan^{-1}\left(\frac{(b-a)x}{x^2+ab}\right)$$

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{(b-a)x}{x^2+ab}\right)^2} \times \frac{(x^2+ab)(b-a) - 2x^2(b-a)}{(x^2+ab)^2}$$

$$= \vdots$$

$\frac{d\theta}{dx} = 0,\quad \frac{(ab-x^2)(b-a)}{(x^2+ab)^2} = 0$

$x = \sqrt{ab}$ or $x = -\sqrt{ab}$ (rej $\because x > 0$)

$$\frac{d\theta}{dx}=\frac{\left(ab-x^2\right)(b-a)}{\left(x^2+ab\right)^2\sec^2\theta}$$

Since $\left(x^2+ab\right)^2>0$, $\sec^2\theta>0$ and $b>a\Rightarrow b-a>0$, it suffices to check $\left(ab-x^2\right)$

Using first derivative test:

x	$\left(\sqrt{ab}\right)^-$	$\sqrt{ab}$	$\left(\sqrt{ab}\right)^+$
$\frac{d\theta}{dx}$	+ve $\because ab-x^2>0$	0	-ve $\because ab-x^2<0$

$\therefore\ \theta$ is a maximum when $x=\sqrt{ab}$.

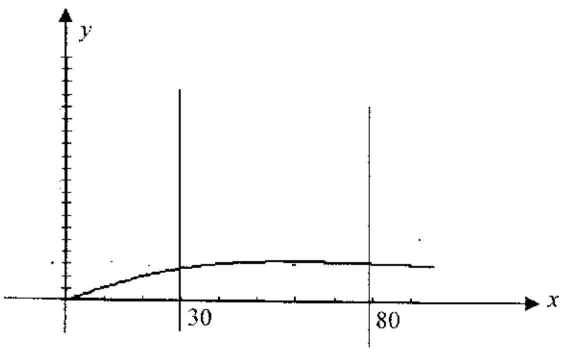
When $x=\sqrt{ab}$,

$$\tan\theta=\frac{(b-a)\sqrt{ab}}{2ab}$$

Question 10(iii)

Given $a=50, b=70$,

$$\tan\theta=\frac{(b-a)x}{x^2+ab}=\frac{20x}{x^2+3500}$$



$$\frac{3}{22}<\tan\theta\leq\frac{1}{\sqrt{35}}$$

Question 10(iv)

$$\frac{dx}{dt} = -3$$

When $a = 50, b = 70, x = 10$,

$$\tan\theta = \frac{(b-a)x}{x^2 + ab}$$

$$= \frac{1}{18}$$

$$\therefore \sec^2\theta = \frac{325}{324}$$

$$\frac{d\theta}{dx} = \frac{(x^2 - ab)(a-b)}{(x^2+ab)^2 \sec^2\theta} = \frac{17}{3250} \text{ or } 0.0052308$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$$

$$= -3 \times \frac{17}{3250}$$

$$= -\frac{51}{3250}$$

The angle θ is decreasing at a rate of $\frac{51}{3250}$ rad/s.

Question 11(i)

$$p: 6x-5y+2z+5=0 \Rightarrow \mathbf{r}\cdot\begin{pmatrix}6\\-5\\2\end{pmatrix}=-5$$

$$l: \frac{x-10}{15}=\frac{y-15}{14}=\frac{z-5}{-10} \Rightarrow \mathbf{r}=\begin{pmatrix}10\\15\\5\end{pmatrix}+\lambda\begin{pmatrix}15\\14\\-10\end{pmatrix}, \lambda\in\mathbb{R}$$

Consider, $\begin{pmatrix}10\\15\\5\end{pmatrix}\cdot\begin{pmatrix}6\\-5\\2\end{pmatrix}=60-75+10=-5 \Rightarrow (10,15,5)$ lies on both l & p.

Consider, $\begin{pmatrix}15\\14\\-10\end{pmatrix}\cdot\begin{pmatrix}6\\-5\\2\end{pmatrix}=90-70-20=0 \Rightarrow l$ is parallel to p.

Therefore, the path l lies on plane p.

Question 11(ii)

When $z=0$, $\frac{x-10}{15}=\frac{1}{2} \Rightarrow x=17.5$ and $\frac{y-15}{14}=\frac{1}{2} \Rightarrow y=22$

Thus fixed point has coordinates $(17.5, 22, 0)$.

Question 11(iii)

$$m: \mathbf{r}=\begin{pmatrix}30\\-10\\0\end{pmatrix}+t\begin{pmatrix}-10\\9\\15\end{pmatrix}, t\in\mathbb{R} \text{ and } p: \mathbf{r}\cdot\begin{pmatrix}6\\-5\\2\end{pmatrix}=-5$$

Consider, $\begin{pmatrix}30-10t\\-10+9t\\15t\end{pmatrix}\cdot\begin{pmatrix}6\\-5\\2\end{pmatrix}=-5$.

$\Rightarrow 180-60t+50-45t+30t=-5$.

$\Rightarrow 75t=235$

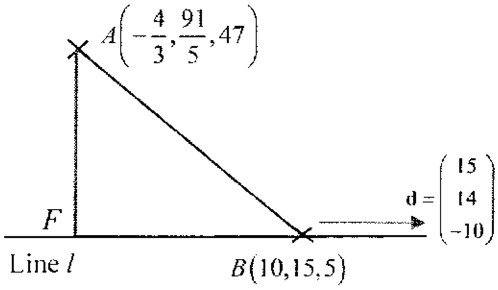
$\Rightarrow t=\frac{47}{15}=3\frac{2}{15}$

$$\mathbf{r}=\begin{pmatrix}30-10\left(\frac{47}{15}\right)\\-10+9\left(\frac{47}{15}\right)\\15\left(\frac{47}{15}\right)\end{pmatrix}=\begin{pmatrix}-\frac{4}{3}\\\frac{91}{5}\\47\end{pmatrix}$$

The unauthorised drone hits the plane p at coordinates $\left(-\frac{4}{3},\frac{91}{5},47\right)$ after taking off for 3 min 8 seconds (or 3.13 min).

Question 11(iv)

Let A(where unauthorised drone hits plane p) and B(point on line l) be points whose coordinates are $\left(-\frac{4}{3},\frac{91}{5},47\right)$ and $(10,15,5)$ respectively.



Let direction vector of line be $\mathbf{d}=\begin{pmatrix}15\\14\\-10\end{pmatrix}$.

Thus, $\overrightarrow{BA}=\begin{pmatrix}-\frac{4}{3}\\\frac{91}{5}\\47\end{pmatrix}-\begin{pmatrix}10\\15\\5\end{pmatrix}=\begin{pmatrix}-\frac{34}{3}\\\frac{16}{5}\\42\end{pmatrix}$.

$$\text{Shortest distance }=\left|\overrightarrow{BA}\times\frac{\mathbf{d}}{|\mathbf{d}|}\right|$$

$$=\left|\begin{pmatrix}-\frac{34}{3}\\\frac{16}{5}\\42\end{pmatrix}\times\frac{1}{\sqrt{521}}\begin{pmatrix}15\\14\\-10\end{pmatrix}\right|$$

$$=\frac{1}{\sqrt{521}}\left|\begin{pmatrix}-620\\\frac{1550}{3}\\\frac{-620}{3}\end{pmatrix}\right|=36.49877=36.5\text{ m (3 sig. fig.)}$$

Question 11(v)

Let F be the foot of the perpendicular from A to line l.

$$\overrightarrow{AF} = \begin{pmatrix} 10+15\lambda \\ 15+14\lambda \\ 5-10\lambda \end{pmatrix} - \begin{pmatrix} -\frac{4}{3} \\ \frac{91}{5} \\ 47 \end{pmatrix} = \begin{pmatrix} 15\lambda+\frac{34}{3} \\ 14\lambda-\frac{16}{5} \\ -10\lambda-42 \end{pmatrix}$$

$$\overrightarrow{AF}\cdot\mathbf{d} = 0 \Rightarrow \begin{pmatrix} 15\lambda+\frac{34}{3} \\ 14\lambda-\frac{16}{5} \\ -10\lambda-42 \end{pmatrix}\cdot\begin{pmatrix} 15 \\ 14 \\ -10 \end{pmatrix} = 0$$

$$\Rightarrow 521\lambda = -545.2 \Rightarrow \lambda = -\frac{2726}{2605} = -1.04645$$

Thus $\overrightarrow{OF} = \begin{pmatrix} -\frac{2968}{521} \\ \frac{911}{2605} \\ \frac{8057}{521} \end{pmatrix} = \begin{pmatrix} -5.70 \\ 0.350 \\ 15.5 \end{pmatrix}$.

Alternatively,

$$\overrightarrow{BA} = \begin{pmatrix} -\frac{4}{3} \\ \frac{91}{5} \\ 47 \end{pmatrix} - \begin{pmatrix} 10 \\ 15 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{34}{3} \\ \frac{16}{5} \\ 42 \end{pmatrix}.$$

$$\overrightarrow{BF} = \left(\overrightarrow{BA}\cdot\frac{\mathbf{d}}{|\mathbf{d}|}\right)\frac{\mathbf{d}}{|\mathbf{d}|}$$

$$= \left[\begin{pmatrix} -\frac{34}{3} \\ \frac{16}{5} \\ 42 \end{pmatrix}\cdot\frac{1}{\sqrt{521}}\begin{pmatrix} 15 \\ 14 \\ -10 \end{pmatrix}\right]\frac{1}{\sqrt{521}}\begin{pmatrix} 15 \\ 14 \\ -10 \end{pmatrix} \qquad = -\frac{2726}{2605}\begin{pmatrix} 15 \\ 14 \\ -10 \end{pmatrix}$$

$$\overrightarrow{OF} = -\frac{2726}{2605}\begin{pmatrix} 15 \\ 14 \\ -10 \end{pmatrix} + \begin{pmatrix} 10 \\ 15 \\ 5 \end{pmatrix} = = \begin{pmatrix} -\frac{2968}{521} \\ \frac{911}{2605} \\ \frac{8057}{521} \end{pmatrix} = \begin{pmatrix} -5.70 \\ 0.350 \\ 15.5 \end{pmatrix}$$

Question 12(i)

$$\frac{dI}{dt} = RI$$

$$\int \frac{1}{I}\,dI = \int R\,dt$$

$\ln I = Rt + c_1$, since $I > 0$

$I = A_1 e^{Rt}$, where $A_1 = e^{c_1}$

When $t = 0$, $I = 5$,

$5 = A_1 e^0 \Rightarrow A_1 = 5$

$\therefore I = 5e^{Rt}$

Question 12(ii)

It means that the number of people being infected will double after every day.

Question 12(iii)

When $t = 1$, model (I) gives $I = 5e^R$

model (II) gives $I_1 = 5\left(2^1\right)$

Equating $5e^R = 5\left(2^1\right)$,

$e^R = 2$

$R = \ln 2$

Question 12(iv)

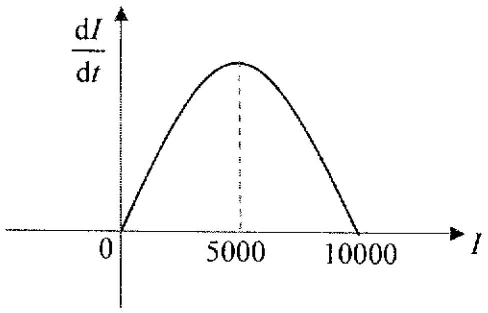
Model (I) is a better model because it can be used to predict the number of infected people at any one time whereas model (II) can only predict so at the start of every new day.

Alternative explanation

The number of infected people must be of integer values. Hence Model (II) is a better model than model (I).

to be awarded for any plausible explanation.

Question 12(v)



From the graph of $\frac{dI}{dt}$ against t, there are 5000 people infected at the instance when the rate of infection is the greatest.

Sub $I = 5000$ and $\frac{dI}{dt} = 2500$ into $\frac{dI}{dt} = aI(10000 - I)$,

$$2500 = 5000(10000 - 5000)a \Rightarrow a = 0.0001$$

Question 12(vi)

$$\frac{dI}{dt} = 0.0001I(10000 - I)$$

$$\int \frac{1}{I(10000 - I)}\, dI = \int 0.00001\, dt$$

$$\int \frac{1}{I^2 - 10000I}\, dI = \int -0.00001\, dt$$

$$\int \frac{1}{(I - 5000)^2 - 5000^2}\, dI = \int -0.00001\, dt$$

$$\frac{1}{2(5000)} \ln\left|\frac{(I - 5000) - 5000}{(I - 5000) + 5000}\right| = -0.00001t + c_2$$

$$\ln\left|\frac{I - 10000}{I}\right| = -t + 10000c_2$$

$$\frac{I - 10000}{I} = A_2 e^{-t}, \text{ where } A_2 = \pm e^{10000c_2}$$

$$I - 10000 = A_2 I e^{-t}$$

$$I\left(1 - A_2 e^{-t}\right) = 10000$$

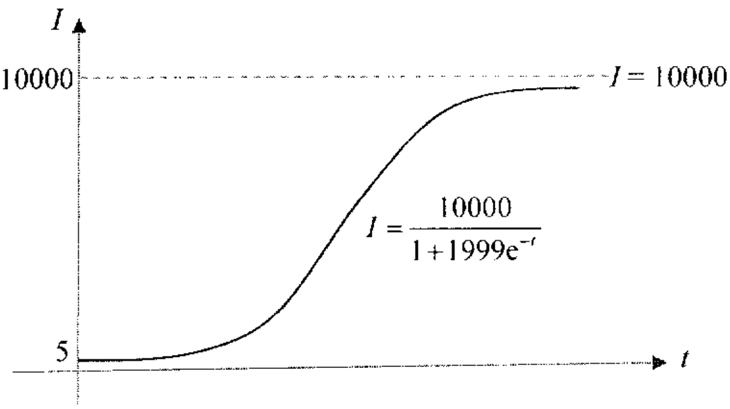
$$I = \frac{10000}{1 - A_2 e^{-t}}$$

When $t = 0$, $I = 5$,

$$5 = \frac{10000}{1 - A_2 e^0} \Rightarrow A_2 = -1999$$

Sub $A_2 = -1999$ in $I = \frac{10000}{1 - A_2 e^{-t}}$, we get $I = \frac{10000}{1 + 1999e^{-t}}$

Question 12(vii)



ANDERSON SERANGOON JUNIOR COLLEGE

H2 MATHEMATICS

9758/02

Preliminary Examination Paper 2
(100 marks)

3 hours

Additional Material(s):

List of Formulae (MF26)

CANDIDATE NAME

CLASS /

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

[Turn over

Section A: Pure Mathematics [40 marks]

1 The complex numbers z and w have moduli k and $3k^2$ respectively and arguments α and 4α respectively, where k is a positive constant and $-\frac{\pi}{7} < \alpha \le \frac{\pi}{7}$.

(i) Express $\frac{z^3}{w^*}$ in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \le \pi$. [2]

(ii) It is given that $\alpha = \frac{\pi}{21}$. Find the integer values of n such that $\left(\frac{z^3}{w^*}\right)^n$ is real. [2]

2 **(i)** Without the use of a calculator, solve the inequality $\frac{17x-9}{2x^2+13x-7} > 1$. [3]

(ii) Hence solve the inequality $\frac{17x+9}{2x^2-13x-7} < -1$. [2]

3 Three distinct vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are each of unit length such that $\lambda\mathbf{a} + \mu\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$, where λ, μ, γ are non-zero scalars.

(i) Show that $\mu(\mathbf{a} \times \mathbf{b}) = \gamma(\mathbf{c} \times \mathbf{a})$ and $\lambda(\mathbf{b} \times \mathbf{a}) = \gamma(\mathbf{c} \times \mathbf{b})$. [3]

(ii) By considering the sines of angles between $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{c}$ and $\mathbf{a}$, and $\mathbf{a}$ and $\mathbf{b}$, show that

$$\frac{|\mu|}{\sin \angle COA} = \frac{|\lambda|}{\sin \angle BOC} = \frac{|\gamma|}{\sin \angle AOB}.$$ [4]

4 (a) The function f is defined as

$$f: x \to \ln|\sec 2x| \text{ for } \frac{\pi}{3} \le x \le \frac{3\pi}{8}.$$

Find $f^{-1}(x)$ and determine its domain. [4]

(b) The function g is one-one and is defined as

$$g: x \to 2x\sin\left(x^2\right) \text{ for } -k \le x \le k,$$

where k is positive constant.

(i) Write down the largest value of k and find the range of g, correct to 3 decimal places. [2]

(ii) Using the value of k found in part **(i)**, sketch the graphs of $y = g(x)$, $y = g^{-1}(x)$ and $y = gg^{-1}(x)$ on the same diagram. Your diagram should illustrate correctly the relationship between the graphs, showing clearly the coordinates of the end-points of all graphs. [3]

(iii) Hence find the exact values of x such that $g(x) = g^{-1}(x)$. [2]

(iv) The function h is defined as

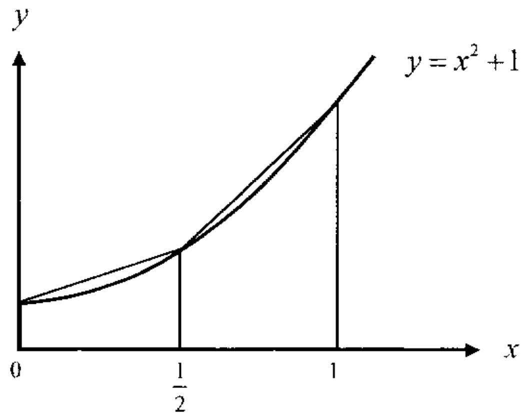
$$h: x \to -2x\sin(x^2), \text{ for } -1 \le x \le 1.$$

With the aid of a diagram, explain clearly why the method used in part **(iii)** to find the exact values of x such that $h(x) = h^{-1}(x)$ would not yield the complete set of solution. [2]

5 It is given that R is the region bounded by the x-axis, the y-axis, the line $x=1$ and the curve $y=x^2+1$.

The area of R can be estimated by calculating the sum of the areas of trapeziums with equal widths. The diagram below (not drawn to scale) shows an example of two such trapeziums.

[Area of trapezium = $\frac{1}{2}\times$sum of parallel sides $\times$height]



(i) Find the total area of the trapeziums shown in the diagram. [2]

To better estimate the area of R, n trapeziums of equal width are drawn.

(ii) State the length of the shorter side of the k^{th} trapezium for $1 \le k \le n$. Hence show that its area is given by $\frac{(k-1)^2+k^2}{2n^3}+\frac{1}{n}$ units^2. [3]

The sum of the areas of the n trapeziums is given as A units^2.

(iii) Find A, giving your answers in terms of n. [4]

[You may use $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$]

(iv) State whether A is an overestimate or an underestimate of the area of region R. [1]

(v) Use your answer in part **(iii)** to find the exact area for region R. [1]

Section B: Probability and Statistics [60 marks]

6 A disease outbreak occurs in a particular city. It is given that p % of the residents from the city has contracted the disease. A test for the disease has recently been developed. It gives an outcome of positive or negative when administered to a person.

The test has a probability 0.9 of giving a positive result when it is administered to a person who has the disease. The test also has the same probability of giving a negative result when it is administered to a person who does not have the disease.

(i) The health authority of the city has been aggressively testing a large number of residents at random and discovered that 20% of the tests return a positive result. Find the proportion of the residents infected with the disease. [2]

(ii) As time goes by, the proportion of people infected with the disease increases to q %. Find, in terms of q, the probability that a person has the disease given that this person has been tested negative. [2]

(iii) Using your answer in part **(ii)**, discuss the effectiveness of this test in the long run. [1]

7 Four friends went for a Christmas party, each bringing a present for gift exchange. During the gift exchange, the presents were randomly distributed, such that each person received exactly one gift, which may be the same or different from the present that he brought along. It is given that the presents brought to the Christmas party were all distinct from one another.

(i) Find the total number of ways to distribute the four presents to the four people if there are no restrictions. [1]

(ii) Find the number of ways the presents can be distributed if there is exactly one person who received back their own presents. [2]

(iii) Find the number of ways the presents can be distributed given that there are exactly two persons who received back his own present. [2]

(iv) Explain why there cannot be a case where there are exactly 3 persons who received back their own gifts. [1]

(v) Hence or otherwise, find the probability that none of the friends received back their own present after the gift exchange. [2]

8 A bag contains n blue cards and $(n+1)$ white cards which are identical in all aspects except for their colour. Two cards are drawn at random without replacement, from the bag.

(i) Find the probability that the two cards drawn are of different colour. [1]

(ii) Find the probability that the two cards drawn are of the same colour. [1]

If the cards are of different colour, two fair coins are then tossed and the number of heads is recorded. If the cards are of the same colour, the two fair coins are each tossed twice and the total number of heads is recorded. The random variable X is the total number of heads recorded when the two cards are drawn without replacement from the bag.

(iii) Show that $\mathrm{P}(X=1)=\dfrac{3n+2}{4(2n+1)}$. [2]

It is now given that there are 3 blue cards.

(iv) Find the probability distribution for X. [4]

(v) Find $\mathrm{P}(X>2 \mid X\leq 3)$. [2]

9 Sleep is important for optimal cognitive functions for students. One particular study found that 85% of students from schools in Singapore obtain less than the 8 hours of nocturnal (night-time) sleep recommended by the US National Sleep Foundation, and average, only 6.5 hours a night on a school-day.

(i) Assuming that the amount of nocturnal sleep for a student per night can be modelled by a normal distribution, find its standard deviation. [2]

Three students in a school are randomly selected by a researcher to participate in a survey on sleep.

(ii) Find the probability that one of the three students has less than 5.5 hours of nocturnal sleep per night and the other two students have more than 7 hours of nocturnal sleep per night.

The researcher opines that the total amount of sleep for students should be considered as the sum of the amount of their nocturnal sleep and afternoon naps. He assumes that the total amount of sleep from afternoon naps on a school-day can be modelled using a normal distribution with mean 65 minutes and standard deviation a minutes.

(iii) Explain why the use of a normal distribution as a model may not be appropriate when $a = 60$. [1]

It is now given that $a = 10$.

(iv) Find the probability that the total amount of sleep for the first student exceeds 8 hours. [3]

(v) Find the probability that the total amount of sleep of the first two students exceeds twice the total amount of sleep of the third student by more than 1 hour. [3]

(vi) State an assumption that you have used in your calculations in part **(iv)** and explain why this assumption may be unrealistic. [2]

10 A firm sells two types of electrical components, A and B.

Type A components are packed in boxes of 50. On average, 2% of the Type A components are faulty.

(i) State, in context, two assumptions needed for the number of faulty components of Type A in a box to be well-modelled by a binomial distribution. [2]

You are now given that the number of faulty components of type A in a box follows a binomial distribution.

(ii) Find the probability that a box of these components contains more than 1 faulty component. [2]

Five boxes of Type A components are selected at random.

(iii) Find the probability that the fifth box is the third box selected which has more than one faulty component. [2]

(iv) Find the probability that there are 3 boxes with more than 1 faulty component each. [2]

(v) Explain why the answer to part **(iv)** is greater than the answer to part **(iii)**. [1]

Type B components are packed in boxes of 20 and on average 0.1% of the Type B components are faulty.

(vi) Thirty boxes of Type A and forty boxes of Type B components are randomly chosen. Find, by using suitable approximations, the probability that the total number of faulty components are no more than 15. [3]

11 An airline which cater various flights within the United States of America has a flight route that travels from New Orleans to Miami. The airline claims that the average flight time is 115 minutes. Brandon, a regular commuter of this flight route, thinks that his journey takes a shorter time on average.

(i) Explain, whether Brandon should carry out a 1-tail test or a 2-tail test. State the hypotheses for the test, defining any symbols you use. [2]

Brandon records the flight time (in minutes) of 8 randomly selected flights as shown below.

113.0 112.4 110.0 113.8 111.3 115.2 114.2 115.5

(ii) Assuming that the flight time from New Orleans to Miami follows a normal distribution that has a variance of 11.3 minutes^2, carry out the test for Brandon at 10% level of significance. Give your conclusion in the context of the question. [3]

(iii) Explain what you understand by the phrase "at 10% level of significance" in the context of the question. [1]

(iv) The same sample of 8 flights is now used to carry out a test at the 10% significance level, to test whether the mean flight times is different from what the airline claims. Using the answer in **(ii)**, state the conclusion of this test. [2]

Another flight route travels from Portland to Los Angeles. Due to bad weather conditions, the flights sometimes take a longer time to arrive at Los Angeles. The airline claims that the mean flight time is k minutes. The operation manager of the airline believes that this number is understated and wants to test this claim. Sixty flights were randomly selected, and the mean flight time of 181 minutes and standard deviation of 13.6 minutes were recorded.

(v) Find the range of values of k for which the claim by the airline is supported at 4% level of significance. Give your answer to the nearest minute. [4]

Question 1(i)

$$\left|\frac{z^3}{w^*}\right| = \frac{|z|^3}{|w|} = \frac{k^3}{3k^2} = \frac{k}{3}$$

$$\arg\left(\frac{z^3}{w^*}\right) = 3\arg(z) - \left[-\arg(w)\right] = 3\alpha + 4\alpha = 7\alpha$$

Thus, $\frac{z^3}{w^*} = \frac{k}{3}\mathrm{e}^{\mathrm{i}(7\alpha)}$

Question 1(ii)

$$\frac{z^3}{w^*} = \frac{k}{3}\mathrm{e}^{\mathrm{i}(\pi/3)} = \frac{k}{3}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$$

$$\arg\left(\frac{z^3}{w^*}\right)^n = n\arg\left(\frac{z^3}{w^*}\right) = \frac{n\pi}{3}$$

For $\left(\frac{z^3}{w^*}\right)^n$ to be real, $\arg\left(\frac{z^3}{w^*}\right)^n = k\pi$, $k \in \mathbb{Z}$

$$\frac{n\pi}{3} = \cdots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \cdots$$

$$n = \cdots, -9, -6, -3, 0, 3, 6, 9, \cdots$$

[Turn over

Question 2(i)

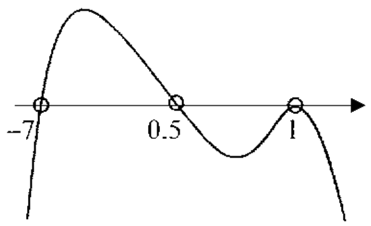
$$\frac{17x-9-2x^2-13x+7}{2x^2+13x-7}>0$$

$$\frac{-2x^2+4x-2}{(2x-1)(x+7)}>0\,;\,x\neq\frac{1}{2}\ ,-7$$

$$\frac{-2(x-1)^2}{(2x-1)(x+7)}>0$$

$$-2(x-1)^2(2x-1)(x+7)>0$$

$$\Rightarrow -7<x<\frac{1}{2}$$



Question 2(ii)

Replace x with $-x$: $\quad \frac{17x+9}{2x^2-13x-7}<-1 \quad \rightarrow \quad \frac{17(-x)+9}{2(-x)^2-13(-x)-7}<-1$

$$\frac{-17x+9}{2x^2+13x-7}<-1$$

$$\frac{17x-9}{2x^2+13x-7}>1$$

From above, $-7<-x<\frac{1}{2}$

$$\Rightarrow -\frac{1}{2}<x<7$$

Question 3(i)

$\lambda\mathbf{a}+\mu\mathbf{b}+\gamma\mathbf{c}=\mathbf{0}$ $\Rightarrow \mathbf{a},\mathbf{b},\mathbf{c}$ lie on the same plane

$$\Rightarrow \mathbf{a}\times(\lambda\mathbf{a}+\mu\mathbf{b}+\gamma\mathbf{c})=\mathbf{a}\times\mathbf{0}$$
$$\Rightarrow \lambda(\mathbf{a}\times\mathbf{a})+\mu(\mathbf{a}\times\mathbf{b})+\gamma(\mathbf{a}\times\mathbf{c})=\mathbf{0}$$
$$\Rightarrow \mathbf{0}+\mu(\mathbf{a}\times\mathbf{b})=-\gamma(\mathbf{a}\times\mathbf{c})$$
$$\Rightarrow \mu(\mathbf{a}\times\mathbf{b})=\gamma(\mathbf{c}\times\mathbf{a}) \text{ (shown) } \dots \text{(A)}$$

Similarly
$$\Rightarrow \mathbf{b}\times(\lambda\mathbf{a}+\mu\mathbf{b}+\gamma\mathbf{c})=\mathbf{b}\times\mathbf{0}$$
$$\Rightarrow \lambda(\mathbf{b}\times\mathbf{a})+\mu(\mathbf{b}\times\mathbf{b})+\gamma(\mathbf{b}\times\mathbf{c})=\mathbf{0}$$
$$\Rightarrow \lambda(\mathbf{b}\times\mathbf{a})=-\gamma(\mathbf{b}\times\mathbf{c})$$
$$\Rightarrow \lambda(\mathbf{b}\times\mathbf{a})=\gamma(\mathbf{c}\times\mathbf{b}) \dots \text{(B)}$$

Question 3(ii)

Consider,
$$|\mathbf{b}\times\mathbf{c}|=|\mathbf{b}||\mathbf{c}|\sin\angle BOC \quad \dots (1)$$
$$|\mathbf{c}\times\mathbf{a}|=|\mathbf{c}||\mathbf{a}|\sin\angle COA \quad \dots (2)$$
$$|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\angle AOB \quad \dots (3)$$

Take $\frac{(2)}{(3)}: \frac{|\mathbf{c}\times\mathbf{a}|}{|\mathbf{a}\times\mathbf{b}|}=\frac{|\mathbf{c}||\mathbf{a}|\sin\angle COA}{|\mathbf{a}||\mathbf{b}|\sin\angle AOB}$

from (A): $\frac{\frac{|\mu|}{|\gamma|}|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}\times\mathbf{b}|}=\frac{\sin\angle COA}{\sin\angle AOB} \Rightarrow \frac{|\mu|}{\sin\angle COA}=\frac{|\gamma|}{\sin\angle AOB}$

Similarly, taking $\frac{(1)}{(3)}: \frac{|\mathbf{b}\times\mathbf{c}|}{|\mathbf{a}\times\mathbf{b}|}=\frac{|\mathbf{b}||\mathbf{c}|\sin\angle BOC}{|\mathbf{a}||\mathbf{b}|\sin\angle AOB}$

from (B): $\frac{\frac{|\lambda|}{|\gamma|}|\mathbf{b}\times\mathbf{a}|}{|\mathbf{a}\times\mathbf{b}|}=\frac{\sin\angle BOC}{\sin\angle AOB} \Rightarrow \frac{|\lambda|}{\sin\angle BOC}=\frac{|\gamma|}{\sin\angle AOB}$

Thus, $\frac{|\mu|}{\sin\angle COA}=\frac{|\lambda|}{\sin\angle BOC}=\frac{|\gamma|}{\sin\angle AOB}$ (proven)

Question 4(a)

Let $y = \ln|\sec 2x| \Rightarrow \pm e^{y} = \sec 2x$

Since $\frac{\pi}{3} \le x \le \frac{3\pi}{8}$, then sec $2x < 0$

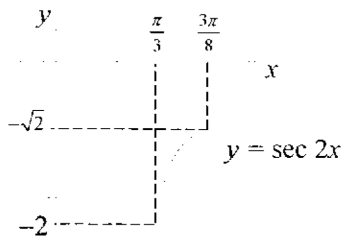
$\Rightarrow \quad \sec 2x = -e^{y}$

Then $\quad \cos 2x = -e^{-y}$

$$x = \frac{1}{2}\cos^{-1}\left(-e^{-y}\right)$$

Hence $f^{-1}(x) = \frac{1}{2}\cos^{-1}\left(-e^{-x}\right)$.

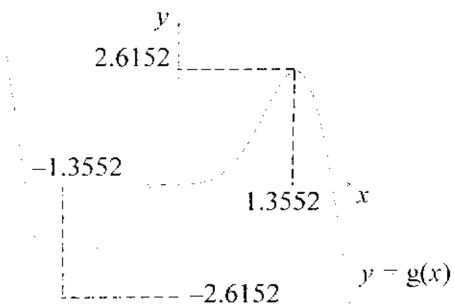
Domain of f^{-1} = $[\ln\sqrt{2}, \ln 2]$ or [0.347, 0.693]



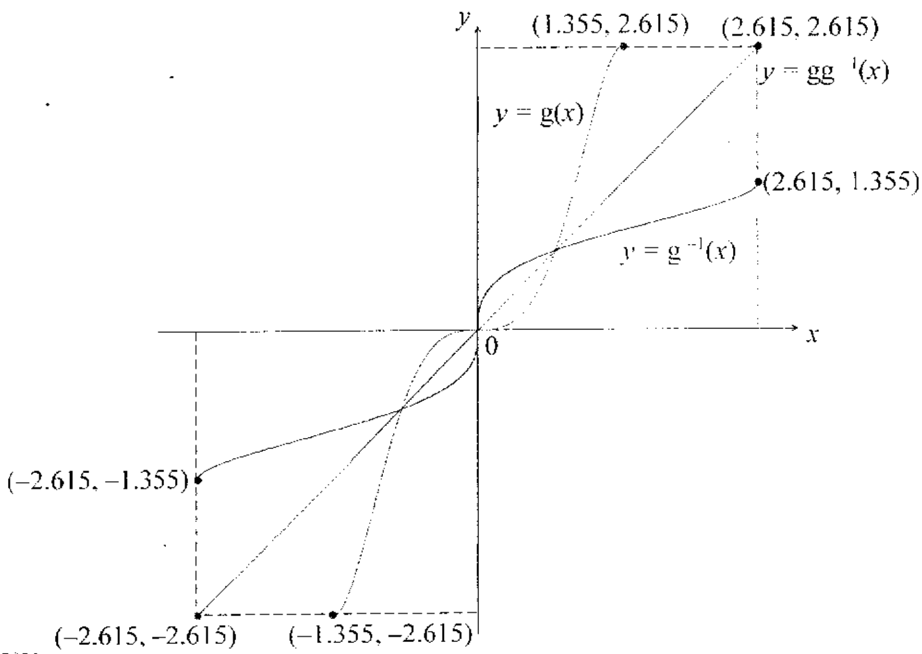
Question 4(b)(i)

From the GC, $k = 1.3552 = 1.355$ (3 dp)

$R_g = [-2.615, 2.615]$



Question 4(b)(ii)



<u>Question 4(b)(iii)</u>

Let $\mathrm{g}(x) = x$.

Then $2x\sin(x^2) = x$

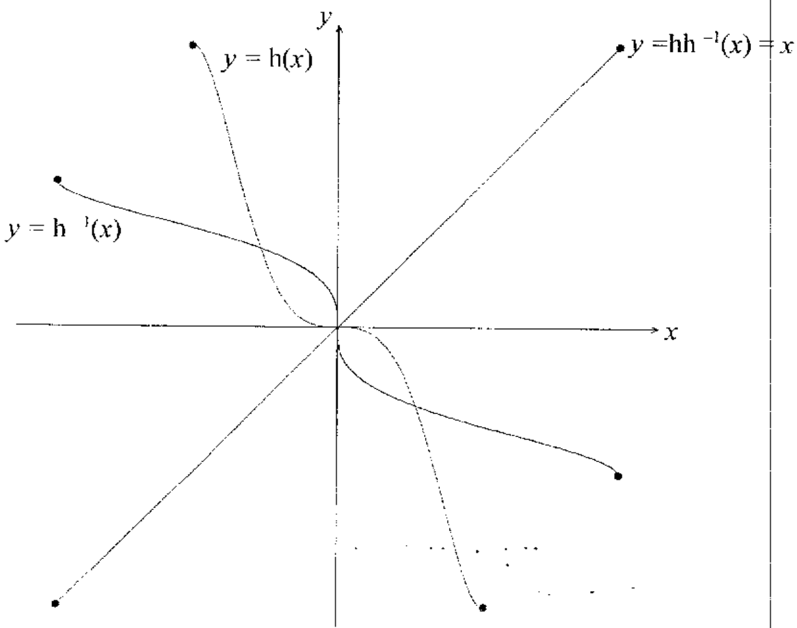
$$x\left[2\sin\left(x^2\right)-1\right]=0$$

$$x=0 \quad \text{or} \quad \sin\left(x^2\right)=\frac{1}{2}$$

$$x^2=\sin^{-1}\frac{1}{2}=\frac{\pi}{6}$$

$$x=\pm\sqrt{\frac{\pi}{6}}$$

<u>Question 4(b)(iv)</u>



From the diagram, there are 3 solutions for $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$.

However, there is only 1 solution for $\mathrm{g}(x) = x$.

Hence the method used in part **(iii)** would not yield the complete set of solutions.

Question 5(i)

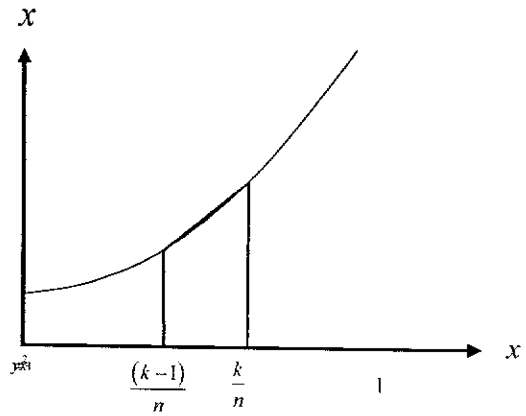
$$\text{Width} = \frac{1}{2} \text{ units}$$

$$\text{Area of first trapezium} = \frac{1}{2}\times\left(0^2+1+\frac{1}{2}^2+1\right)\times\frac{1}{2}=\frac{9}{16} \text{ units}^2$$

$$\text{Area of second trapezium} = \frac{1}{2}\times\left(\frac{1}{2}^2+1+1^2+1\right)\times\frac{1}{2}=\frac{13}{16} \text{ units}^2$$

$$\text{Total area of trapeziums} = \frac{9}{16}+\frac{13}{16}=\frac{11}{8} \text{ units}^2$$

Question 5(ii)



Width $=\dfrac{1}{n}$ and x-value of the shorter side of the k^{th} trapezium $=\dfrac{(k-1)}{n}$

Hence the length of the shorter side of the kth trapezium is $\left[\dfrac{(k-1)}{n}\right]^2+1$

$$\text{Area of } k^{\text{th}} \text{ trapezium } =\frac{1}{2}\times\left[\left(\frac{k-1}{n}\right)^2+1+\left(\frac{k}{n}\right)^2+1\right]\times\frac{1}{n}$$

$$=\frac{1}{2n^3}\times\left[(k-1)^2+k^2+2n^2\right]$$

$$=\frac{(k-1)^2+k^2}{2n^3}+\frac{1}{n} \text{ units}^2.$$

Question 5(iii)

$$A=\sum_{k=1}^{n}\left[\frac{(k-1)^2+k^2}{2n^3}+\frac{1}{n}\right]$$

$$=\frac{1}{2n^3}\sum_{k=1}^{n}\left[k^2-2k+1+k^2\right]+\frac{1}{n}\sum_{k=1}^{n}1$$

$$=\frac{1}{2n^3}\left[2\sum_{k=1}^{n}k^2-2\sum_{k=1}^{n}k+\sum_{k=1}^{n}1\right]+\frac{n}{n}$$

$$=\frac{1}{2n^3}\left[\frac{n(n+1)(2n+1)}{3}-2n\left(\frac{n+1}{2}\right)+n\right]+1$$

$$=\frac{1}{6n^3}\left[2n^3+3n^2+n-3n^2-3n+3n\right]+1$$

$$=\frac{1}{3}+\frac{1}{6n^2}+1$$

$$=\frac{4}{3}+\frac{1}{6n^2}$$

Question 5(iv)

It is an overestimate.

As the number of trapezium increase to infinity, the total of trapezium will tend to the exact area of region R.

$\therefore$ Area of region $R=\int_0^1\left(x^2+1\right)\mathrm{d}x=\lim_{n\to\infty}\left(\frac{4}{3}+\frac{1}{6n^2}\right)=\frac{4}{3}$

Question 6(i)

$$(p)(0.9) + (100 - p)(0.1) = 20$$

$$(p)(0.9) + (100 - p)(0.1) = 20$$

$$0.9p + 10 - 0.1p = 20$$

$$p = 12.5$$

The proportion of the residents infected is $\frac{12.5}{100} = \frac{1}{8}$ or 12.5 %

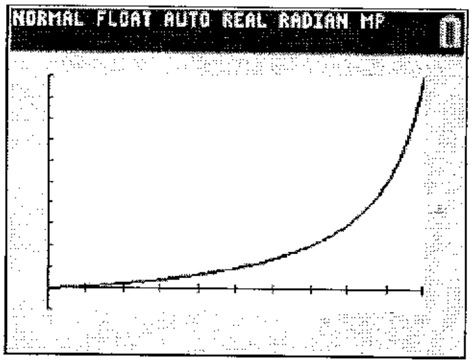
Question 6(ii)

$$\text{P(has disease} \mid \text{tested negative)} = \frac{\text{P(has disease \& tested negative)}}{\text{P(tested negative)}}$$

$$= \frac{0.1q}{0.1q + 0.9(100-q)}$$

$$= \frac{q}{900 - 8q}$$

Question 6(iii)



As the proportion of people getting infected (q) increases, the probability that a person has the disease given that he has been tested negative also increases as seen in the graph. So, the test is not effective.

Question 7(i)

No of ways required = 4! = 24

Question 7(ii)

No of ways required = ${}^4C_1 \times {}^2C_1 \times {}^1C_1 = 8$

Alternative solution

Correct present	Listing	No of ways
A	A,C,D,B or A,D,B,C	2
B	C,B,D,A or D,B,A,C	2
C	B,D,C,A or D,A,C,B	2
D	B,C,A,D or C,A,B,D	2

Question 7(iii)

No of ways required = ${}^4C_2 \times {}^1C_1 \times {}^1C_1 = 6$

Alternative solution

Correct presents	Listing	No of ways
A, B	A,B,D,C	1
A, C	A,D,C,B	1
A, D	A,C,B,D	1
B, C	D,B,C,A	1
B, D	C,B,A,D	1
C, D	B,A,C,D	1

Question 7(iv)

If 3 persons have received back their own gifts, there will be one remaining gift and one person who have yet to receive a gift. But this last person is the exact person who brought this last gift to the party. If he were to receive the remaining gift, he would have received back his own gift.

Question 7(v)

No of ways required $= 24 - (8 + 6) - 1 = 9$

Probability required $= \frac{9}{24} = \frac{3}{8}$

Question 8(i)

$$P(\text{different colour}) = P(B,W) = \frac{n}{2n+1}\cdot\frac{n+1}{2n}\cdot 2 = \frac{n+1}{2n+1}.$$

Question 8(ii)

$$P(\text{same colour}) = P(B,B) + P(W,W)$$

$$= \frac{n}{2n+1}\cdot\frac{n-1}{2n} + \frac{n+1}{2n+1}\cdot\frac{n}{2n} = \frac{n-1}{2(2n+1)} + \frac{n+1}{2(2n+1)} = \frac{n}{2n+1}$$

Question 8(iii)

$$P(X=1) = \left(\frac{n+1}{2n+1}\right)\left(\frac{1}{2}\right)^2(2) + \left(\frac{n}{2n+1}\right)\left(\frac{1}{2}\right)^4\frac{4!}{3!} = \frac{3n+2}{4(2n+1)}$$

Alternatively,

$$P(X=1) = P(B,W,H,T) + P(B,B,H,T,T,T) + P(W,W,H,T,T,T)$$

$$= \left[\frac{n+1}{2n+1}\right]\left(\frac{1}{2}\right)^2(2) + \left[\frac{n-1}{2(2n+1)}\right]\left(\frac{1}{2}\right)^4\frac{4!}{3!} + \left[\frac{n+1}{2(2n+1)}\right]\left(\frac{1}{2}\right)^4\frac{4!}{3!} = \frac{3n+2}{4(2n+1)}$$

Question 8(iv)

If there are 3 blue cards $\Rightarrow n = 3$

$$P(B,W) = \frac{n+1}{2n+1} = \frac{4}{7};\ P(B,B) = \frac{n-1}{2(2n+1)} = \frac{1}{7}; P(W,W) = \frac{n+1}{2(2n+1)} = \frac{2}{7}$$

$$P(X=1) = \frac{3n+2}{4(2n+1)} = \frac{11}{28}$$

$$P(X=4) = P(B,B,H,H,H,H) + P(W,W,H,H,H,H)$$

$$= \frac{1}{7}\left(\frac{1}{2}\right)^4 + \frac{2}{7}\left(\frac{1}{2}\right)^4 = \frac{3}{112}$$

$$P(X=3) = P(B,B,H,H,H,T) + P(W,W,H,H,H,T)$$

$$= \frac{1}{7}\left(\frac{1}{2}\right)^4\left(\frac{4!}{3!}\right) + \frac{2}{7}\left(\frac{1}{2}\right)^4\left(\frac{4!}{3!}\right) = \frac{3}{28}$$

$$P(X=0) = P(B,W,T,T) + P(B,B,T,T,T,T) + P(W,W,T,T,T,T)$$

$$= \frac{4}{7}\left(\frac{1}{2}\right)^2 + \frac{1}{7}\left(\frac{1}{2}\right)^4 + \frac{2}{7}\left(\frac{1}{2}\right)^4 = \frac{19}{112}$$

$$P(X=2)=1-P(X=0)-P(X=1)-P(X=3)-P(X=4)$$
$$=1-\left(\frac{19}{112}+\frac{11}{28}+\frac{3}{28}+\frac{3}{112}\right)$$
$$=1-\frac{39}{56}=\frac{17}{56}$$

Alternatively,

x	0	1	2	3	4
$P(X=x)$	$\frac{4}{7}\left(\frac{1}{2}\right)^2+\frac{3}{7}\left(\frac{1}{2}\right)^4$ $=\frac{19}{112}$	$\frac{3(3)+2}{4(2(3)+1)}$ $=\frac{11}{28}$	$\frac{4}{7}\left(\frac{1}{2}\right)^2+{}^4C_2\frac{3}{7}\left(\frac{1}{2}\right)^4$ $=\frac{17}{56}$	${}^4C_1\frac{3}{7}\left(\frac{1}{2}\right)^4$ $=\frac{3}{28}$	$\frac{3}{7}\left(\frac{1}{2}\right)^4$ $=\frac{3}{112}$

<u>Question 8(v)</u>

When $n=3$, $P(X>2\mid X\le 3)=\frac{P(2<X\le 3)}{P(X\le 3)}$

$$=\frac{P(X=3)}{1-P(X=4)}$$
$$=\frac{\frac{3}{28}}{1-\frac{3}{112}}$$
$$=\frac{12}{109}$$

Question 9(i)

Let X denote the random variable representing the amount of nocturnal sleep (hours) for a randomly selected student on a school-day night. Then, $X \sim N(6.5, \sigma^2)$.

$P(X < 8) = 0.85 \quad \Rightarrow \quad P\left(Z < \frac{8-6.5}{\sigma}\right) = 0.85$

From the GC, $\frac{1.5}{\sigma} = 1.03643338$

$\therefore \sigma = 1.4473 = 1.45$ (3 sf)

Hence, the required standard deviation is 1.45 hrs.

Question 9(ii)

Probability required $= 3P(X_1 < 5.5) \times P(X_2 > 7) \times P(X_3 > 7)$
$= 3(0.24480)(0.36487)^2$
$= 0.097771$
$= 0.0978$ (3 sf)

Question 9(iii)

Let Y denote the random variable representing the amount of sleep (in minutes) from daily afternoon naps on a school-day for a student. Then $Y \sim N(65, a^2)$.

When $a = 60$, $P(Y < 0) = 0.139$. Since 0.139 is not negligible, the amount of sleep from daily afternoon naps for a student on a school-day may not be appropriately modelled by a normal distribution.

Question 9(iv)

Let T and W denote the random variables representing the total amount of sleep (in hours) on a school-day for a student and the amount of sleep from daily afternoon naps on a school-day for a student. $\therefore T = X + W$ where $X \sim N(6.5, 1.4473^2)$ and $W \sim N\left(\frac{65}{60}, \left(\frac{10}{60}\right)^2\right)$

$E(T) = E(X+W) = E(X) + E(W) = 6.5 + \frac{65}{60} = \frac{91}{12} = 7.5833$

$Var(T) = Var(X+W) = Var(X) + Var(W) = 1.44727^2 + \left(\frac{1}{6}\right)^2 = 2.1224$

$\therefore T \sim N(7.5833, 2.1224)$

$\Rightarrow P(T > 8) = 0.38743 = 0.387$ (3 sf)

Question 9(iv)

Probability required = $P(T_1 + T_2 - 2T_3 > 1)$

$E(T_1 + T_2 - 2T_3) = E(T_1) + E(T_2) - 2E(T_3) = 0$

$Var(T_1 + T_2 - 2T_3) = Var(T_1) + Var(T_2) + 4Var(T_3) = 2.1224 + 2.1224 + 4(2.1224) = 12.735$

$\therefore T_1 + T_2 - 2T_3 \sim N(0, 12.735)$

$P(T_1 + T_2 - 2T_3 > 1) = 0.38965 = 0.390$ (3 sf)

Question 9(v)

Assumptions: The amount of nocturnal sleep on a school-day night (X) is independent of the amount of sleep from afternoon naps (W) on the same day.

The assumption may be unrealistic because a student who has slept longer in the afternoon may require less nocturnal sleep. Hence X and W may not be independent.

Question 10(i)

The event that a Type A component is faulty is independent of the event of any other Type A component that is faulty.

The probability that a Type A component is faulty is constant at 0.02.

Question 10(ii)

Let X be the random variable denoting the number of faulty Type A components in a box of 50.

$$X \sim B(50, 0.02)$$

$$\begin{aligned} P(X > 1) &= 1 - P(X \leq 1) \\ &= 0.26423 \text{ (to 5 s.f.)} \\ &= 0.264 \text{ (to 3 s.f.)} \end{aligned}$$

Question 10(iii)

$$\text{Required probability} = (0.26423)^2 (1 - 0.26423)^2 (0.26423) \times \frac{4!}{2!2!}$$

$$= 0.0599 \text{ (to 3 s.f.)}$$

OR

Let Q be the random variable denoting the number of boxes with more than 1 faulty component out of 4.

$$Q \sim B(4, 0.26423)$$

$$P(Q=2) \times (0.26423) = 0.0599$$

Question 10(iv)

Let Y be the random variable denoting the number of boxes with more than 1 faulty component out of 5.

$$Y \sim B(5, 0.26423)$$

$$P(Y=3) = 0.0999$$

Question 10(v)

Event in part (iii) is a subset of event in part (iv).

Question 10(vi)

Let W be the random variable denoting the number of faulty Type B components in a box of 20.

$W \sim \text{B}(20, 0.001)$

Since sample size is large for both types of components, by Central Limit Theorem,

$W_1 + W_2 + ... + W_{40} \sim \text{N}(0.8, 0.7992)$ approx

$X_1 + X_2 + ... + X_{30} \sim \text{N}(30, 29.4)$ approx.

Let $T = X_1 + X_2 + ... + X_{30} + W_1 + W_2 + ... + W_{40}$

$T \sim \text{N}(30.8, 30.1992)$ approximately

$\text{P}(T \leq 15) = 0.00202$ (to 3 s.f.)

Question 11(i)

Since Brandon suspects that the average flight time from New Orleans to Miami is shorter than 115 minutes, he should carry out a 1-tail test.

Let X be the random variable denoting the flight time from New Orleans to Miami in minutes and μ be the mean flight time.

$H_0 : \mu = 115$

$H_1 : \mu < 115$

Question 11(ii)

Using one-tailed test at 10% significance level.

Under H_0, $\overline{X} \sim N\left(115, \frac{11.3}{8}\right)$.

Sample readings : $\overline{x} = 113.175$

Using Z-test, p- value = 0.062322.

Since the p-value < 0.1 (the significance level), we reject H_0 and conclude that there is sufficient evidence at the 10% level of significance to conclude that the mean flight time from New Orleans to Miami is less than 115 minutes.

Question 11(iii)

10% level of significance refers to a probability of 0.1 that we conclude that the average flight time from New Orleans to Miami is less than 115 minutes when in fact it is 115 minutes.

Question 11(iv)

p-value = $2 \times 0.062322 = 0.124644$

Since the p-value > 0.1 (the significance level), we do not reject H_0 and conclude that there is insufficient evidence at the 10% level of significance to conclude that the mean flight time from New Orleans to Miami is not 115 minutes.

Question 11(v)

Let Y be the random variable denoting the flight time from Portland to Los Angeles in minutes and μ be the mean time.

$$H_0 : \mu = k$$

$$H_1 : \mu > k$$

Using one-tailed test at 4% significance level.

Under H_0, since $n = 60$ is large, by central limit theorem,

$$\overline{Y} \sim \mathrm{N}\left(k, \frac{\frac{60}{59}(13.6)^2}{60} \right) \text{ approximately.}$$

$$\Rightarrow \frac{\overline{Y} - k}{\frac{13.6}{\sqrt{59}}} = Z \sim \mathrm{N}(0,1)$$

Since $\overline{y} = 181$, test statistics $= \dfrac{181-k}{\frac{13.6}{\sqrt{59}}}$

Critical value = 1.7507 (by InvNorm)

Since H_0 is not rejected,

$$\frac{181-k}{\frac{13.6}{\sqrt{59}}} < 1.7507$$

$\Rightarrow k > 177.90$

$\Rightarrow k > 178$ (3 s.f.) or $k \geq 178$