2021 NJC H2 Mathematics Prelim Paper 1

1 (i) Show that $\dfrac{\sin\left[(n+1)\theta - n\theta\right]}{\cos n\theta \cos(n+1)\theta} = \tan(n+1)\theta - \tan n\theta$. [2]

(ii) Hence find, in terms of n and θ, an expression for

$$\sec\theta\sec 2\theta + \sec 2\theta\sec 3\theta + \sec 3\theta\sec 4\theta + \ldots + \sec n\theta\sec(n+1)\theta, \text{ where } n \in \mathbb{Z}^+.$$ [3]

2 The number of bacteria (in millions) in culture A at the start of the n^{th} day is denoted by u_n, for $n \in \mathbb{Z}^+$. After the start of each day, a researcher subjects culture A to high temperatures that kill 60% of the existing bacteria. At the end of each day, 3 million new bacteria are produced in culture A. There were 5 million bacteria at the start of the first day.

(i) Write down a sequence in the form $u_{n+1} = au_n + b$, where a and b are constants. [1]

(ii) Describe the behavior of the number of bacteria in culture A in the long run. [1]

In culture B, the number of bacteria (in millions) at the start of the n^{th} day is denoted by

$$v_n = \frac{pn}{n^2 + qn + r}, \text{ for } n \in \mathbb{Z}^+ \text{ and where } p, q, r \text{ are constants.}$$

The researcher started the experiment for culture B on 1 April and collected the following data:

At the start of	Number of bacteria (in millions) present in culture B
1 April	2
2 April	2.4
4 April	1.6

(iii) Find the value of p, q and r. [3]

(iv) On which date will the researcher first record the number of bacteria in culture B to be below half a million? [1]

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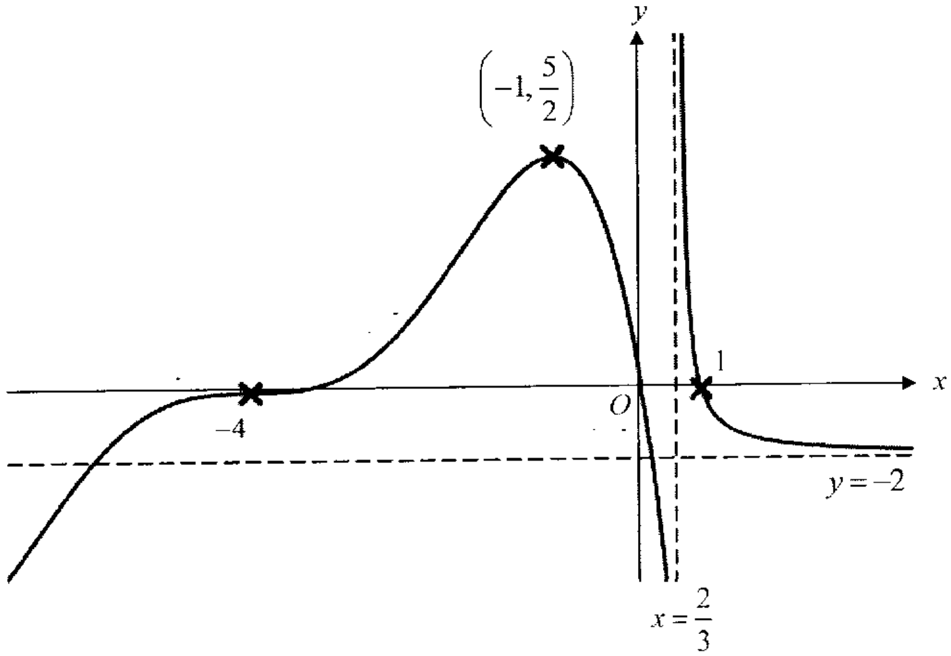

The diagram shows the curve with equation $y = \mathrm{f}(x)$, for $x \in \mathbb{R}, x \neq \frac{2}{3}$. The curve crosses the axes at $x = -4$, $x = 1$ and the origin, and has asymptotes with equations $x = \frac{2}{3}$ and $y = -2$. The curve has a stationary point of inflexion at $x = -4$ and a turning point with coordinates $\left(-1, \frac{5}{2}\right)$.

(i) Sketch the curve $y = \frac{1}{\mathrm{f}(x)}$, labelling any axial intercepts and coordinates of turning points, and the equations of any asymptotes. [3]

(ii) Sketch the curve $y = \mathrm{f}'(x)$, labelling any x-intercepts and the equations of any asymptotes. [3]

4 **(i)** Differentiate $\mathrm{e}^{\cos 2x}$ with respect to x. [1]

(ii) Find $\int \mathrm{e}^{\cos 2x} \sin 4x \, \mathrm{d}x$. [3]

(iii) Hence find $\int \mathrm{e}^{\cos 2x} (\cos 3x \sin x) \, \mathrm{d}x$. [3]

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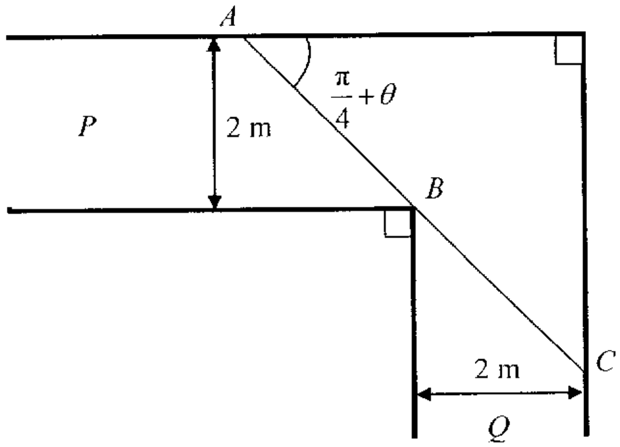


Two straight corridors, P and Q, each of width 2 m, meet at right angles. A banner is hung across the ceiling of the corridors using a taut string such that the string is parallel to the ground and always touches the inside corner of the wall at point B. The string also touches the outer walls at variable points A and C respectively. In the position shown in the diagram, the acute angle between AC and the wall of corridor P is $\frac{\pi}{4}+\theta$, where θ is a sufficiently small angle.

(i) Show that $AC = 2\left[\frac{1}{\sin\left(\frac{\pi}{4}+\theta\right)}+\frac{1}{\cos\left(\frac{\pi}{4}+\theta\right)}\right]$. [2]

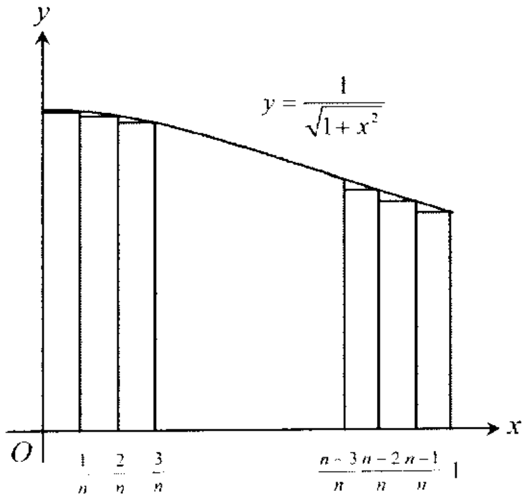
(ii) Hence show that

$$AC \approx r + s\theta^2$$

where r and s are constants to be determined. [5]

6 **(i)** Using the substitution $x=\tan\theta$, find the exact value of $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$. [4]

(ii) The graph of $y=\frac{1}{\sqrt{1+x^2}}$, for $0\le x\le 1$, is shown in the diagram. Rectangles, each of width $\frac{1}{n}$, are drawn under the curve.



Show that the total area A of all n rectangles is given by

$$A=\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\frac{1}{\sqrt{n^2+3^2}}+\ldots+\frac{1}{\sqrt{n^2+(n-1)^2}}+\frac{1}{\sqrt{2n^2}}.$$

State the limit of A as $n\to\infty$. [3]

7 A sequence of positive numbers $u_1, u_2, u_3, \ldots$ is a strictly increasing arithmetic progression. It is given that the first term is a and the ninth term is b.

(i) Find u_5 in terms of a and b and show that $u_3+u_5+u_7=\frac{3}{2}(b+a)$. [3]

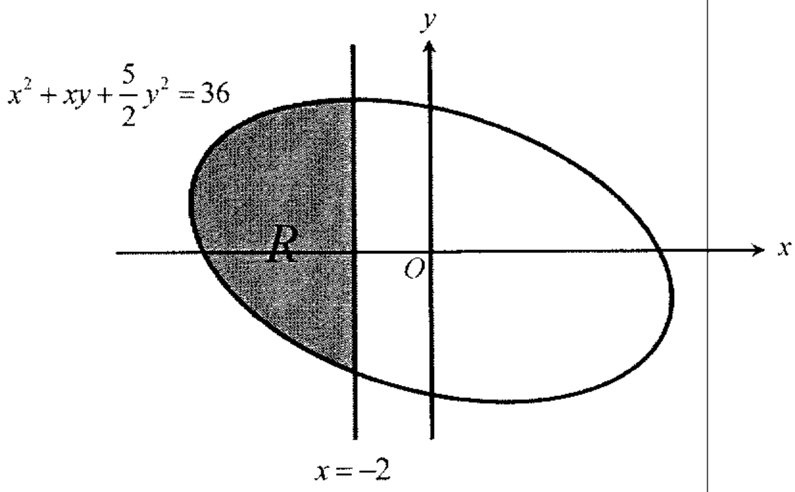
(ii) Given also that a, u_3 and b are consecutive terms of a geometric progression, express b in terms of a. [3]

(iii) Hence, determine if a sequence that consists of consecutive terms $\ln(u_3), \ln(u_5)$ and $\ln(u_7)$ is an arithmetic progression. [2]

8 The curve C has equation $x^2 + xy + ay^2 = 36$, where a is a constant such that $a > \frac{1}{4}$.

(i) Find the x-coordinates of the points on C where the normal is parallel to the y-axis, leaving your answers in terms of a. [4]

(ii) For $a = \frac{5}{2}$, the region R is bounded by C and the line $x = -2$ as shown in the diagram. It is also given that all points in the region R are such that $x \le -\frac{y}{2}$.



Find the volume formed when R is rotated completely about the y-axis, leaving your answers correct to 2 decimal places. [4]

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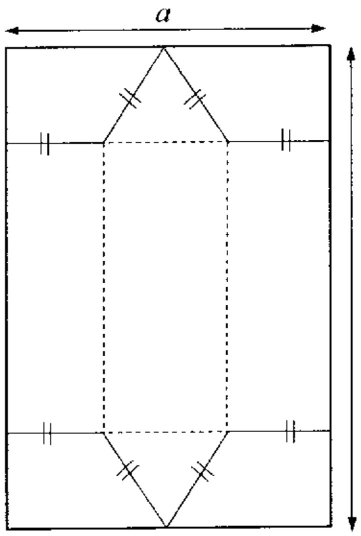


Figure 1

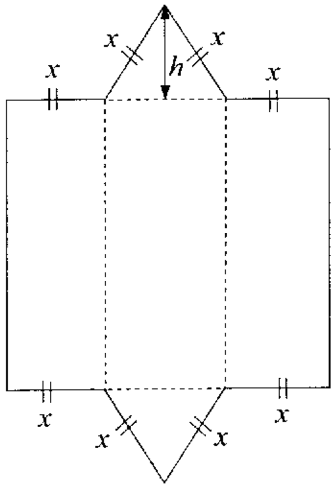


Figure 2

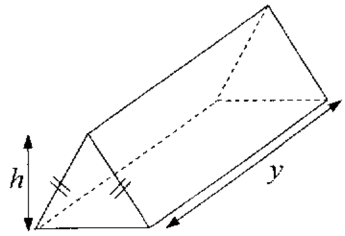


Figure 3

Figure 1 shows a piece of card in the shape of a rectangle with sides a metres and $2a$ metres. A trapezium is cut from each corner, to give the shape shown in **Figure 2** which consists of two identical isosceles triangles and three rectangles. For the triangles, the two equal sides are of length x metres each and the height is h metres. The remaining card shown in **Figure 2** is then folded along the dotted lines to form a closed triangular prism with height y metres as shown in **Figure 3**. The volume of the closed triangular prism is denoted by V.

(i) Find a formula for x in terms of h and a. Hence show that the value of h that gives a stationary value of V satisfies the equation $-16h^3+12ah^2+2a^2h-a^3=0$. [6]

(ii) Suppose $a=5$. Find the value of h that gives a stationary value of V, and explain why there is only one answer. Hence prove that this stationary value of V is a maximum. [4]

10 Antibiotics are used to treat bacterial infections. The rate at which the amount of antibiotics in a patient's body decays is proportional to the amount of antibiotics in the patient's body, x, at any time t in hours. It is given that an initial dose of antibiotics with amount x_0 is administered to a patient. After 6 hours, the amount of antibiotics in the patient's body is $\frac{x_0}{1000}$.

(i) Write down a differential equation relating x and t. [1]

(ii) Solve this differential equation to find an expression for x in the form $\frac{x_0}{P^t}$, where P is an exact constant to be determined. Hence find the time taken for the amount of antibiotics in the patient's body to reach 25% of the initial dose. [6]

As the amount of antibiotics in the patient's body decays with time, a pharmacist recommends administering the antibiotics every T hours with a dosage of x_0, for an extended period of time.

(iii) State the amount of antibiotics in the patient's body immediately after the second dose. Hence show that the amount of antibiotics in the patient's body at any time, t, after the second dose and before the third dose is $x_0\left(10^{\frac{1}{2}(T-t)}+10^{\left(-\frac{t}{2}\right)}\right)$, for $T \le t < 2T$. [3]

11 **(i)** Sketch the curve with equation $y=\frac{1}{2}+\frac{1}{|x-2|-3}$, stating the equations of the asymptotes.

Hence solve the inequality $\frac{1}{2}+\frac{1}{|x-2|-3}\geq\frac{1}{x}-\frac{1}{3}$. [4]

The functions f and g are defined by

$$\mathrm{f}:x\mapsto\frac{1}{2}+\frac{1}{|x-2|-3},\ x\in\mathbb{R},\ -1<x<1,$$

$$\mathrm{g}:x\mapsto\sin\left(\frac{\pi x}{c}\right),\ x\in\mathbb{R},\ \frac{5c}{3}\leq x\leq\frac{14c}{5}\ \text{where } c\in\mathbb{R}^{+}.$$

(ii) Find f^{-1} and state its domain. [3]

(iii) Find the exact range of g. [2]

The function h is given by

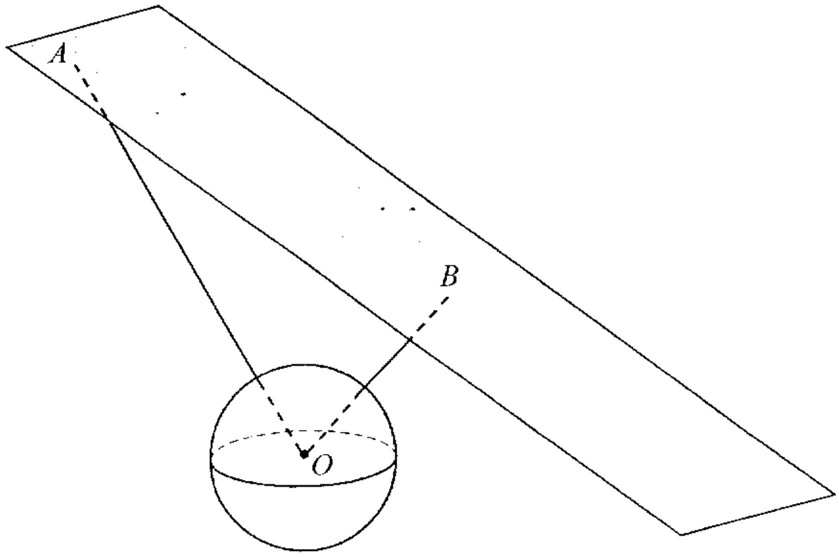
$$\mathrm{h}(x)=\mathrm{g}(x),\ x\in\mathbb{R},\ \frac{3c}{2}<x<\frac{5c}{2}$$

where $c\in\mathbb{R}^{+}$.

(iv) Find $(\mathrm{fh})^{-1}\left(-\frac{1}{2}\right)$ in terms of c. [3]

12 **(a)** The lines l_1 has equation $\mathbf{r} = 10\mathbf{i} + 8\mathbf{j} + 8\mathbf{k} + \lambda(\mathbf{i} + 14\mathbf{j} + h\mathbf{k})$, where λ is a parameter and h is a constant. Another line l_2 has equation $\mathbf{r} = s\mathbf{i} - 10\mathbf{j} + 12\mathbf{k} + \mu(2\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$, where μ is a parameter and s is a constant. Given that l_1 and l_2 are skew lines that are perpendicular, find the possible values of h and s. [4]

(b)



In an exhibition hall, an advertisement ball in the shape of a sphere with radius 1 unit is suspended from the roof of a building using hanging cords. Points (x, y, z) are defined relative to the centre of the ball at $(0,0,0)$, where units are in metres. Cords connecting the ball to the roof are straight lines and the thickness of the cords can be neglected.

The roof can be modelled by a plane with equation $6x + 8z = 25$. Cord OA starts at the centre of the ball and the coordinate of A is $(-2.5, 0, 5)$. Cord OB also starts at the centre of the ball and it is the shortest possible cord from the centre of the ball to the roof.

(i) Find the coordinates of B and hence find the shortest distance between the surface of the advertisement ball and the roof. [3]

To further secure the suspended advertisement ball, a third hanging cord OD is added such that cord OD is the reflection of cord OA in cord OB.

(ii) Find an equation of the line representing cord OD. [3]

A square LED light panel that is part of a plane is to be installed between the advertisement ball and the roof. The distance between the plane containing the LED light panel and the roof is 0.8 metres. Assume that the thickness of the LED light panel is negligible.

(iii) Find a cartesian equation of the plane which represents the LED light panel. [2]

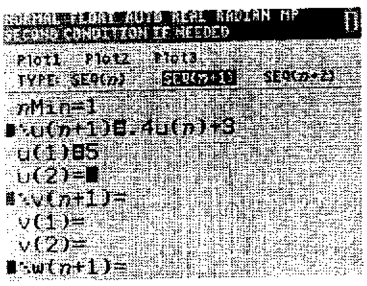
(iv) It is given further that the square LED light panel has sides of length n metres and its centre passes through cord OB. Find the largest possible integer n such that the panel will not touch cord OA. [2]

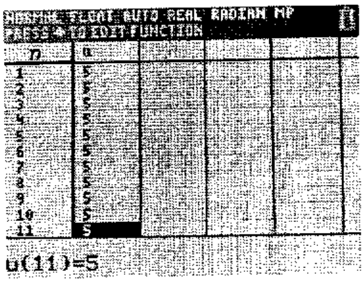
Q1	Suggested Solutions
(i)	$\frac{\sin\left[(n+1)\theta-n\theta\right]}{\cos n\theta\cos(n+1)\theta}$ $=\frac{\sin(n+1)\theta\cos n\theta-\cos(n+1)\theta\sin n\theta}{\cos n\theta\cos(n+1)\theta}$ $=\frac{\sin(n+1)\theta}{\cos(n+1)\theta}-\frac{\sin n\theta}{\cos n\theta}$ $=\tan(n+1)\theta-\tan n\theta$ (shown)
(ii)	$\sec\theta\sec 2\theta+\sec 2\theta\sec 3\theta+\sec 3\theta\sec 4\theta+...+\sec n\theta\sec(n+1)\theta$ $=\sum_{r=1}^{n}\sec r\theta\sec(r+1)\theta$ $=\sum_{r=1}^{n}\frac{1}{\cos r\theta\cos(r+1)\theta}$ $=\sum_{r=1}^{n}\left(\frac{\tan(r+1)\theta-\tan r\theta}{\sin\left[(r+1)\theta-r\theta\right]}\right)$ $=\frac{1}{\sin\theta}\sum_{r=1}^{n}\left(\tan(r+1)\theta-\tan r\theta\right)$ $=\frac{1}{\sin\theta}\begin{bmatrix}\tan 2\theta-\tan\theta\\+\tan 3\theta-\tan 2\theta\\+...\\+\ \tan n\theta-\tan(n-1)\theta\\+\ \tan(n+1)\theta-\tan n\theta\end{bmatrix}$ $=\frac{\tan(n+1)\theta-\tan\theta}{\sin\theta}$

Q2	Suggested Solutions
(i)	$u_{n+1} = 0.4u_n + 3$

(ii)

The number of bacteria in culture A remains constant at 5 million.





(iii)

$$v_n = \frac{pn}{n^2 + qn + r}$$

At the start of the 1st day, $n = 1$:

$$2 = \frac{p}{1+q+r} \Rightarrow 2 + 2q + 2r = p$$

$p - 2q - 2r = 2$

At the start of the 2nd day, $n = 2$:

$$2.4 = \frac{2p}{4+2q+r} \Rightarrow 9.6 + 4.8q + 2.4r = 2p$$

$2p - 4.8q - 2.4r = 9.6$ $(\text{or } p - 2.4q - 1.2r = 4.8)$

At the start of the 4th day, $n = 4$:

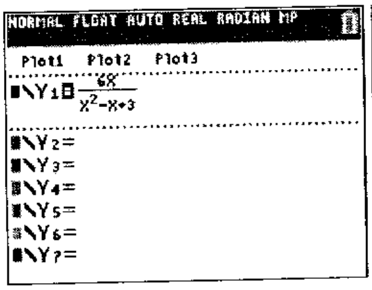
$$1.6 = \frac{4p}{16+4q+r} \Rightarrow 25.6 + 6.4q + 1.6r = 4p$$

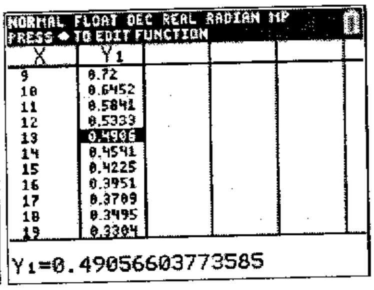
$4p - 6.4q - 1.6r = 25.6$ $(\text{or } p - 1.6q - 0.4r = 6.4)$

By solving the system of linear equations using GC,
$p = 6,\ q = -1,\ r = 3$.

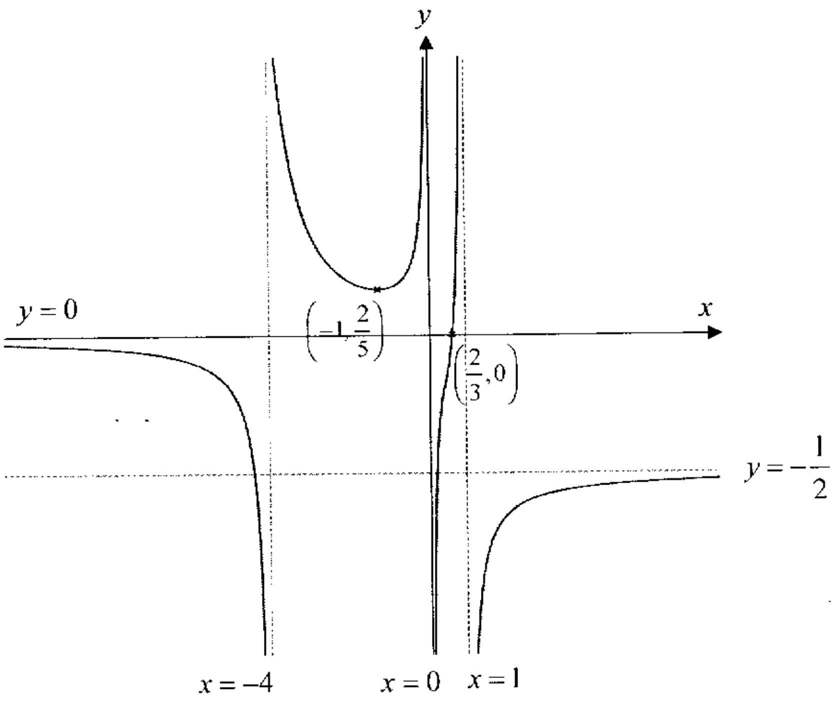
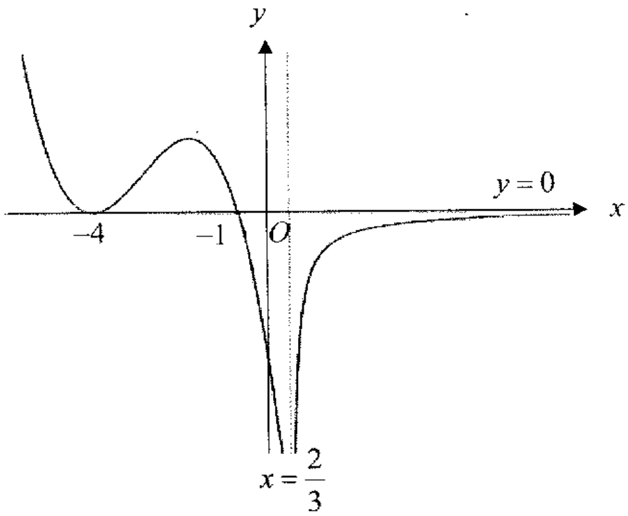
(iv)

Find least n such that $v_n < 0.5$.





From table, least $n = 13$
Therefore, the researcher first record the number of bacteria in culture B to be below half a million on 13 April.

Q3	Suggested Solutions
(i)	
(ii)	

Q4	Suggested Solutions
(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\cos 2x}\right)=-2\mathrm{e}^{\cos 2x}\sin 2x$
(ii)	$\int \mathrm{e}^{\cos 2x}\sin 4x\ \mathrm{d}x=-\int\left(-2\mathrm{e}^{\cos 2x}\sin 2x\right)\cos 2x\ \mathrm{d}x$ $=-\mathrm{e}^{\cos 2x}\cos 2x+\int \mathrm{e}^{\cos 2x}\left(-2\sin 2x\right)\mathrm{d}x$ $=-\mathrm{e}^{\cos 2x}\cos 2x+\mathrm{e}^{\cos 2x}+c$ $=\mathrm{e}^{\cos 2x}\left(1-\cos 2x\right)+c$
(iii)	$\int \mathrm{e}^{\cos 2x}\left(\cos 3x\sin x\right)\ \mathrm{d}x$ $=\frac{1}{2}\int \mathrm{e}^{\cos 2x}\left(\sin 4x-\sin 2x\right)\ \mathrm{d}x$ $=\frac{1}{2}\int \mathrm{e}^{\cos 2x}\left(\sin 4x\right)\ \mathrm{d}x+\frac{1}{2}\int \mathrm{e}^{\cos 2x}\left(-\sin 2x\right)\ \mathrm{d}x$ $=\frac{1}{2}\mathrm{e}^{\cos 2x}\left(1-\cos 2x\right)+\frac{1}{4}\mathrm{e}^{\cos 2x}+c$ $=\mathrm{e}^{\cos 2x}\left(\frac{3}{4}-\frac{1}{2}\cos 2x\right)+c$

Q5	**Suggested Solutions**
(i)	$\sin\left(\frac{\pi}{4}+\theta\right)=\frac{2}{AB}\Rightarrow AB=\frac{2}{\sin\left(\frac{\pi}{4}+\theta\right)}$ $\cos\left(\frac{\pi}{4}+\theta\right)=\frac{2}{BC}\Rightarrow BC=\frac{2}{\cos\left(\frac{\pi}{4}+\theta\right)}$ OR $\sin\left(\frac{\pi}{4}-\theta\right)=\frac{2}{BC}$ $BC=\frac{2}{\sin\left(\frac{\pi}{4}-\theta\right)}=\frac{2}{\cos\left(\frac{\pi}{2}-\left(\frac{\pi}{4}-\theta\right)\right)}=\frac{2}{\cos\left(\frac{\pi}{4}+\theta\right)}$ $AC=AB+BC$ $=\frac{2}{\sin\left(\frac{\pi}{4}+\theta\right)}+\frac{2}{\cos\left(\frac{\pi}{4}+\theta\right)}$ $=2\left(\frac{1}{\sin\left(\frac{\pi}{4}+\theta\right)}+\frac{1}{\cos\left(\frac{\pi}{4}+\theta\right)}\right)$

Q5	Suggested Solutions
(ii)	$AC=2\left(\dfrac{1}{\sin\left(\dfrac{\pi}{4}+\theta\right)}+\dfrac{1}{\cos\left(\dfrac{\pi}{4}+\theta\right)}\right)$ $=2\left(\dfrac{1}{\sin\dfrac{\pi}{4}\cos\theta+\cos\dfrac{\pi}{4}\sin\theta}+\dfrac{1}{\cos\dfrac{\pi}{4}\cos\theta-\sin\dfrac{\pi}{4}\sin\theta}\right)$ $=2\sqrt{2}\left(\dfrac{1}{\cos\theta+\sin\theta}+\dfrac{1}{\cos\theta-\sin\theta}\right)$ $\approx 2\sqrt{2}\left(\dfrac{1}{1-\dfrac{\theta^2}{2}+\theta}+\dfrac{1}{1-\dfrac{\theta^2}{2}-\theta}\right)$ $=2\sqrt{2}\left[\left(1+\theta-\dfrac{\theta^2}{2}\right)^{-1}+\left(1-\theta-\dfrac{\theta^2}{2}\right)^{-1}\right]$ $=2\sqrt{2}\left\{\left[1+(-1)\left(\theta-\dfrac{\theta^2}{2}\right)+\dfrac{(-1)(-2)}{2}\left(\theta-\dfrac{\theta^2}{2}\right)^2+\ldots\right]\right.$ $\left.+\left[1+(-1)\left(-\theta-\dfrac{\theta^2}{2}\right)+\dfrac{(-1)(-2)}{2}\left(-\theta-\dfrac{\theta^2}{2}\right)^2+\ldots\right]\right\}$ $=2\sqrt{2}\left[\left(1-\theta+\dfrac{\theta^2}{2}+\theta^2+\ldots\right)+\left(1+\theta+\dfrac{\theta^2}{2}+\theta^2+\ldots\right)\right]$ $\approx 4\sqrt{2}+6\sqrt{2}\theta^2$

Q5	Suggested Solutions
	Alternative Method (Cosine double angle formula) $AC = 2\left(\dfrac{1}{\sin\left(\dfrac{\pi}{4}+\theta\right)}+\dfrac{1}{\cos\left(\dfrac{\pi}{4}+\theta\right)}\right)$ $= 2\left(\dfrac{1}{\sin\dfrac{\pi}{4}\cos\theta+\cos\dfrac{\pi}{4}\sin\theta}+\dfrac{1}{\cos\dfrac{\pi}{4}\cos\theta-\sin\dfrac{\pi}{4}\sin\theta}\right)$ $= 2\sqrt{2}\left(\dfrac{1}{\cos\theta+\sin\theta}+\dfrac{1}{\cos\theta-\sin\theta}\right)$ $= 2\sqrt{2}\left(\dfrac{2\cos\theta}{\cos^2\theta+\sin^2\theta}\right)$ $= 2\sqrt{2}\left(\dfrac{2\cos\theta}{\cos 2\theta}\right)$ $\approx 4\sqrt{2}\left(\dfrac{1-\dfrac{\theta^2}{2}}{1-\dfrac{(2\theta)^2}{2}}\right)$ $= 4\sqrt{2}\left(1-\dfrac{\theta^2}{2}\right)\left(1-2\theta^2\right)^{-1}$ $= 4\sqrt{2}\left(1-\dfrac{\theta^2}{2}\right)\left(1+2\theta^2+\ldots\right)$ $= 4\sqrt{2}\left(1+\dfrac{3}{2}\theta^2+\ldots\right)$ $\approx 4\sqrt{2}+6\sqrt{2}\theta^2$

Q6	Suggested Solutions

(i)

$$x=\tan\theta \Rightarrow \frac{dx}{d\theta}=\sec^2\theta$$

When $x=0$, $\tan\theta=0 \quad\Rightarrow\quad \theta=0$

When $x=1$, $\tan\theta=1 \quad\Rightarrow\quad \theta=\frac{\pi}{4}$

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx &= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta \\ &= \int_0^{\frac{\pi}{4}} \sec\theta\,d\theta \\ &= \Big[\ln\left|\sec\theta+\tan\theta\right|\Big]_0^{\frac{\pi}{4}} \\ &= \ln\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\ln\left(\sec 0+\tan 0\right) \\ &= \ln\left(\sqrt{2}+1\right)-\ln\left(1+0\right) \\ &= \ln\left(\sqrt{2}+1\right)\end{aligned}$$

(ii)

$$A=\frac{1}{n}\left[\frac{1}{\sqrt{1+\left(\frac{1}{n}\right)^2}}+\frac{1}{\sqrt{1+\left(\frac{2}{n}\right)^2}}+\ldots+\frac{1}{\sqrt{1+\left(\frac{n-1}{n}\right)^2}}+\frac{1}{\sqrt{1+(1)^2}}\right]$$

$$=\frac{1}{\sqrt{n^2}}\left[\frac{1}{\sqrt{1+\left(\frac{1}{n}\right)^2}}+\frac{1}{\sqrt{1+\left(\frac{2}{n}\right)^2}}+\ldots+\frac{1}{\sqrt{1+\left(\frac{n-1}{n}\right)^2}}+\frac{1}{\sqrt{1-(1)^2}}\right]$$

$$=\frac{1}{\sqrt{n^2\left(1+\left(\frac{1}{n}\right)^2\right)}}+\frac{1}{\sqrt{n^2\left(1+\left(\frac{2}{n}\right)^2\right)}}+\ldots+\frac{1}{\sqrt{n^2\left(1+\left(\frac{n-1}{n}\right)^2\right)}}+\frac{1}{\sqrt{n^2\left(1+(1)^2\right)}}$$

$$=\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\ldots+\frac{1}{\sqrt{n^2+(n-1)^2}}+\frac{1}{\sqrt{n^2+n^2}}$$

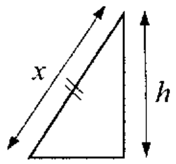
As $n\to\infty$, $A\to\ln\left(\sqrt{2}+1\right)$.

Q7	Suggested Solutions
(i)	Let d be the common difference. $a+(9-1)d=b$ $d=\frac{b-a}{8}$ $u_3=a+\frac{2(b-a)}{8}=\frac{3a+b}{4}$ $u_3+u_5+u_7$ $=\left(a+\frac{2(b-a)}{8}\right)+\left(a+\frac{4(b-a)}{8}\right)+\left(a+\frac{6(b-a)}{8}\right)$ $=3a+\frac{12(b-a)}{8}$ $=\frac{3}{2}(b+a)$
(ii)	$\frac{u_3}{a}=\frac{b}{u_3}$ $ab=(u_3)^2$ $=(a+2d)^2$ $=\left(a+\frac{b-a}{4}\right)^2$ $=\left(\frac{3a+b}{4}\right)^2$ $=\frac{9a^2+6ab+b^2}{16}$ $\frac{9a^2+6ab+b^2}{16}-ab=0$ $\frac{9a^2+6ab+b^2-16ab}{16}=0$ $9a^2-10ab+b^2=0$ $(9a-b)(a-b)=0$ Since the arithmetic progression is strictly increasing, $b\neq a$. Hence $b=9a$.

Q7	Suggested Solutions
(iii)	$\ln(u_5)-\ln(u_3)=\ln\left(\dfrac{a+\dfrac{4(9a-a)}{8}}{a+\dfrac{2(9a-a)}{8}}\right)$ $=\ln\left(\dfrac{a+4a}{a+2a}\right)$ $=\ln\left(\dfrac{5}{3}\right)$ $\ln(u_7)-\ln(u_5)=\ln\left(\dfrac{a+\dfrac{6(9a-a)}{8}}{a+\dfrac{4(9a-a)}{8}}\right)$ $=\ln\left(\dfrac{a+6a}{a+4a}\right)$ $=\ln\left(\dfrac{7}{5}\right)$ Since $\ln(u_7)-\ln(u_5)\neq\ln(u_5)-\ln(u_3)$, the terms are not consecutive terms of an arithmetic progression.

Q8	**Suggested Solutions**
(i)	$x^2+xy+ay^2=36$ $2x+x\frac{dy}{dx}+y+2ay\frac{dy}{dx}=0$ $(x+2ay)\frac{dy}{dx}=-y-2x$ $\frac{dy}{dx}=-\frac{y+2x}{x+2ay}$ For the normal to be parallel to y-axis, the tangent will be parallel to the x-axis. Hence $\frac{dy}{dx}=0$. Therefore, $y=-2x$. Substituting $y=-2x$: $x^2+xy+ay^2=36$ $x^2+x(-2x)+a(-2x)^2=36$ $x^2(4a-1)=36$ $x^2=\frac{36}{4a-1}$ $x=\frac{6}{\sqrt{4a-1}}$ or $-\frac{6}{\sqrt{4a-1}}$
(ii)	When $x=-2$, $(-2)^2-2y+\frac{5}{2}y^2=36$ $\frac{5}{2}y^2-2y-32=0$ $y=4$ or -3.2 by GC $x^2+xy+\frac{5}{2}y^2=36$ $x^2+xy+\left(\frac{5}{2}y^2-36\right)=0$ $x=\frac{-y\pm\sqrt{y^2-4(1)(2.5y^2-36)}}{2}$ $=\frac{-y\pm\sqrt{144-9y^2}}{2}$ Since R is in the region where $x\le-\frac{y}{2}$, $x=\frac{-y-\sqrt{144-9y^2}}{2}$

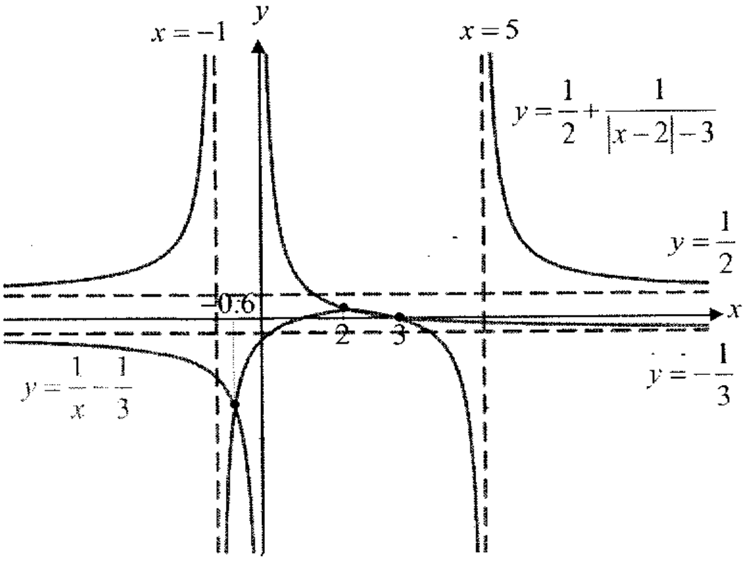
Q8	Suggested Solutions
	Required volume $=\pi\int_{-3.2}^{4}\left[\left(\frac{-y-\sqrt{144-9y^2}}{2}\right)^2-(-2)^2\right]\mathrm{d}y$ $=542.8672117$ $=542.87$ (2 d.p.)

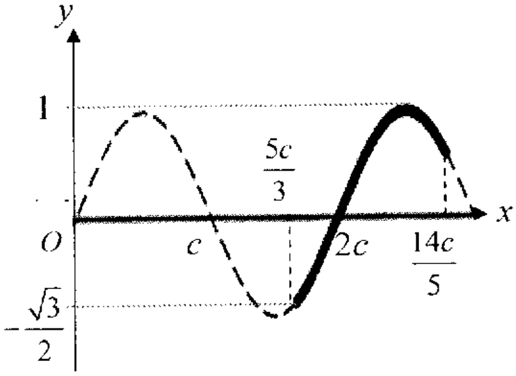
Q9	Suggested Solutions
(i)	In Figure 1, considering the breadth of the rectangle, Base of the isosceles triangle $= a - 2x$ and consider the triangle on the right and half the breadth of the rectangle, we have $x^2 - h^2 = \left(\frac{a}{2} - x\right)^2$ $= \frac{a^2}{4} - ax + x^2$ $x = \frac{a}{4} + \frac{h^2}{a}$ 
	Consider the length of the rectangle. We have $2a = y + 2h \quad \Rightarrow \quad y = 2a - 2h$ $V = (\text{Area of isosceles triangle}) \times y$ $= \frac{1}{2}(h)(a-2x)(y)$ $= \frac{1}{2}(h)\left[a - 2\left(\frac{a}{4} + \frac{h^2}{a}\right)\right](2a - 2h)$ $= \left(\frac{ah}{2} - \frac{2h^3}{a}\right)(a - h)$ $= \frac{a^2}{2}h - \frac{a}{2}h^2 - 2h^3 + \frac{2}{a}h^4$ $\frac{dV}{dh} = \frac{a^2}{2} - ah - 6h^2 + \frac{8}{a}h^3$ For stationary V, $\frac{dV}{dh} = 0$. $\frac{a^2}{2} - ah - 6h^2 + \frac{8}{a}h^3 = 0$ $16h^3 - 12ah^2 - 2a^2h + a^3 = 0$ $-16h^3 + 12ah^2 + 2a^2h - a^3 = 0$
(ii)	If $a = 5$ $-16h^3 + 12ah^2 + 2a^2h - a^3 = 0$ $h = 4.0451, -1.5451$ or 1.25 Since $h > 0$, we reject $h = -1.5451$. Also if $h = 4.0451$, $x \approx \frac{5}{4} + \frac{4.0451^2}{5} \approx 4.522566$ and $2x \approx 9.04513 > 5 = a$. Thus the only possible value of h is 1.25.

Q9	Suggested Solutions
	$\frac{dV}{dh}=\frac{a^2}{2}-ah-6h^2+\frac{8}{a}h^3$ $\frac{d^2V}{dh^2}=-a-12h+\frac{24}{a}h^2$ When $a=5$ and $h=1.25$, $\frac{d^2V}{dh^2}=-5-12(1.25)+\frac{24}{5}(1.25)^2$ $=-12.5<0$ Thus, V is maximum at $h=1.25$

Q10	Suggested Solutions
(i)	$\frac{dx}{dt}=-kx$, where k is a positive constant
(ii)	$\frac{dx}{dt}=-kx$ $\frac{1}{x}\frac{dx}{dt}=-k$ $\int\frac{1}{x}\,dx=\int -k\,dt$ $\ln\lvert x\rvert=-kt+C$ $\lvert x\rvert=e^{-kt+C}$ $x=\pm e^{-kt}.e^{C}$ $x=Ae^{-kt}$, where $A=\pm e^{C}$ When $t=0$, $x=x_0$, $A=x_0$. When $t=6$, $x=\frac{x_0}{1000}$. $\frac{x_0}{1000}=x_0e^{-6k}$ $e^{-6k}=\frac{1}{1000}$ $k=\frac{\ln 1000}{6}=\frac{1}{2}\ln 10=\ln\sqrt{10}$ $x=x_0e^{\left(-\ln\sqrt{10}\right)t}$ $=x_0e^{\left(\ln\frac{1}{\sqrt{10}}\right)t}$ $=\frac{x_0}{\left(\sqrt{10}\right)^t}$
	$\frac{x_0}{\left(\sqrt{10}\right)^t}=\frac{1}{4}x_0$ $\left(\sqrt{10}\right)^t=4$ $\ln\left(\sqrt{10}\right)^t=\ln 4$ $t\left(\ln\sqrt{10}\right)=\ln 4$ $t=\frac{1}{\left(\ln\sqrt{10}\right)}\ln 4=1.20$ hours (3 s.f.)

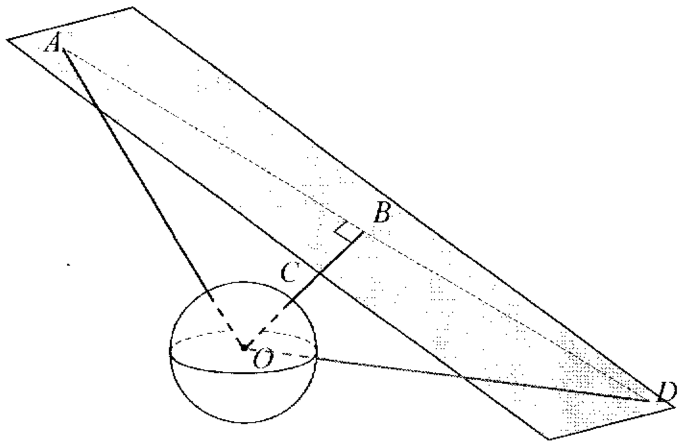
Q10	Suggested Solutions
(iii)	From part **(i)**, we have $x=\frac{x_0}{\left(\sqrt{10}\right)^t}$. Let the time from 2nd dose be R. Then $R=t-T$. Just after 2nd dose, amount of antibiotics in patient's body is $x_0+\frac{x_0}{\left(\sqrt{10}\right)^T}$. Replace initial dose '$x_o$' with '$x_0+\frac{x_0}{\left(\sqrt{10}\right)^T}$'. Therefore, we have the amount of antibiotics in the patient's body after the second dose $\frac{\text{amount of antibiotics in body just after 2nd dose}}{\left(\sqrt{10}\right)^R}$ $=\frac{x_0+\frac{x_0}{\left(\sqrt{10}\right)^T}}{\left(\sqrt{10}\right)^{t-T}}$ and before the third dose is $=\frac{x_0}{\left(\sqrt{10}\right)^{t-T}}+\frac{x_0}{\left(\sqrt{10}\right)^{t-T+T}}$ $=x_0\left(10^{-\frac{1}{2}}\right)^{t-T}+x_0\left(10^{-\frac{1}{2}}\right)^t$ $=x_0\left(10^{\frac{1}{2}(T-t)}+10^{\left(-\frac{t}{2}\right)}\right)$

Q11	Suggested Solutions
(i)	We sketch the curves $y=\frac{1}{2}+\frac{1}{\lvert x-2\rvert-3}$ and $y=\frac{1}{x}-\frac{1}{3}$.  From graph, for $\frac{1}{2}+\frac{1}{\lvert x-2\rvert-3}\geq\frac{1}{x}-\frac{1}{3}$, $x<-1$ or $-0.6\leq x<0$ or $2\leq x\leq 3$ or $x>5$.
(ii)	Since $x<2$ (given domain is $-1<x<1$), $\frac{1}{2}+\frac{1}{\lvert x-2\rvert-3}=\frac{1}{2}+\frac{1}{-(x-2)-3}=\frac{1}{2}-\frac{1}{x+1}$ $y=\frac{1}{2}-\frac{1}{x+1}$ $\frac{1}{x+1}=\frac{1}{2}-y$ $\frac{1}{x+1}=\frac{1-2y}{2}$ $x+1=\frac{2}{1-2y}$ $x=\frac{2}{1-2y}-1$ $f^{-1}(x)=\frac{2}{1-2x}-1$ Consider graph of $f(x)=\frac{1}{2}-\frac{1}{x+1}$ for $-1<x<1$ in part **(i)**. $D_{f^{-1}}=R_f=(-\infty,0)$

Q11	Suggested Solutions
(iii)	$\mathrm{g}(x)=\sin\left(\frac{\pi x}{c}\right),\ x\in\mathbb{R},\ \frac{5c}{3}\le x<\frac{14c}{5}$ Period of g = $\frac{2\pi}{\pi/c}=2c$  When $x=\frac{5c}{3}$, $\sin\left(\frac{\pi x}{c}\right)=\sin\left(\frac{5\pi}{3}\right)=\sin\left(-\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$ $\mathrm{R_g}=\left[-\frac{\sqrt{3}}{2},1\right]$
(iv)	Let $(\mathrm{fh})^{-1}\left(-\frac{1}{2}\right)=x$ $\mathrm{fh}\left((\mathrm{fh})^{-1}\left(-\frac{1}{2}\right)\right)=\mathrm{fh}(x)$ $-\frac{1}{2}=\mathrm{fh}(x)$ $\mathrm{f}^{-1}\left(-\frac{1}{2}\right)=\mathrm{f}^{-1}\mathrm{fh}(x)$ $\mathrm{f}^{-1}\left(-\frac{1}{2}\right)=\mathrm{h}(x)$ $\mathrm{h}(x)=\frac{2}{1-2(-0.5)}-1=0$ $\sin\left(\frac{\pi x}{c}\right)=0$ $\frac{\pi x}{c}=0\pm 2k\pi$ where $k\in\mathbb{Z}$ Since $\frac{5c}{3}\le x<\frac{7c}{3}$, therefore $\frac{5\pi}{3}\le\frac{\pi x}{c}<\frac{7\pi}{3}$. $\frac{\pi x}{c}=2\pi$ $x=2c$

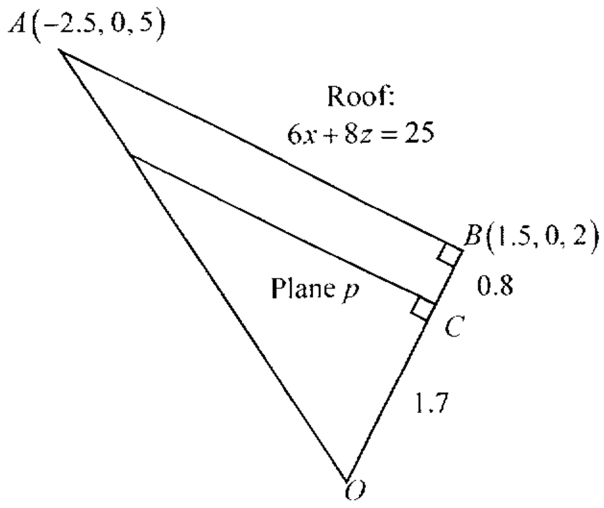
Q12	Suggested Solutions
(a)	$l_1: \mathbf{r} = \begin{pmatrix} 10 \\ 8 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 14 \\ h \end{pmatrix}, \lambda \in \mathbb{R}$ $l_2: \mathbf{r} = \begin{pmatrix} s \\ -10 \\ 12 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}, \mu \in \mathbb{R}$ l_1 and l_2 are perpendicular: $\begin{pmatrix} 1 \\ 14 \\ h \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix} = 0$ $2 + 28 - 5h = 0$ $h = 6$ Suppose l_1 and l_2 intersect, $\begin{pmatrix} 10+\lambda \\ 8+14\lambda \\ 8+6\lambda \end{pmatrix} = \begin{pmatrix} s+2\mu \\ -10+2\mu \\ 12-5\mu \end{pmatrix}$ $\lambda \quad -2\mu \quad -s = -10$ $14\lambda \quad -2\mu \quad = -18$ $6\lambda \quad +5\mu \quad = 4$ By GC, $s = 5$ Since the two lines do not intersect, $s \neq 5$. Hence $h = 6$ and $s \in \mathbb{R}, s \neq 5$.
(b)(i)	Given: OB is the 'shortest possible cord from the centre of the ball to the roof'. This implies that line $OB \perp$ plane.

Q12	Suggested Solutions
	$\overrightarrow{OB}$ = projection of $\overrightarrow{OA}$ onto the normal of the roof $=\left[\begin{pmatrix}-2.5\\0\\5\end{pmatrix}\cdot\frac{1}{\sqrt{6^2+8^2}}\begin{pmatrix}6\\0\\8\end{pmatrix}\right]\frac{1}{\sqrt{6^2+8^2}}\begin{pmatrix}6\\0\\8\end{pmatrix}$ $=\frac{1}{6^2+8^2}\left[\begin{pmatrix}-2.5\\0\\5\end{pmatrix}\cdot\begin{pmatrix}6\\0\\8\end{pmatrix}\right]\begin{pmatrix}6\\0\\8\end{pmatrix}$ $=\frac{-15+40}{100}\begin{pmatrix}6\\0\\8\end{pmatrix}$ $=\begin{pmatrix}1.5\\0\\2\end{pmatrix}$ $\therefore B(1.5,0,2)$ **Alternative Method** $l_{OB}: \mathbf{r}=\lambda\begin{pmatrix}6\\0\\8\end{pmatrix}, \lambda\in\mathbb{R}$ Plane: $\mathbf{r}\cdot\begin{pmatrix}6\\0\\8\end{pmatrix}=25$ $\begin{pmatrix}6\lambda\\0\\8\lambda\end{pmatrix}\cdot\begin{pmatrix}6\\0\\8\end{pmatrix}=25$ $36\lambda+64\lambda=25$ $\lambda=0.25$ $\overrightarrow{OB}=0.25\begin{pmatrix}6\\0\\8\end{pmatrix}=\begin{pmatrix}1.5\\0\\2\end{pmatrix}$ $\therefore B(1.5,0,2)$

(b)(i)	Shortest distance from O to roof, OB $=\left\|\begin{pmatrix}1.5\\0\\2\end{pmatrix}\right\|$ $=\sqrt{(1.5)^2+2^2}$ $=2.5$ Since the radius of the ball is 1 unit. Shortest distance between surface of ball to roof $=2.5-1$ $=1.5$ metres
(b)(ii)	 Point B is the foot of perpendicular of A onto line OB. $\overrightarrow{OB}=\dfrac{\overrightarrow{OA}+\overrightarrow{OD}}{2}$ $\overrightarrow{OD}=2\overrightarrow{OB}-\overrightarrow{OA}$ $=2\begin{pmatrix}1.5\\0\\2\end{pmatrix}-\begin{pmatrix}-2.5\\0\\5\end{pmatrix}$ $=\begin{pmatrix}5.5\\0\\-1\end{pmatrix}$ $l_{OD}:\mathbf{r}=\mu\begin{pmatrix}5.5\\0\\-1\end{pmatrix},\mu\in\mathbb{R}$

(b) (iii)

Side view



Let C be the point on the ball that is nearest to the roof.

$$\overrightarrow{OC} = 1.7\frac{\overrightarrow{OB}}{\left|\overrightarrow{OB}\right|} = 1.7\frac{\begin{pmatrix}1.5\\0\\2\end{pmatrix}}{\sqrt{1.5^2+2^2}} = 0.68\begin{pmatrix}1.5\\0\\2\end{pmatrix} = \begin{pmatrix}1.02\\0\\1.36\end{pmatrix}$$

$$\begin{pmatrix}1.02\\0\\1.36\end{pmatrix}\cdot\begin{pmatrix}6\\0\\8\end{pmatrix} = 17$$

Equation of plane p is $6x+8z=17$.

Alternative Method

For any plane with equation $\mathbf{r}\cdot\mathbf{n} = D$:

Shortest distance from O to plane $= \dfrac{|D|}{|\mathbf{n}|}$

Shortest distance from O to plane p $= 2.5-0.8 = 1.7$

Since plane p is parallel to the roof, $\mathbf{n} \,//\, \begin{pmatrix}6\\0\\8\end{pmatrix}$.

Therefore,

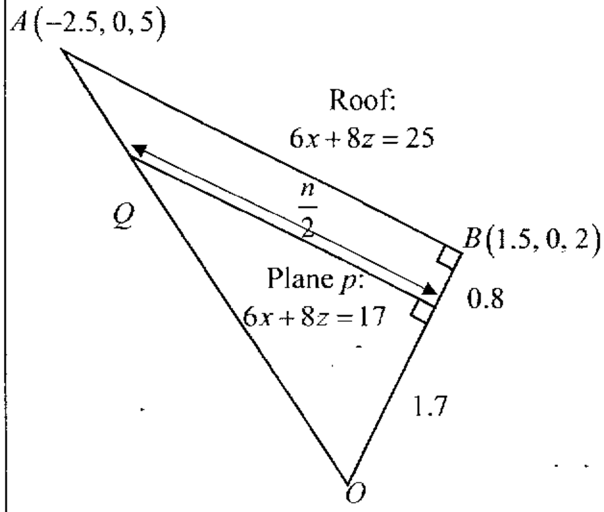
$$1.7 = \frac{|D|}{\sqrt{6^2+8^2}} \Rightarrow |D| = 17$$

Since plane p are the roof are on the same side of O, $D=17$ (same sign as '25' from equation of the roof).

Equation of plane p is $6x+8z=17$.

(b) (iv)

Side view



$AB=\sqrt{(1.5+2.5)^2+(2-5)^2}=5$

By similar triangles,

$$\frac{n/2}{5}=\frac{1.7}{1.7+0.8}$$

$$n=6.8$$

Therefore, largest **integer** n is 6.

Alternative Method

Let Q be the point of intersection between plane p and OA.

$$\alpha\begin{pmatrix}-2.5\\0\\5\end{pmatrix}\cdot\begin{pmatrix}6\\0\\8\end{pmatrix}=17$$

$$\alpha(-15+40)=17$$

$$\alpha=\frac{17}{25}$$

$$\overrightarrow{OQ}=\frac{17}{25}\begin{pmatrix}-2.5\\0\\5\end{pmatrix}=\begin{pmatrix}-1.7\\0\\3.4\end{pmatrix}$$

$$OQ=\sqrt{(-1.7)^2+(3.4)^2}=\sqrt{14.45}$$

By Pythagoras Theorem,

$$OQ^2=\left(\frac{n}{2}\right)^2+(1.7)^2$$

$$\frac{n^2}{4}=14.45-(1.7)^2$$

$$n^2=46.24$$

$$n=6.8 \text{ since } n>0$$

Therefore, largest **integer** n is 6.

2021 NJC H2 Math Prelim P2

Section A: Pure Mathematics [40 marks]

1 A curve C has parametric equations

$$x = t^3 - 12t, \quad y = t - 2 \quad \text{for } t \le 2.$$

(i) Sketch C, labelling the coordinates of any end points. [2]

A line l has equation $y = m(x+16)$, where m is positive. It is given that l intersects C at the points where $t = 2$ and $t = k$, where $k \le -4$.

(ii) Show that the area of the region bounded by C and l is $6k^2 - \dfrac{k^4}{4} - 16k + 12 + \dfrac{(k-2)^2}{2m}$. [5]

2 An Art teacher teaches her students to create patterns using squares of different sizes. One possible pattern is to begin with the first square with sides of length 2 mm. The first square is inscribed in the second square, where the corners of the first square coincide with the midpoints of the second square. She continues inscribing squares in this manner where the n^{th} square is inscribed in the $(n+1)^{\text{th}}$ square. **Figure 1** shows a piece of artwork after 4 squares are drawn.

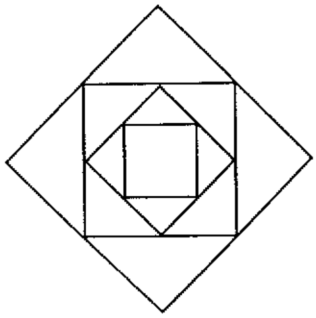


Figure 1

By using this pattern, Student A begins his artwork.

(i) Find, in terms of n, the length of the sides of the n^{th} square. [2]

(ii) A standard A4 paper measures 210 mm by 297 mm. Find the maximum number of complete squares that he can draw on the paper. [2]

Student B uses a giant drawing board and decides to make his artwork more eye-catching. He uses the same pattern and measurements as Student A, but he shades the 1^{st} square and also shades on any protruding areas covered by the 4^{th}, 7^{th} ,..., $(3N+1)^{\text{th}}$ squares, where N is a non-negative integer. A protruding area is defined by the region bounded by the newly drawn square and the square immediately preceding it. **Figure 2** shows a piece of artwork if he draws 4 squares.

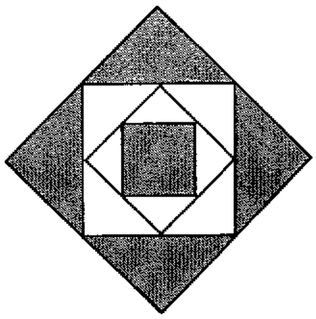


Figure 2

(iii) Find, in mm^2, the total shaded area as shown in **Figure 2.** [2]

(iv) Hence or otherwise, find the total shaded area if he draws 30 squares. Give your answer in m^2. [3]

3 Referred to the origin O, three distinct and non-collinear points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Point L is the mid-point of BC. The position vector of a point P is given by $(1-k)\mathbf{a}+\frac{k}{2}(\mathbf{b}+\mathbf{c})$, where k is a non-zero constant and $k \neq 1$.

(i) Show that A, L and P are collinear. [3]

(ii) Show that $\frac{1}{2}\left|\overrightarrow{CP}\times\overrightarrow{CB}\right| = \frac{|1-k|}{2}\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}\right|$. [3]

For the rest of the question, let $k=\frac{1}{2}$.

Let point Q be a point on the line passing through A and L. P and Q are distinct points and the areas of triangle CPB and triangle CQB are equal.

(iii) By considering part **(ii)**, find the position vector of Q in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. [4]

(iv) Given that $\left|\overrightarrow{BC}\right|=1$, interpret geometrically $\left|\overrightarrow{LP}\cdot\overrightarrow{BC}\right|$. [1]

4 **(a)** A quartic equation

$$\mathrm{i}z^4+(-3-7\mathrm{i})z^3+(21+17\mathrm{i})z^2+(-51-15\mathrm{i})z+45=0$$

has 4 distinct roots, z_1, z_2, z_3 and z_4 which are represented by points A, B, C and D respectively. It is given that $z_1=-3\mathrm{i}$, $z_3=3$ and $\mathrm{Im}(z_4)>0$.

(i) Find z_2 and z_4. [3]

(ii) Sketch the points A, B, C and D on an Argand diagram. [2]

(iii) Point E represents the complex number wz_3 such that $ABDE$ forms a parallelogram. Find w in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r>0$ and $0\le\theta<2\pi$. [2]

(b) **Do not use a graphing calculator in answering this question.**

Express $\dfrac{(-4-4\mathrm{i})^5}{\left(-2\sqrt{3}+2\mathrm{i}\right)^7}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r>0$ and $-\pi<\theta\le\pi$. [3]

(c) **Do not use a graphing calculator in answering this question.**

Given that $q=1-\mathrm{i}$. Find the three smallest positive integers n for which $\left(\mathrm{i}q^n\right)^*$ is real and positive. [3]

Section B: Probability and Statistics [60 marks]

5 Twelve books, consisting of 5 identical Geography books, 4 identical Mathematics books and 3 identical Literature books, are arranged on a bookshelf that has a top rack and a bottom rack. Six books are chosen and arranged on each rack. Let

A be the event that **all** the Mathematics books are together,
B be the event that **all** the Literature books are on the same rack and separated.

(i) Find the number of ways to arrange the books if A and B occur. [2]

(ii) Find the number of ways to arrange the books if at least one Mathematics book is on the top rack. [4]

6 **(a)** For events F and G, it is given that $\mathrm{P}(F)=\frac{2}{5}$ and $\mathrm{P}(G)=\frac{2}{3}$. Find the greatest and least possible values of $\mathrm{P}(F'\cap G)$. [3]

(b) For events A, B and C, it is given that $\mathrm{P}(A)=\frac{3}{8}$, $\mathrm{P}(B)=\frac{2}{3}$, $\mathrm{P}(C)=\frac{5}{8}$, $\mathrm{P}(A\cap C)=\frac{1}{3}$ and $\mathrm{P}(A\cup B\cup C)=\frac{3}{4}$. It is also given that events A and B are independent, and that events B and C are independent.

(i) Find $\mathrm{P}(A'\cap B' \mid C')$. [3]

(ii) Find the exact value of $\mathrm{P}(A\cap B\cap C)$. [3]

7 The medical director of a hospital knows that the mean systolic blood pressure of patients who suffer from high blood pressure is 140 mmHg. He wishes to carry out a clinical trial to evaluate whether a new drug is effective in reducing the systolic blood pressure of patients who suffer from high blood pressure. The systolic blood pressure, x mmHg, of a random sample of 60 patients are summarized as follows.

$$\sum(x-140)=-37.6 \qquad \sum(x-140)^2=1012.17$$

(i) Calculate unbiased estimates of the population mean and variance of the systolic blood pressure of patients who suffer from high blood pressure [2]

(ii) Carry out the test, at 5% level of significance, for the medical director. You should state your hypotheses and define any symbols that you use. [5]

(iii) Upon closer inspection of the data of the sample of 60 patients, the director noted that the value of $\sum(x-140)$ is correct but the value of $\sum(x-140)^2$ should be larger instead. If a new test is carried out using this information at the 5% level of significance, explain whether the result of this test will differ from the result of the test in part **(ii)**. [3]

8 National Fruit Company owns a large tomato farm. The tomatoes produced are harvested and sold in boxes of 25. It is known that $100p\%$ of the tomatoes are rotten. For these boxes, the mean number of rotten tomatoes in a box is 1.

(i) Explain why the context above may not be well-modelled by a binomial distribution. [1]

Assume now that the context above is well-modelled by a binomial distribution.

(ii) State the value of p. [1]

(iii) Find the probability that a box chosen at random has less than 2 rotten tomato. [2]

(iv) A customer chose a box and inspected the contents individually. Find the probability that the twenty-first tomato is the fourth rotten tomato and no rotten tomatoes are found subsequently. [3]

Boxes that contains at least 24 tomatoes that are not rotten are deemed satisfactory.

(v) A customer first picks 3 boxes of tomatoes, of which at least 2 boxes are satisfactory. The customer then decides to buy another 5 boxes. Find the probability that exactly 6 of the 8 boxes are satisfactory. [4]

9 A shop sells two models of ovens produced by Factory A and Factory B. The lifespans of ovens produced by Factory A have the normal distribution with mean 13 years and standard deviation 6 months, while the lifespans of ovens produced by Factory B have the normal distribution with mean 15 years and standard deviation k months. The lifespan of any oven is independent of one another.

(i) Given that 90% of the ovens produced by Factory B exceeds a lifespan of 14 years. Show that $k = 9.3636$, correct to 5 significant figures. [3]

(ii) Find the probability that the lifespan of a randomly chosen oven produced by Factory B exceeds the lifespan of a randomly chosen oven produced by Factory A by less than 3 years. [2]

(iii) There is a probability of at least 0.4 that the lifespan of a randomly chosen oven produced by Factory A is within n years of 13 years. Find the least value of n, correct to 3 decimal places. [3]

(iv) Every oven produced by Factory A is wrapped in a box. A carton contains 20 of such boxes. If there are at least 3 ovens in a carton with lifespans of less than 12 years, the carton will be rejected. Find the probability that a carton is rejected. [3]

10 A circular card is divided into 3 sectors with values 0, 1, 2 and having angles $180°, (360p)°, (360q)°$ respectively where p and q are non-zero constants. The card has a pointer pivoted at its centre. After being set in motion, the pointer comes to rest randomly in one of the sectors.

In a game, a player gets to spin the pointer twice. The player's score is denoted by X. The player's score is

- the greater of the two values if the values shown on both spins are different.
- the sum of the two values obtained if the values shown on both spins are equal.

(i) Show that the probability that a player's score in a game is 2 is $0.5 - p^2$. [3]

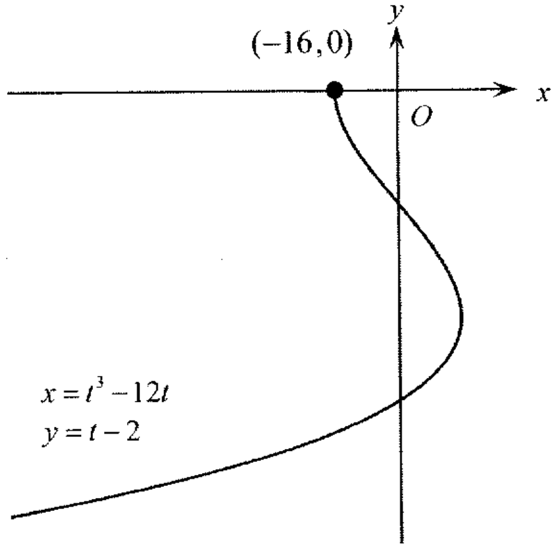
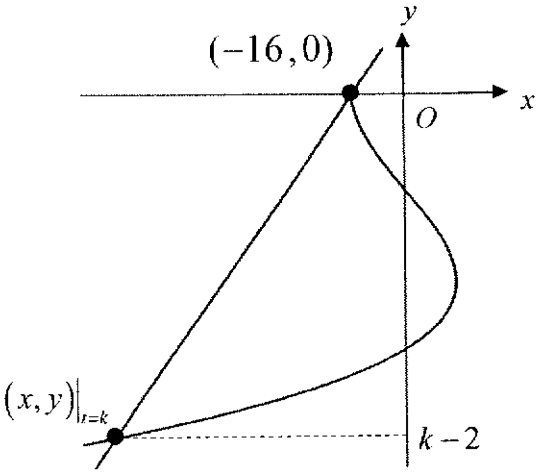
(ii) Find, in terms of p, the probability distribution of X. [2]

(iii) Given that $\mathrm{E}(X) = \frac{11}{9}$, find the exact value of $\mathrm{Var}(X)$. [4]

(iv) Find the probability that a player's mean score in 50 games is less than 1.5. [2]

A player plays 3 games. Let
A be the event that a player's total score in the 3 games is more than 5.
B be the event that a player's score is at least 2 in each of the 3 games,

(v) Without doing any calculation, explain why $\mathrm{P}(B)$ is less than $\mathrm{P}(A)$. [1]

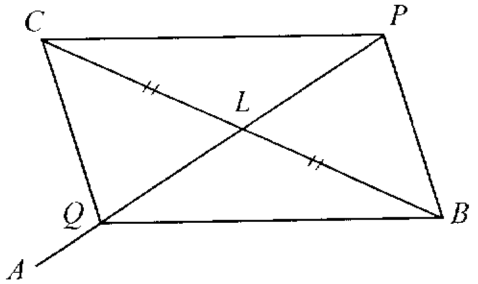
Q1	Suggested Solutions
(i)	
(ii)	 $y=m(x+16)\Rightarrow x=\dfrac{y}{m}-16$ $A=\int_{k-2}^{0}x\,\mathrm{d}y-\int_{k-2}^{0}\left(\dfrac{y}{m}-16\right)\mathrm{d}y$ $=\int_{k}^{2}\left(t^3-12t\right)(1\,\mathrm{d}t)-\left[\dfrac{y^2}{2m}-16y\right]_{k-2}^{0}$ $=\left[\dfrac{t^4}{4}-6t^2\right]_{k}^{2}-\left[0-\left(\dfrac{(k-2)^2}{2m}-16(k-2)\right)\right]$ $=\left[\left(\dfrac{16}{4}-24\right)-\left(\dfrac{k^4}{4}-6k^2\right)\right]+\left[\dfrac{(k-2)^2}{2m}-16(k-2)\right]$ $=6k^2-\dfrac{k^4}{4}-16k+12+\dfrac{(k-2)^2}{2m}$

Q2	Suggested Solutions
(i)	<table><tr><th>n</th><th>Length of n^{th} square</th></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>$\left(2^{\frac{1}{2}}\right)2 = 2^{\frac{3}{2}}$</td></tr><tr><td>3</td><td>$\left(2^{\frac{2}{2}}\right)2 = 2^{\frac{4}{2}}$</td></tr><tr><td>⋮</td><td>⋮</td></tr><tr><td>n</td><td>$2^{\frac{n+1}{2}}$</td></tr></table> The length of the n^{th} square is $2^{\frac{n+1}{2}}$ mm.
(ii)	$2^{\frac{n+1}{2}} < 210$ $\frac{n+1}{2} < \frac{\ln(210)}{\ln 2}$ $n < 14.428$ Hence maximum number of square is 14.
(iii)	From part **(i)**, length of the n^{th} square is $2^{\frac{n+1}{2}}$. Therefore, area of the n^{th} square $= \left(2^{\frac{n+1}{2}}\right)^2 = 2^{n+1}$. Area of the 1^{st} square $= 2^2$ Area of the 4^{th} square – Area of the 3^{rd} square $= 2^5 - 2^4$ $= 2^4(2-1)$ $= 2^4$ Total shaded area in Figure 2 $= 2^2 + 2^4 = 20$
(iv)	<table><tr><th>n</th><th>Area of n^{th} square</th><th>Protruding area of n^{th} square</th></tr><tr><td>1</td><td>2^2</td><td>2^2</td></tr><tr><td>2</td><td>2^3</td><td>$2^3 - 2^2 = 2^2(2-1) = 2^2$</td></tr><tr><td>3</td><td>2^4</td><td>$2^4 - 2^3 = 2^3(2-1) = 2^3$</td></tr><tr><td>4</td><td>2^5</td><td>2^4</td></tr><tr><td>⋮</td><td>⋮</td><td>⋮</td></tr><tr><td>7</td><td>2^8</td><td>2^7</td></tr><tr><td>⋮</td><td>⋮</td><td>⋮</td></tr><tr><td>28</td><td>2^{29}</td><td>2^{28}</td></tr></table> He will only shade up to the 28^{th} square if he draws 30 squares.

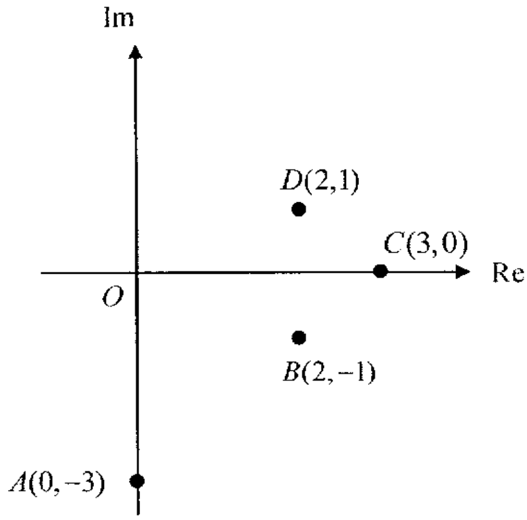
Q2	Suggested Solutions
	Total shaded area $= 2^2 + 2^4 + 2^7 + \ldots + 2^{28}$ $= 4 + \dfrac{2^4(2^{3(9)} - 1)}{(2^3 - 1)}$ $= 306,783,380 \text{ mm}^2$ $= 307 \text{ m}^2$ (3 s.f.).

Q3	Suggested Solutions
(i)	$\overrightarrow{AL}$ $=\overrightarrow{OL}-\overrightarrow{OA}$ $=\dfrac{\mathbf{b}+\mathbf{c}}{2}-\mathbf{a}$ $\overrightarrow{AP}$ $=\overrightarrow{OP}-\overrightarrow{OA}$ $=(1-k)\mathbf{a}+k\dfrac{\mathbf{b}+\mathbf{c}}{2}-\mathbf{a}$ $=k\left(\dfrac{\mathbf{b}+\mathbf{c}}{2}-\mathbf{a}\right)$ $=k\overrightarrow{AL}$ OR $\overrightarrow{PL}$ $=\overrightarrow{OL}-\overrightarrow{OP}$ $=\dfrac{\mathbf{b}+\mathbf{c}}{2}-(1-k)\mathbf{a}-k\dfrac{\mathbf{b}+\mathbf{c}}{2}$ $=(1-k)\left(\dfrac{\mathbf{b}+\mathbf{c}}{2}-\mathbf{a}\right)$ $=(1-k)\overrightarrow{AL}$ Since $\overrightarrow{AP}$ is **parallel** to $\overrightarrow{AL}$ with a **common point** L, A, L and P are collinear.
(ii)	$\dfrac{1}{2}\left\|\overrightarrow{CP}\times\overrightarrow{CB}\right\|$ $=\dfrac{1}{2}\left\|\left(\overrightarrow{OP}-\overrightarrow{OC}\right)\times\left(\overrightarrow{OB}-\overrightarrow{OC}\right)\right\|$ $=\dfrac{1}{2}\left\|\left((1-k)\mathbf{a}+k\dfrac{(\mathbf{b}+\mathbf{c})}{2}-\mathbf{c}\right)\times(\mathbf{b}-\mathbf{c})\right\|$ $=\dfrac{1}{2}\left\|(1-k)\mathbf{a}\times(\mathbf{b}-\mathbf{c})+\dfrac{k}{2}(\mathbf{b}+\mathbf{c})\times(\mathbf{b}-\mathbf{c})-\mathbf{c}\times(\mathbf{b}-\mathbf{c})\right\|$ $=\dfrac{1}{2}\left\|\begin{array}{l}(1-k)\mathbf{a}\times(\mathbf{b}-\mathbf{c})\\+\dfrac{k}{2}(\mathbf{b}\times\mathbf{b}-\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{b}-\mathbf{c}\times\mathbf{c})\\-\mathbf{c}\times\mathbf{b}+\mathbf{c}\times\mathbf{c}\end{array}\right\|$

Q3	Suggested Solutions
	$=\frac{1}{2}\left\| \begin{array}{l}(1-k)\mathbf{a}\times(\mathbf{b}-\mathbf{c}) \\ +\frac{k}{2}(\mathbf{0}-\mathbf{b}\times\mathbf{c}-\mathbf{b}\times\mathbf{c}-\mathbf{0}) \\ +\mathbf{b}\times\mathbf{c}+\mathbf{0}\end{array}\right\|$ $=\frac{1}{2}\left\|(1-k)\mathbf{a}\times(\mathbf{b}-\mathbf{c})+(1-k)(\mathbf{b}\times\mathbf{c})\right\|$ $=\frac{\|1-k\|}{2}\|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}\|$
(iii)	$k=\frac{1}{2}$ $\overrightarrow{OP}=(1-k)\mathbf{a}+\frac{k}{2}(\mathbf{b}+\mathbf{c})=\frac{1}{2}\mathbf{a}+\frac{1}{4}(\mathbf{b}+\mathbf{c})$ The points A, L, P and Q are collinear. Let l be the line passing through points A and L. $l:\mathbf{r}=(1-k)\mathbf{a}+\frac{k}{2}(\mathbf{b}+\mathbf{c})$ where $k\in\mathbb{R}, k\neq 1$ Since P and Q lie on the line passing through A and L, hence $\overrightarrow{OQ}=(1-\lambda)\mathbf{a}+\frac{\lambda}{2}(\mathbf{b}+\mathbf{c})$ for some real constant λ. Hence, $\overrightarrow{OQ}$ has the same form as $\overrightarrow{OP}$. This implies that area of triangle CQB $=\frac{\|1-\lambda\|}{2}\|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}\|$ by **(ii)**. Area of triangle CQB = Area of triangle CPB $\frac{\|1-\lambda\|}{2}\|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}\|=\frac{\|1-0.5\|}{2}\|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}\|$ $\frac{\|1-\lambda\|}{2}=\frac{\|1-0.5\|}{2}$ $\|1-\lambda\|=0.5$ $1-\lambda=0.5$ or -0.5 $\lambda=0.5$ (value of k that gives point P) or 1.5. $\overrightarrow{OQ}=(1-1.5)\mathbf{a}+\frac{1.5}{2}(\mathbf{b}+\mathbf{c})$ $=-0.5\mathbf{a}+0.75(\mathbf{b}+\mathbf{c})$

Q3	Suggested Solutions
	Alternative Method (Geometrical)  $CPQB$ is a parallelogram. $k=\frac{1}{2}$ $\overrightarrow{OP}=(1-k)\mathbf{a}+\frac{k}{2}(\mathbf{b}+\mathbf{c})=\frac{1}{2}\mathbf{a}+\frac{1}{4}(\mathbf{b}+\mathbf{c})$ $\overrightarrow{QC}=\overrightarrow{BP}$ $\overrightarrow{OC}-\overrightarrow{OQ}=\overrightarrow{OP}-\overrightarrow{OB}$ $\overrightarrow{OQ}=\overrightarrow{OB}+\overrightarrow{OC}-\overrightarrow{OP}$ $=\mathbf{b}+\mathbf{c}-\left[\frac{1}{2}\mathbf{a}+\frac{1}{4}(\mathbf{b}+\mathbf{c})\right]$ $=-0.5\mathbf{a}+0.75(\mathbf{b}+\mathbf{c})$
(iv)	$\left\|\overrightarrow{LP}\cdot\overrightarrow{BC}\right\|$ is the length of projection of $\overrightarrow{LP}$ onto $\overrightarrow{BC}$.

Note that the formula for length of projection of is $\|h\cdot\hat{a}\|$ where $\hat{a}$ is an unit vector. Hence this works because $\left\|\overrightarrow{BC}\right\|=1$.

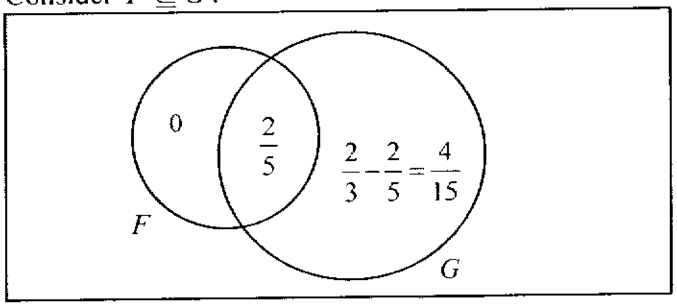
Q4	Suggested Solutions
(a) (i)	$\mathrm{i}z^4+(-3-7\mathrm{i})z^3+(21+17\mathrm{i})z^2+(-51-15\mathrm{i})z+45=0$ $(z-3)(z+3\mathrm{i})(\mathrm{i}z^2+az+b)=0$ $(z^2+(3\mathrm{i}-3)z-9\mathrm{i})(\mathrm{i}z^2+az+b)=0$ Note that the coefficients *a* and *b* may not be real. The conjugate root theorem also do not apply in this case because the equation is not one with all coefficients real. By comparing coefficient of constant term: $(-9\mathrm{i})(b)=45 \Rightarrow b=5\mathrm{i}$ *z* term: $-9a\mathrm{i}+b(3\mathrm{i}-3)=-51-15\mathrm{i}$ $-9a\mathrm{i}-15-15\mathrm{i}=-51-15\mathrm{i}$ $-9a\mathrm{i}=-36$ $a=\frac{4}{\mathrm{i}}$ $a=-4\mathrm{i}$ $(z^2+(3\mathrm{i}-3)z-9\mathrm{i})(\mathrm{i}z^2-4\mathrm{i}z+5\mathrm{i})=0$ $(z^2+(3\mathrm{i}-3)z-9\mathrm{i})(\mathrm{i})(z^2-4z+5)=0$ Solving $(z^2-4z+5)=0$ by GC, $z_2=2-\mathrm{i}$ and $z_4=2+\mathrm{i}$ since $\mathrm{Im}(z_4)>0$.
(ii)	
(iii)	From the Argand diagram, for *ABDE* to form a parallelogram, $E(0,-1)$. Therefore, $wz_3=-\mathrm{i}$. $wz_3=-\mathrm{i}$ $3w=\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}$ $w=\frac{1}{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}$

Q4	Suggested Solutions
	$w=\frac{1}{3}e^{i\left(\frac{3\pi}{2}\right)}$
(b)	$(-4-4i)^5=\left[\sqrt{4^2+4^2}\right]^5 e^{i\left(-\frac{3\pi}{4}\right)(5)}$ $=\left(\sqrt{32}\right)^5 e^{i\frac{\pi}{4}}$ $\left(-2\sqrt{3}+2i\right)^7=\left(\sqrt{12+4}\right)^7 e^{i\left(\frac{5\pi}{6}\right)(7)}$ $=4^7 e^{i\left(-\frac{\pi}{6}\right)}$ $\frac{(-4-4i)^5}{\left(-2\sqrt{3}+2i\right)^7}=\frac{\left(\sqrt{32}\right)^5 e^{i\frac{\pi}{4}}}{4^7 e^{i\left(-\frac{\pi}{6}\right)}}$ $=\frac{1}{2\sqrt{2}}e^{i\frac{5\pi}{12}}$
(c)	$\arg\left(iq^n\right)^*=-\arg\left(iq^n\right)$ $=-\left[\arg(i)+n\arg(q)\right]$ $=-\left(\frac{\pi}{2}-\frac{n\pi}{4}\right)$ Real and positive implies that $\arg\left(iq^n\right)^*=2k\pi,\ k\in\mathbb{Z}$ $-\left(\frac{\pi}{2}-\frac{n\pi}{4}\right)=2k\pi$ $\frac{n}{4}-\frac{1}{2}=2k$ $n=8k+2$ The three smallest positive integers are 2, 10 and 18.

Q5	Suggested Solutions
(i)	No of ways $= 2\times 3\times {}^4C_3 = 24$
(ii)	**Using complement method** Total number of ways to arrange the 12 books $=\dfrac{12!}{3!4!5!}=27720$ 3 cases if there is no Mathematics book on the top rack. Case 1: 4 Mathematics and 2 Literature on the bottom rack. No of ways $=\left(\dfrac{6!}{4!2!}\right)\left(\dfrac{6!}{5!}\right)=90$ Case 2: 4 Mathematics and 2 Geography books on the bottom rack. No of ways $=\left(\dfrac{6!}{4!2!}\right)\left(\dfrac{6!}{3!3!}\right)=300$ Case 3: 4 Mathematics and 1 Geography and 1 Literature book on the bottom rack. No of ways $=\left(\dfrac{6!}{4!}\right)\left(\dfrac{6!}{4!2!}\right)=450$ Total no of ways $= 27720-90-300-450=26880$

Q6	Suggested Solutions

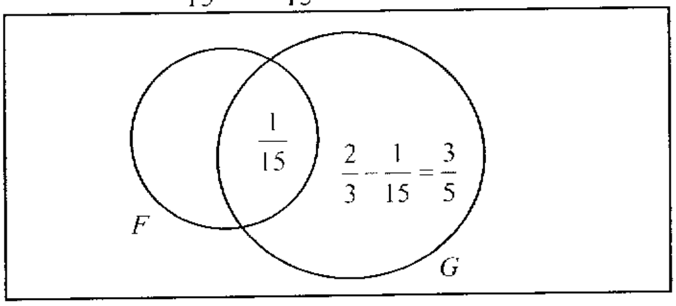
(a) Consider $F \subseteq G$:



Least $\mathrm{P}(F' \cap G)$ is $\frac{4}{15}$.

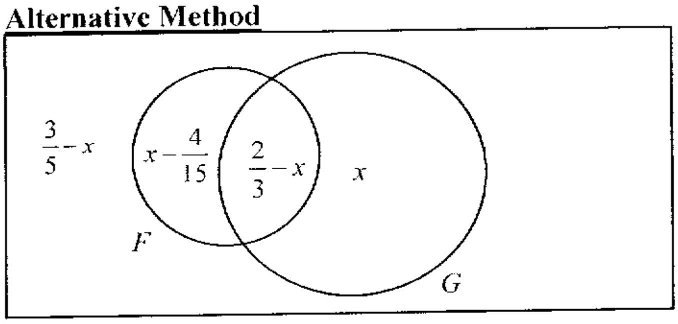
$\frac{2}{3}+\frac{2}{5}=\frac{16}{15}>1$

$\therefore \mathrm{P}(F \cap G)=\frac{16}{15}-1=\frac{1}{15}$



Greatest $\mathrm{P}(F' \cap G)$ is $\frac{3}{5}$.

Alternative Method



Let $\mathrm{P}(F' \cap G)=x$.

$\mathrm{P}(F \cap G)=\mathrm{P}(G)-\mathrm{P}(F' \cap G)=\frac{2}{3}-x$.

$\mathrm{P}(F \cap G')=\mathrm{P}(F)-\mathrm{P}(F \cap G)=\frac{2}{5}-\left(\frac{2}{3}-x\right)=x-\frac{4}{15}$.

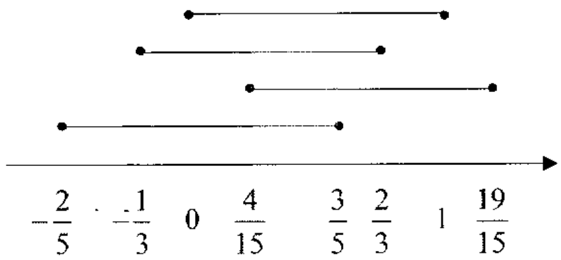
$\mathrm{P}(F' \cap G')=1-\mathrm{P}(F \cup G)=1-\left(x-\frac{4}{15}+\frac{2}{3}\right)=\frac{3}{5}-x$.

Therefore, $0 \le x \le 1$ and

Q6	Suggested Solutions

$0 \le \frac{2}{3} - x \le 1$ and $0 \le x - \frac{4}{15} \le 1$ and $0 \le \frac{3}{5} - x \le 1$.

This is equivalent to $0 \le x \le 1$ and
$-\frac{1}{3} \le x \le \frac{2}{3}$ and $\frac{4}{15} \le x \le \frac{19}{15}$ and $-\frac{2}{5} \le x \le \frac{3}{5}$.



$\frac{4}{15} \le x \le \frac{3}{5}$

Greatest $\mathrm{P}(F' \cap G)$ is $\frac{3}{5}$. Least $\mathrm{P}(F' \cap G)$ is $\frac{4}{15}$.

(b)(i)

$$\mathrm{P}(A' \cap B' | C') = \frac{\mathrm{P}(A' \cap B' \cap C')}{\mathrm{P}(C')}$$
$$= \frac{1 - \mathrm{P}(A \cup B \cup C)}{1 - \mathrm{P}(C)}$$
$$= \frac{1 - \frac{3}{4}}{1 - \frac{5}{8}}$$
$$= \frac{2}{3}$$

(b)(ii)

Since events A and B are independent,
$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) = \frac{1}{4}$.
Since events B and C are independent,
$\mathrm{P}(B \cap C) = \mathrm{P}(B)\mathrm{P}(C) = \frac{5}{12}$.

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) + \mathrm{P}(A \cap B \cap C)$$

$$\frac{3}{4} = \frac{3}{8} + \frac{2}{3} + \frac{5}{8} - \frac{1}{4} - \frac{1}{3} - \frac{5}{12} + \mathrm{P}(A \cap B \cap C)$$

$$\mathrm{P}(A \cap B \cap C) = \frac{1}{12}$$

Q7	Suggested Solutions
(i)	$\bar{x} = \frac{-37.6}{60} + 140 = 139.3733 \approx 139$ (3 s.f.) $s^2 = \frac{1}{59}\left[1012.17 - \frac{(-37.60)^2}{60}\right]$ $= 16.7560565$ $= 16.8$ (3 s.f.)
(ii)	Let μ be the **population mean** systolic blood pressure of patients who suffer from high blood pressure. $H_0 : \mu = 140$ vs $H_1 : \mu < 140$ Level of significance: 5% (lower tailed). Under H_0, $\overline{X}$ is approximately normal by Central Limit Theorem since $n = 60 (\geq 30)$ is large. Hence, $Z = \frac{\overline{X} - 140}{\sqrt{S^2/60}} \sim N(0,1)$ approximately. **Method 1 : Using *p*-value** By GC, p-value $= 0.11783 > 0.05$ **Method 2: Using critical region and test statistic, z** Critical region: $z < -1.64485$ $z = \frac{139.3733 - 140}{\sqrt{16.7560565/60}}$ $= -1.18590 > -1.6449$ Do not reject H_0. We conclude that there is insufficient evidence at 5% level of significance to claim that the new drug is effective in reducing the systolic blood pressure of patients suffering from high blood pressure.
(iii)	Since $\sum(x-140)^2$ is larger, the value of s^2 increases. The new observed test statistic value gets smaller (closer to zero) and thus less negative than the original test statistic value. Therefore, new observed z-value > original observed z-value > $z_{critical}$. The new test statistic value remains to be outside the critical region (new p-value gets larger which exceeds the critical region). So the result of this test will not differ from the result of the test in part **(ii)**.
(iii)	**Alternative Method** Since $\sum(x-140)^2 > 1012.17$,

Q7	**Suggested Solutions**
	then $s^2 = \frac{1}{59}\left[\sum(x-140)^2 - \frac{(-37.60)^2}{60}\right] > 16.75601$ Test statistic: $Z = \frac{\overline{X}-140}{\sqrt{s^2/60}} = \sqrt{\frac{60}{s^2}}\left(\overline{X}-140\right)$ Therefore, new $z = \sqrt{\frac{60}{s^2}}(139.3733-140) > -1.6448$ Do not reject H_0. Hence, the result of the new test remains unchanged from the result of the test carried out in part **(ii)**. Detailed manipulation: $\frac{1}{s^2} < \frac{1}{16.75601}$ $\sqrt{\frac{60}{s^2}} < \sqrt{\frac{60}{16.75601}}$ $\sqrt{\frac{60}{s^2}}(139.3733-140) > \sqrt{\frac{60}{16.75601}}(139.3733-140) > -1.6449$ Therefore, new $z = \sqrt{\frac{60}{s^2}}(139.3733-140) > -1.6448$

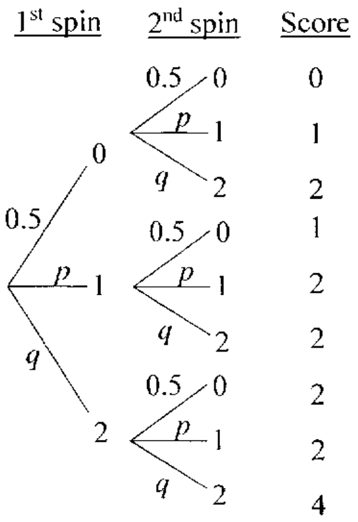
Q8	Suggested Solutions
(i)	The **probability** that a tomato is rotten **may not be the same** for all tomatoes because the tomatoes from the farm may be subjected to different treatment, thereby affecting the quality of the tomatoes. OR Whether a tomato is rotten may not be **independent** of whether another tomato is rotten because a rotten tomato may **affect** the quality of other tomatoes from the same plant.
(ii)	$25(p)=1$ $p=\frac{1}{25}=0.04$
(iii)	Let X be the number of rotten tomatoes out of 25. $X \sim \mathrm{B}(25, 0.04)$ $\mathrm{P}(X<2)=\mathrm{P}(X \leq 1)$ $=0.735810$ $=0.736(3\text{ s.f.})$
(iv)	Let Y be the number of rotten tomatoes out of 20. $Y \sim \mathrm{B}(20, 0.04)$ Required probability $=\mathrm{P}(Y=3)(0.04)(0.96)^4$ $=(0.0364499)(0.04)(0.96)^4$ $=0.00124$ (3 s.f.)
(v)	Let Q be the number of satisfactory boxes out of 3. $Q \sim \mathrm{B}(3, 0.73581)$ Let R be the number of satisfactory boxes out of 5 $R \sim \mathrm{B}(5, 0.73581)$ $\mathrm{P}(\text{exactly 6 boxes satisfactory} \mid Q \geq 2)$ $=\frac{\mathrm{P}(\text{exactly 6 boxes satisfactory and } Q \geq 2)}{\mathrm{P}(Q \geq 2)}$ $=\frac{\mathrm{P}(Q=2)\mathrm{P}(R=4)+\mathrm{P}(Q=3)\mathrm{P}(R=3)}{\mathrm{P}(Q \geq 2)}$ $=\frac{\mathrm{P}(Q=2)\mathrm{P}(R=4)+\mathrm{P}(Q=3)\mathrm{P}(R=3)}{1-\mathrm{P}(Q \leq 1)}$ $=0.335\ (3\text{ s.f.})$

Q9	**Suggested Solutions**
(i)	Let B denote the lifespan (in **years**) of an oven produced by Factory B. $B \sim \mathrm{N}\left(15, \left(\frac{k}{12}\right)^2\right)$ $\mathrm{P}(B > 14) = 0.9$ $\mathrm{P}\left(Z > \frac{14-15}{\left(\frac{k}{12}\right)}\right) = 0.9$ $-\frac{12}{k} \approx -1.281551567$ $k \approx 9.3636497$ $= 9.3636$ (5 s.f.) <u>OR</u> Let B denote the lifespan (in **months**) of an oven produced by Factory B. $B \sim \mathrm{N}\left(180, k^2\right)$ $\mathrm{P}(B > 14 \times 12) = 0.9$ $\mathrm{P}\left(Z > \frac{168-180}{k}\right) = 0.9$ $-\frac{12}{k} \approx -1.281551567$ $k \approx 9.3636497$ $= 9.3636$ (5 s.f.)
(ii)	$\mathrm{E}(B-A) = 15 - 13 = 2$ $\mathrm{Var}(B-A) = \left(\frac{9.3636}{12}\right)^2 + 0.5^2 \approx 0.85886809$ $B - A \sim \mathrm{N}(2, 0.85887)$ $\mathrm{P}(0 < B - A < 3) = 0.844$ (3 s.f.)
(iii)	Let A denote the lifespan (in **years**) of an oven produced by Factory A. $A \sim \mathrm{N}\left(13, 0.5^2\right)$

Q9	Suggested Solutions
	$\mathrm{P}(13-n \le A \le 13+n) \ge 0.4$ $\mathrm{P}(A \le 13-n) \le 0.3$ Let $\mathrm{P}(A \le a) = 0.3$ From GC, $a \approx 12.7377997$ $13-n \le 12.7378$ $n \ge 0.2622$ Least $n = 0.263$ (3 d.p.)
(iv)	$\mathrm{P}(A < 12) \approx 0.022750062$ Let X denote the number of oven, out of 20, with lifespan less than 12 years. $X \sim \mathrm{B}(20, 0.022750)$ $\mathrm{P}(X \ge 3) = 1 - \mathrm{P}(X \le 2)$ $\approx 1 - 0.9899538$ $= 0.0100$ (3 s.f.)

Q10 **Suggested Solutions**

(i)



$$P(X=2) = 0.5q + p^2 + pq + 0.5q + pq$$
$$= q + 2pq + p^2$$
$$= (0.5-p) + 2p(0.5-p) + p^2 \quad \because 0.5 + p + q = 1$$
$$= 0.5 - p^2$$

(ii)

$P(X=0) = (0.5)^2 = 0.25$

$P(X=1) = 0.5p + p(0.5) = p$

$P(X=2) = 0.5 - p^2$ from part **(i)**

$P(X=4) = q^2 = (0.5-p)^2$

x	0	1	2	4
$P(X=x)$	0.25	p	$0.5-p^2$	$(0.5-p)^2$

(iii)

$$E(X) = \frac{11}{9}$$
$$0 + p + 2(0.5-p^2) + 4(0.5-p)^2 = \frac{11}{9}$$
$$p + 1 - 2p^2 + 4(0.25 - p + p^2) = \frac{11}{9}$$
$$2 - 3p - 2p^2 = \frac{11}{9}$$
$$18p^2 - 27p + 7 = 0$$
$$(3p-1)(6p-7) = 0$$
$$p = \frac{1}{3} \text{ or } p = \frac{7}{6} \text{ (Rej } \because 0 \le p \le 1)$$

Q10	Suggested Solutions
	<table><tr><td>x</td><td>0</td><td>1</td><td>2</td><td>4</td></tr><tr><td>$\mathrm{P}(X=x)$</td><td>0.25</td><td>$\frac{1}{3}$</td><td>$\frac{7}{18}$</td><td>$\frac{1}{36}$</td></tr></table> $\mathrm{E}(X^2)=0+\frac{1}{3}+2^2\left(\frac{7}{18}\right)+4^2\left(\frac{1}{36}\right)$ $=\frac{7}{3}$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$ $=\frac{7}{3}-\left(\frac{11}{9}\right)^2$ $=\frac{68}{81}$
(iv)	Since $n=50$ is large, $\bar{X}\sim \mathrm{N}\left(\frac{11}{9},\ \frac{68}{81(50)}\right)$ approximately by Central Limit Theorem. $\mathrm{P}(\bar{X}<1.5)=0.984$
(v)	B is a proper **subset** of A. (If a player scores at least 2 in each of the three games, then a player's total score in the three games will be at least 6 which is more than 5. Therefore, all the possible outcomes of event B are also outcomes of event A.) Furthermore, there are outcomes in event A that are not outomes of event B. For example, a player can score a combination of 0, 2, and 4 in each of the 3 games. The total score is 6 which is more than 5, but the player did not score at least 2 in each of the 3 games. Therefore, $\mathrm{P}(B)$ is less than $\mathrm{P}(A)$.