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2021 TJC Prelim Paper 1

1 The curve with equation $y = p + \frac{1}{x+q}$, where p and q are constants, is transformed by a scaling of factor 4 parallel to the x-axis, followed by a translation of 3 units in the negative y-direction, followed by a reflection about the y-axis.
The resulting curve has asymptotes $x = 2$ and $y = 1$. Find the values of p and q. [5]

2 **(i)** Sketch the graphs of $y = e^{-x}$ and $y = \ln(x+k)$, where $k > e$. Indicate clearly the coordinates of intersection(s) with axes and the equations of any asymptotes. [2]

(ii) Given that $e^{-a} = \ln(a+k)$ and $e^{-b} = -\ln(b+k)$, solve the following inequalities in terms of a, b and k.

(a) $e^{-x} \le \ln(x+k)$ [1]

(b) $e^{-x} \le |\ln(x+k)|$ [2]

3 **(a)** Find $\int \frac{1}{4x^2+9}\,dx$ [2]

(b) Using the substitution $x = \tan\theta$, find the exact value of $\int_1^2 \frac{1}{x^2\sqrt{1+x^2}}\,dx$. [5]

4 **(i)** Expand $\frac{1+ax}{(a+x)^2}$ in ascending powers of x up to and including the term in x^3, where a is a positive constant.
Given that there is no term in x, show that $a = \sqrt{2}$. [5]

(ii) The coefficient of x^n in the expansion of $\frac{1+\sqrt{2}x}{\left(\sqrt{2}+x\right)^2}$ where $n \in \mathbb{Z}^+$, is denoted by A_n. It is found that $A_n = (-1)^{n+1}\left(\frac{n-1}{2\left(\sqrt{2}\right)^n}\right)$. Find the value of n such that $|A_n|$ has the largest value. [2]

5 A curve C has parametric equations

$$x = \theta + \cos\theta, \qquad y = (1-k)\theta + \frac{1}{2}\sin 2\theta,$$

where $0 \le \theta \le \pi$.

(i) Show that C has stationary points when $\cos^2\theta = \frac{k}{2}$. Hence find the range of values of k such that C has stationary points. [4]

(ii) In the case where $k = 1$, find in non-trigonometric form the exact coordinates of the turning points and sketch C. [4]

6 **(a)** The sum of the first 5 terms of an arithmetic series is $\frac{1}{3}$ times the sum of its next 5 terms. Given that the common difference of this series is 3, find the first term of the series. [4]

(b) A teacher makes an initial payment of \$$k$ to a retirement fund on 1 January 2021, and thereafter, she contributes \$12 000 per year to it on the first day of each year. The retirement fund guarantees a compound annual interest rate of 4% on the last day of the year.

(i) Find the minimum value of k, correct to the nearest dollars such that the total value of her retirement fund at the end of the 10th year when interest is applied exceeds \$500 000. [3]

(ii) It is given that $k = 105\,000$. Find the day in which the total amount in the retirement fund will first exceed \$1 million. [3]

7 It is given that the equation of a curve C is $4x^2 - 8x^2y - 32 = 5y^3$.

(i) Show that there is no point on C where the tangent is parallel to the y-axis. [4]

(ii) Find the equation of the tangent which is parallel to the x-axis. [3]

(iii) The point $P(x, y)$ moves along C in a way such that the x-coordinate of P is increasing at a constant rate of 2 units per second. Find the exact rate of increase of the y-coordinate at the instant when $x = 1.5$. [3]

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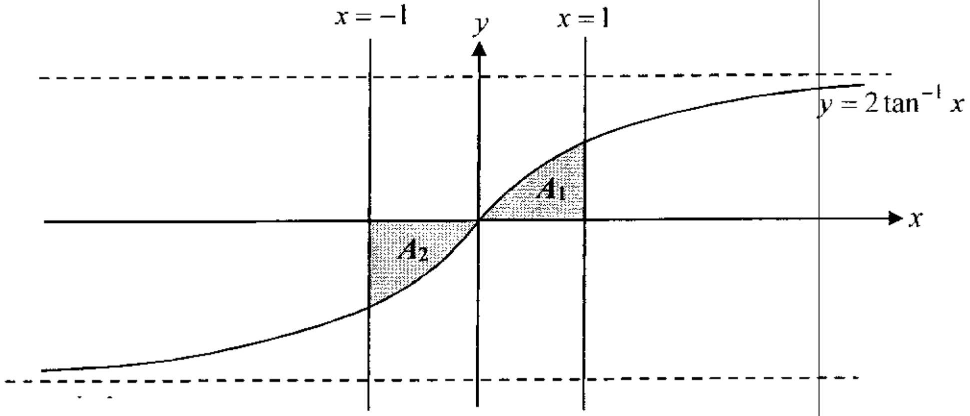


The diagram shows the curve with equation $y = 2\tan^{-1} x$ and the lines with equations $x = -1$ and $x = 1$.

(i) Write down the equations of the two horizontal asymptotes of the curve $y = 2\tan^{-1} x$. [1]

Region A_1 and A_2 are the shaded regions shown in the diagram. A_1 is bounded by the curve $y = 2\tan^{-1} x$, the x-axis and the lines $x = 1$ while A_2 is bounded by the curve $y = 2\tan^{-1} x$, the x-axis and the lines $x = -1$.

Region B is bounded by the curve $y = 2\tan^{-1} x$, the y-axis and the line $y = \frac{\pi}{2}$.

(ii) Find the exact area of region B and show that total area of regions A_1 and A_2 is larger than area of region B by $a\pi - b\ln 2$, where a and b are constants to be determined. [4]

(iii) Find the exact volume generated if region B is rotated completely about the y-axis. [3]

(iv) By considering a suitable translation of the graph, or otherwise, find the volume generated when region B is rotated about the line $x = 1$ through 4 right-angles, giving your answer correct to 2 decimal places. [3]

9 **(i)** Verify that one of the roots of the equation $z^3 - (1+2i)z^2 + (a-1+i)z - a(1+i) = 0$ where a is real, is $1 + i$. [2]

(ii) Show that the other 2 roots z_1 and z_2 can be expressed as $z_1 = \dfrac{\sqrt{-1-4a}+i}{2}$ and $z_2 = \dfrac{-\sqrt{-1-4a}+i}{2}$. [3]

(iii) Find the range of a such that z_1 and z_2 are purely imaginary. [2]

(iv) Given that $\arg(z_1) = \dfrac{\pi}{3}$, find a. [3]

Hence find $|z_2|$. [2]

10 On a mangrove swamp, scientists are investigating the population of mudskippers and crabs. Initially, there are 800 mudskippers and 800 crabs on the mangrove swamp. At time t years, the number of mudskippers and crabs are M and C respectively.

(a) For the mudskippers, the scientists discovers that every year, the growth rate is 0.4% of the population size and the death rate is 5 mudskippers per year.

(i) Write down a differential equation relating M and t. [1]

(ii) Solve the differential equation in part (i) and determine what happens to the mudskipper population in the future. [4]

(b) For the crabs, the scientists propose that C and t are related by the differential equation $\frac{\mathrm{d}C}{\mathrm{d}t} = 8C - 0.005C^2$.

(i) Find the number of crabs in the mangrove swamp when the rate of change of C is a maximum. [2]

(ii) Find C in terms of t. Sketch a graph of C against t. [5]

11 Coordinates axes $Oxyz$ are set up with the origin O at the base of an airport control tower. The x-axis is due East, the y-axis due North and the z-axis vertical. The units of distances are kilometres. An airplane A takes off from the point X. For the first 4 minutes, the position vector of A at time t minutes after take-off, is given by

$$\mathbf{r} = (2+t)\mathbf{i} + (1+2t)\mathbf{j} + 3t\mathbf{k}, \quad 0 \le t \le 4.$$

(i) State the coordinates of X. [1]

(ii) Find the acute angle the flight path makes with the horizontal. [2]

(iii) The airplane enters a cloud at a height of 5 km. Find the coordinates of the point where it enters the cloud. [2]

A second airplane B takes off from the point $(-2,-1,0)$ at the same time as the first airplane A and is traveling at a constant speed in a straight line for the first 4 minutes. Two minutes after take-off, B is at the point $(1,5,\alpha)$.

(iv) Find in terms of α, the position vector of B after t minutes where $0 \le t \le 4$. Explain if it is possible for the two airplanes to collide in the first 4 minutes. [4]

(v) At $t = 4$, a third airplane C was spotted to be equidistant from the first two airplanes. At the same instant, two buildings on the ground D and E are such that A and B are equidistant from both D and E, i.e. $AD = BD$ and $AE = BE$. Find the Cartesian equation of the plane in terms of α in which C, D and E lie. [4]

2021 TJC Prelim Paper 1 Suggested Solutions

1 The curve with equation $y = p + \dfrac{1}{x+q}$, where p and q are constants, is transformed by a scaling of factor 4 parallel to the x-axis, followed by a translation of 3 units in the negative y-direction, followed by a reflection about the y-axis.

The resulting curve has asymptotes $x = 2$ and $y = 1$. Find the values of p and q. [5]

[Solution]

[Solution]

$y = p + \dfrac{1}{x+q}$ 1st : x replaced by $\dfrac{x}{4}$ to get $y = p + \dfrac{1}{\frac{x}{4}+q} = p + \dfrac{4}{x+4q}$ 2nd: y replaced by $y+3$ to get $y + 3 = p + \dfrac{4}{x+4q} \Rightarrow y = p - 3 + \dfrac{4}{x+4q}$ 3rd : x replaced by $-x$ to get $y = p - 3 + \dfrac{4}{-x+4q}$ Horizontal asymptote: $y = p - 3 = 1 \Rightarrow p = 4$ Vertical asymptote: $-x + 4q = 0$ $\Rightarrow x = 4q = 2 \Rightarrow q = \dfrac{1}{2}$	

2 **(i)** Sketch the graphs of $y=\mathrm{e}^{-x}$ and $y=\ln(x+k)$, where $k>\mathrm{e}$. Indicate clearly the coordinates of intersection(s) with axes and the equations of any asymptotes. [2]

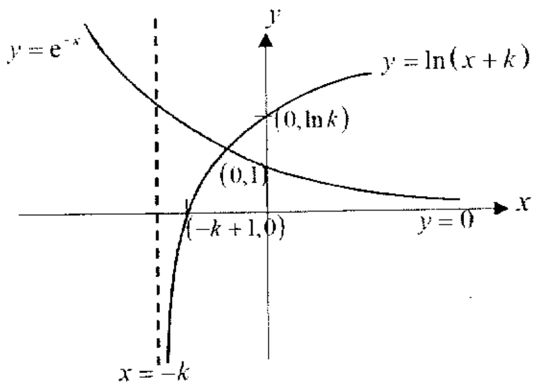
(ii) Given that $\mathrm{e}^{-a}=\ln(a+k)$ and $\mathrm{e}^{-b}=-\ln(b+k)$, solve the following inequalities in terms of a, b and k.

(a) $\mathrm{e}^{-x}\le\ln(x+k)$ [1]

(b) $\mathrm{e}^{-x}\le|\ln(x+k)|$ [2]

[Solution]

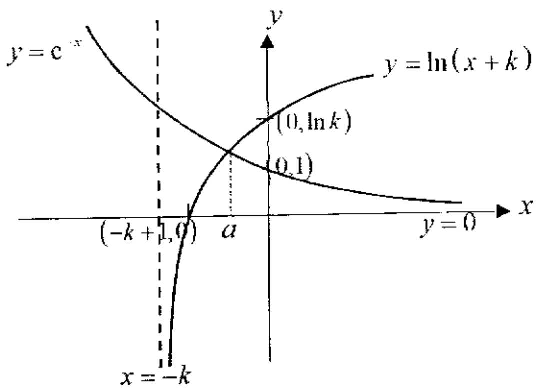
(i) Note that for $k>\mathrm{e}$, $\ln k>1$.



(ii) Given: $\mathrm{e}^{-a}=\ln(a+k)$

(a) $\mathrm{e}^{-x}\le\ln(x+k)$

From diagram,

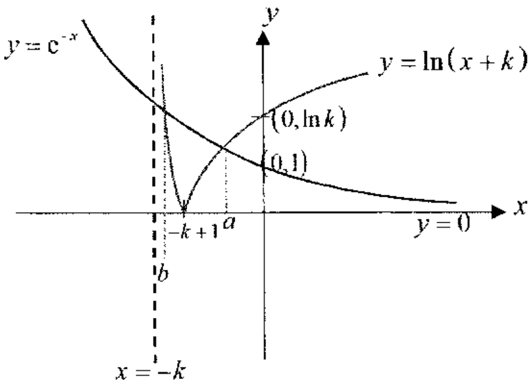


solution is $x\ge a$.

Given: $e^{-a} = \ln(a+k)$ and $e^{-b} = -\ln(b+k)$

(b) $e^{-x} \le |\ln(x+k)|$

From diagram,



Solution is $x \ge a$ or $-k < x \le b$.

3 **(a)** Find $\int \frac{1}{4x^2+9}\,dx$ [2]

(b) Using the substitution $x = \tan\theta$, find the exact value of $\int_1^2 \frac{1}{x^2\sqrt{1+x^2}}\,dx$. [5]

[Solution]

(a)
$$\int \frac{1}{4x^2+9}\,\mathrm{d}x = \int \frac{1}{4(x^2+(\frac{3}{2})^2)}\,\mathrm{d}x$$
$$= \frac{1}{4}\left(\frac{2}{3}\right)\tan^{-1}\frac{2x}{3} + C = \frac{1}{6}\tan^{-1}\frac{2x}{3} + C$$

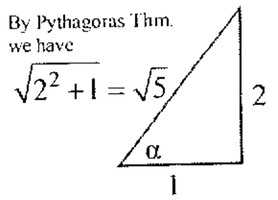
Alternatively,
$$\int \frac{1}{4x^2+9}\,\mathrm{d}x = \frac{1}{2}\int \frac{2}{(2x)^2+3^2}\,\mathrm{d}x$$
$$= \frac{1}{2}\left(\frac{1}{3}\tan^{-1}\frac{2x}{3}\right) + c = \frac{1}{6}\tan^{-1}\frac{2x}{3} + c$$

(b) $x = \tan\theta$, $\frac{dx}{d\theta} = \sec^2\theta$.

When $x = 1$, $\theta = \frac{\pi}{4}$ and $x = 2$, $\theta = \tan^{-1} 2$

$$\int_1^2 \frac{1}{x^2\sqrt{1+x^2}}\,dx = \int_{\frac{\pi}{4}}^{\tan^{-1}2} \frac{1}{\tan^2\theta\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\tan^{-1}2} \frac{1}{\tan^2\theta(\sec\theta)}\sec^2\theta\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\tan^{-1}2} \frac{\sec\theta}{\tan^2\theta}\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\tan^{-1}2} \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\tan^{-1}2} \cot\theta\,\operatorname{cosec}\theta\,d\theta$$
$$= -[\operatorname{cosec}\theta]_{\frac{\pi}{4}}^{\alpha} \qquad \text{where } \tan\alpha = 2$$

As $\tan\alpha = 2$, we get the right-angled triangle



$$= -[\operatorname{cosec}\alpha - \operatorname{cosec}\tfrac{\pi}{4}]$$
$$= -\left[\frac{\sqrt{5}}{2} - \sqrt{2}\right] = \sqrt{2} - \frac{\sqrt{5}}{2}$$

4 **(i)** Expand $\dfrac{1+ax}{(a+x)^2}$ in ascending powers of x up to and including the term in x^3, where a is a positive constant.
Given that there is no term in x, show that $a=\sqrt{2}$. [5]

(ii) The coefficient of x^n in the expansion of $\dfrac{1+\sqrt{2}x}{\left(\sqrt{2}+x\right)^2}$ where $n \in \mathbb{Z}^+$, is denoted by A_n. It is found that $A_n = (-1)^{n+1}\left(\dfrac{n-1}{2\left(\sqrt{2}\right)^n}\right)$. Find the value of n such that $|A_n|$ has the largest value. [2]

[Solution]

(i) $\dfrac{1+ax}{(a+x)^2}=(1+ax)(a+x)^{-2}$ $=a^{-2}(1+ax)\left(1+\dfrac{x}{a}\right)^{-2}$ $=a^{-2}(1+ax)\left(1-2\left(\dfrac{x}{a}\right)+\dfrac{(-2)(-3)}{2!}\left(\dfrac{x}{a}\right)^2+\dfrac{(-2)(-3)(-4)}{3!}\left(\dfrac{x}{a}\right)^3+\cdots\right)$ $=a^{-2}(1+ax)\left(1-\dfrac{2}{a}x+\dfrac{3}{a^2}x^2-\dfrac{4}{a^3}x^3+\cdots\right)$ $=a^{-2}\left(1-\dfrac{2}{a}x+\dfrac{3}{a^2}x^2-\dfrac{4}{a^3}x^3+ax-2x^2+\dfrac{3}{a}x^3+\cdots\right)$ $=\dfrac{1}{a^2}+\left(-\dfrac{2}{a^3}+\dfrac{1}{a}\right)x+\left(\dfrac{3}{a^4}-\dfrac{2}{a^2}\right)x^2+\left(-\dfrac{4}{a^5}+\dfrac{3}{a^3}\right)x^3+\cdots$ $=\dfrac{1}{a^2}+\left(\dfrac{-2+a^2}{a^3}\right)x+\left(\dfrac{3-2a^2}{a^4}\right)x^2+\left(\dfrac{-4+3a^2}{a^5}\right)x^3+\cdots$ Since there is no term in x, we have $\dfrac{-2+a^2}{a^3}=0$ $a=\sqrt{2}$ (since a is positive)	

We have $A_n = (-1)^{n+1}\left(\dfrac{n-1}{2\left(\sqrt{2}\right)^n}\right)$ $\therefore \lvert A_n \rvert = \left\lvert \dfrac{n-1}{2\left(\sqrt{2}\right)^n} \right\rvert$ Using GC, $\lvert A_1 \rvert < \lvert A_2 \rvert < \lvert A_3 \rvert = 0.3536 < \lvert A_4 \rvert = 0.375 > \lvert A_5 \rvert = 0.3536 > \lvert A_6 \rvert > \ldots$ The value of n such that $\lvert A_n \rvert$ has the largest value is 4.	

5 A curve C has parametric equations

$$x = \theta + \cos\theta, \qquad y = (1-k)\theta + \frac{1}{2}\sin 2\theta,$$

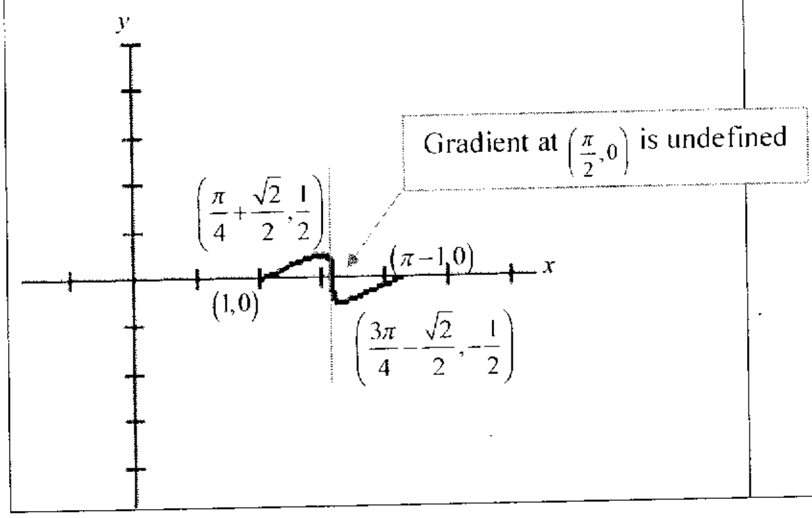
where $0 \le \theta \le \pi$.

(i) Show that C has stationary points when $\cos^2\theta = \frac{k}{2}$. Hence find the range of values of k such that C has stationary points. [4]

(ii) In the case where $k = 1$, find in non-trigonometric form the exact coordinates of the turning points and sketch C. [4]

[Solution]

$x = \theta + \cos\theta, \qquad y = (1-k)\theta + \frac{1}{2}\sin 2\theta,$ $\frac{dx}{d\theta} = 1 - \sin\theta,$ $\frac{dy}{d\theta} = (1-k) + \frac{1}{2}2\cos 2\theta = 1 - k + 2\cos^2\theta - 1 = 2\cos^2\theta - k$ $\frac{dy}{dx} = \frac{2\cos^2\theta - k}{1-\sin\theta}$ **(i)** C has stationary points $\Rightarrow \frac{dy}{dx} = 0$ $\Rightarrow 2\cos^2\theta - k = 0$ and $1 - \sin\theta \ne 0$ $\therefore \cos^2\theta = \frac{k}{2}$ For $0 \le \theta \le \pi \Rightarrow -1 \le \cos\theta \le 1, \quad \cos\theta \ne 0$ $\Rightarrow 0 < \cos^2\theta \le 1$ $\Rightarrow 0 < \frac{k}{2} \le 1$ $\therefore 0 < k \le 2$	

(ii) $\cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$ $\theta = \frac{\pi}{4} \Rightarrow x = \frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\pi}{4} + \frac{\sqrt{2}}{2}, y = \frac{1}{2}\sin\frac{\pi}{2} = \frac{1}{2}$ $\theta = \frac{3\pi}{4} \Rightarrow x = \frac{3\pi}{4} + \cos\frac{3\pi}{4} = \frac{3\pi}{4} - \frac{\sqrt{2}}{2}, y = \frac{1}{2}\sin\frac{3\pi}{2} = -\frac{1}{2}$ Therefore, the coordinates of turning points are $\left(\frac{\pi}{4} + \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ and $\left(\frac{3\pi}{4} - \frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$.  	

6 **(a)** The sum of the first 5 terms of an arithmetic series is $\frac{1}{3}$ times the sum of its next 5 terms. Given that the common difference of this series is 3, find the first term of the series. [4]

(b) A teacher makes an initial payment of $k to a retirement fund on 1 January 2021, and thereafter, she contributes $12 000 per year to it on the first day of each year. The retirement fund guarantees a compound annual interest rate of 4% on the last day of the year.

(i) Find the minimum value of k, correct to the nearest dollars such that the total value of her retirement fund at the end of the 10^{th} year when interest is applied exceeds $500 000. [3]

(ii) It is given that $k = 105\,000$. Find the day in which the total amount in the retirement fund will first exceed $1 million. [3]

(a) $S_5 = \frac{1}{3}(S_{10} - S_5) \Rightarrow \quad 4S_5 = S_{10}$ $\Rightarrow \quad 4\left[\frac{5}{2}(2a+4(3))\right] = \frac{10}{2}(2a+9(3))$ $\Rightarrow \quad 20a+120 = 10a+135$ $\Rightarrow \quad a = \frac{3}{2}$	

(b)(i)

	Year	First Day of Year	Last day of Year
1Jan 2021	1	k	$1.04k$
	2	$1.04k+12000$	$1.04(1.04k+12000)$
	3	$1.04(1.04k+12000)$ $+12000$	$1.04(1.04(1.04k+12000)+12000)$ $=1.04^3k+12000(1.04^2+1.04)$
	n		$1.04^nk+12000(1.04^{n-1}+\cdots+1.04^2+1.04)$

For $n=10$,

$$1.04^{10}k+12000(1.04^9+\cdots+1.04^2+1.04)>500000$$

$$1.04^{10}k+12000\frac{1.04(1.04^9-1)}{0.04}>500000$$

$k>248558.11$

Minimum $k=248559$ (correct to nearest dollars)

(b)(ii)

For $k=105\,000$, let T be total amount in the retirement fund.

$$T=1.04^n(105000)+12000\frac{1.04(1.04^{n-1}-1)}{0.04}>1000000$$

From GC, when $n=30$ (31 Dec 2051), $T=1001575.99$

The day in which the total amount in the retirement fund will first exceed \$1 million is <u>31 Dec 2050.</u>

7 It is given that the equation of a curve C is $4x^2-8x^2y-32=5y^3$.

(i) Show that there is no point on C where the tangent is parallel to the y-axis. [4]

(ii) Find the equation of the tangent which is parallel to the x-axis. [3]

(iii) The point $P(x,y)$ moves along C in a way such that the x-coordinate of P is increasing at a constant rate of 2 units per second. Find the exact rate of increase of the y-coordinate at the instant when $x=1.5$. [3]

[Solution]

(i) $4x^2-8x^2y-32=5y^3$

Differentiating w.r.t. x,

$$8x-8x^2\frac{dy}{dx}-16xy=15y^2\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{8x-16xy}{8x^2+15y^2}$$

For tangent to be parallel to y-axis, $\frac{dy}{dx}$ is undefined.

$\Rightarrow\ 8x^2+15y^2=0$

$\Rightarrow\ x=y=0$

Check (0, 0) is not a point on C.

So there is no point on C where the tangent is parallel to the y-axis.

(ii) For tangent to be parallel to x-axis, $\frac{dy}{dx}=0$.

$8x-16xy=0\ \Rightarrow\ x(1-2y)=0\ \Rightarrow\ x=0$ or $y=\frac{1}{2}$

When $x=0$, $5y^3=-32\ \Rightarrow\ y=\left(-\frac{32}{5}\right)^{\frac{1}{3}}$

Check: When $y=\frac{1}{2}$,

$4x^2-8x^2\left(\frac{1}{2}\right)-32=5\left(\frac{1}{2}\right)^3\ \Rightarrow\ -32=\frac{5}{8}$

which is invalid

The equation of the tangent parallel to the x-axis is $y=\left(-\frac{32}{5}\right)^{\frac{1}{3}}$.

(iii) Given $\frac{dx}{dt}=2$.

When $x=1.5$, $y=-1$.

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=\frac{8x-16xy}{8x^2+15y^2}\cdot\frac{dx}{dt}=\frac{8(1.5)-16(1.5)(-1)}{8(1.5)^2+15(-1)^2}\cdot(2)=\frac{24}{11}$$

The y-coordinate is increasing at a rate of $\frac{24}{11}$ units/s

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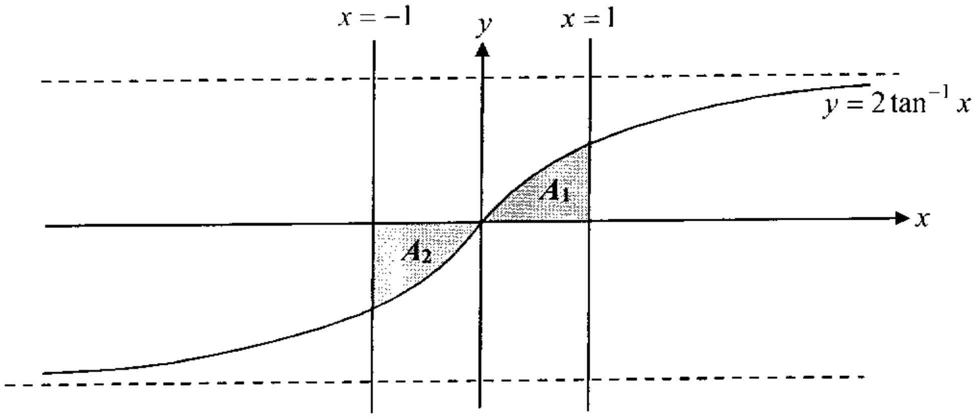


The diagram shows the curve with equation $y = 2\tan^{-1} x$ and the lines with equations $x = -1$ and $x = 1$.

(i) Write down the equations of the two horizontal asymptotes of the curve $y = 2\tan^{-1} x$. [1]

Region A_1 and A_2 are the shaded regions shown in the diagram. A_1 is bounded by the curve $y = 2\tan^{-1} x$, the x-axis and the lines $x = 1$ while A_2 is bounded by the curve $y = 2\tan^{-1} x$, the x-axis and the lines $x = -1$.

Region B is bounded by the curve $y = 2\tan^{-1} x$, the y-axis and the line $y = \dfrac{\pi}{2}$.

(ii) Find the exact area of region B and show that total area of regions A_1 and A_2 is larger than area of region B by $a\pi - b\ln 2$, where a and b are constants to be determined. [4]

(iii) Find the exact volume generated if region B is rotated completely about the y-axis. [3]

(iv) By considering a suitable translation of the graph, or otherwise, find the volume generated when region B is rotated about the line $x = 1$ through 4 right-angles, giving your answer correct to 2 decimal places. [3]

[Solution]

(i) The horizontal asymptotes are $y = \pm\pi$

(ii)

Area B $= \int_0^{\frac{\pi}{2}} \tan\frac{y}{2}\,dy$

$= \left[2\ln\left|\sec\left(\frac{y}{2}\right)\right|\right]_0^{\frac{\pi}{2}}$

$= 2\ln\sec\frac{\pi}{4} - \ln\sec 0$

$= 2\ln\sqrt{2}$

$= \ln 2$

Area of regions A_1 and $A_2 = 2\,(\frac{\pi}{2} - \ln 2) = \pi - 2\ln 2$

Area A_1 + Area A_2 – Area B

$= \pi - 2\ln 2 - \ln 2$

$= \pi - 3\ln 2.$

So $a = 1$ and $b = 3$

Method 2

Area of regions A_1 and A_2

$= 2\int_0^1 2\tan^{-1} x\,dx$ by symmetry

$= \left[4x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{4x}{1+x^2}\,dx$

$= \pi - \left[2\ln(1+x^2)\right]_0^1$

$= \pi - 2\ln 2$

Area B $= \frac{\pi}{2} - \frac{1}{2}(\pi - 2\ln 2) = \ln 2$

Area A_1 + Area A_2 – Area B

$= \pi - 2\ln 2 - \ln 2$

$= \pi - 3\ln 2.$

So $a = 1$ and $b = 3$

(iii) Volume for B rotated about y-axis

$= \pi\int_0^{\frac{\pi}{2}} x^2\,dy$

$= \pi\int_0^{\frac{\pi}{2}} \tan^2\left(\frac{y}{2}\right)dy$

$= \pi \int_0^{\frac{\pi}{2}} (\sec^2\left(\frac{y}{2}\right) - 1)dy$

$= \pi \left[2\tan\frac{y}{2} - y\right]_0^{\frac{\pi}{2}}$

$= \pi(2 - \frac{\pi}{2})$

(iv) Translating the curve $y = 2\tan^{-1} x$ by -1 unit along x-axis, we have $y = 2\tan^{-1}(x+1)$,

i.e. $x = \tan\frac{y}{2} - 1$

Volume of region A_1 rotated about the line $x = 1$ is

$\pi\int_0^1 \left(\tan\frac{y}{2} - 1\right)^2 dy$

Volume of the cylinder formed by A_1 and B when rotated about $x = 1$ is $\pi(1^2)\frac{\pi}{2} = \frac{\pi^2}{2}$

The required volume

$= \frac{\pi^2}{2} - \pi\int_0^{\frac{\pi}{2}} \left(\tan\frac{y}{2} - 1\right)^2 dy = 3.01$ (correct to 2 dps)

9 **(i)** Verify that one of the roots of the equation $z^3-(1+2\mathrm{i})z^2+(a-1+\mathrm{i})z-a(1+\mathrm{i})=0$ where a is real, is $1+\mathrm{i}$. [2]

(ii) Show that the other 2 roots z_1 and z_2 can be expressed as $z_1=\dfrac{\sqrt{-1-4a}+\mathrm{i}}{2}$ and $z_2=\dfrac{-\sqrt{-1-4a}+\mathrm{i}}{2}$. [3]

(iii) Find the range of a such that z_1 and z_2 are purely imaginary. [2]

(iv) Given that $\arg(z_1)=\dfrac{\pi}{3}$, find a. [3]

Hence find $|z_2|$. [2]

[Solution]

(i) Sub $z=1+\mathrm{i}$, we have

$$LHS=(1+\mathrm{i})^3-(1+2\mathrm{i})(1+\mathrm{i})^2+(a-1+\mathrm{i})(1+\mathrm{i})-a(1+\mathrm{i})$$
$$=-2+2\mathrm{i}+4-2\mathrm{i}+a(1+\mathrm{i})-2-a(1+\mathrm{i})$$
$$=0=RHS$$

(ii) **From (i),** one of the factors of $z^3-(1+2\mathrm{i})z^2+(a-1+\mathrm{i})z-a(1+\mathrm{i})$ is $z-1-\mathrm{i}$

Method 1

Let $(z-(1+\mathrm{i}))(z^2+Az+B)=$
$$z^3-(1+2\mathrm{i})z^2+(a-1+\mathrm{i})z-a(1+\mathrm{i})$$

Comparing coefficient of z^2 : $A-(1+\mathrm{i})=-(1+2\mathrm{i})\Rightarrow A=-\mathrm{i}$

Comparing the constant term: $-B(1+\mathrm{i})=-a(1+\mathrm{i})\Rightarrow B=a$

Method 2: Using Long Division, we have

$$z^3-(1+2\mathrm{i})z^2+(a-1+\mathrm{i})z-a(1+\mathrm{i})=(z-1-\mathrm{i})(z^2-\mathrm{i}z+a)$$

Therefore, $z^3-(1+2\mathrm{i})z^2+(a-1+\mathrm{i})z-a(1+\mathrm{i})=0$

$$\Rightarrow\ (z-1-\mathrm{i})(z^2-\mathrm{i}z+a)=0$$

$\Rightarrow$ $z=1-\mathrm{i}$ or $z^2-\mathrm{i}z+a=0$

The other 2 roots are $z=\frac{\mathrm{i}\pm\sqrt{(-\mathrm{i})^2-4(1)(a)}}{2(1)}=\frac{\mathrm{i}\pm\sqrt{-1-4a}}{2}$

Therefore, we have $z_1=\frac{\sqrt{-1-4a}+\mathrm{i}}{2}$ and $z_2=\frac{-\sqrt{-1-4a}+\mathrm{i}}{2}$

(iii) For z_1 and z_2 to be purely imaginary, $-1-4a\le 0$.

Therefore, $a\ge -\frac{1}{4}$.

(iv) Since $\arg(z_1)=\frac{\pi}{3}$, it means that real part of z_1

must be $\frac{\sqrt{-1-4a}}{2}$, i.e. $-1-4a>0$

Therefore, $\frac{\frac{1}{2}}{\frac{\sqrt{-1-4a}}{2}}=\tan\frac{\pi}{3}=\sqrt{3}$

$\sqrt{-1-4a}=\frac{1}{\sqrt{3}}$

Squaring both sides, we have

$-1-4a=\frac{1}{3}\Rightarrow a=-\frac{1}{3}$

Therefore,

$|z_2|=\left|\frac{-\sqrt{-1-4\left(\frac{-1}{3}\right)}+\mathrm{i}}{2}\right|$

$=\left|\frac{-\sqrt{\frac{1}{3}}+\mathrm{i}}{2}\right|=\frac{1}{2}\sqrt{\left(-\sqrt{\frac{1}{3}}\right)^2+1}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$

10 On a mangrove swamp, scientists are investigating the population of mudskippers and crabs. Initially, there are 800 mudskippers and 800 crabs on the mangrove swamp. At time t years, the number of mudskippers and crabs are M and C respectively.

(a) For the mudskippers, the scientists discovers that every year, the growth rate is 0.4% of the population size and the death rate is 5 mudskippers per year.

(i) Write down a differential equation relating M and t. [1]

(ii) Solve the differential equation in part (i) and determine what happens to the mudskipper population in the future. [4]

(b) For the crabs, the scientists propose that C and t are related by the differential equation $\frac{dC}{dt} = 8C - 0.005C^2$.

(i) Find the number of crabs in the mangrove swamp when the rate of change of C is a maximum. [2]

(ii) Find C in terms of t. Sketch a graph of C against t. [5]

(i)	Growth rate $= \frac{0.4}{100}M = 0.004M$ Death rate = 5 $\frac{dM}{dt}$ = Growth rate – death rate $= 0.004M - 5$	
(ii)	$\frac{dM}{dt} = 0.004M - 5 = 0.004(M - 1250)$ $\frac{1}{M-1250}\frac{dM}{dt} = 0.004$ Integrate wrt t: $\int \frac{1}{M-1250}\,dM = \int 0.004\,dt$ $\Rightarrow \ln\lvert M-1250\rvert = 0.004t + c$ $\Rightarrow \lvert M-1250\rvert = e^{0.004t+c}$ $\Rightarrow M - 1250 = \pm e^{0.004t+c}$ $\Rightarrow M - 1250 = Ae^{0.004t}, A = \pm e^c$ At $t = 0, M = 800$, $\Rightarrow A = -450$ $\therefore M = 1250 - 450e^{0.004t}$ As $t \to \infty$, $450e^{0.004t} \to \infty$, $M = 1250 - 450e^{0.004t} \to -\infty$ This means that the mudskipper population will become extinct in the future. $M = 1250 - 450e^{0.004t} = 0 \Rightarrow e^{0.004t} = \frac{25}{9} \Rightarrow t = 255.4$ The mudskipper will be extinct in the 256th year.	•
b(i)	Method 1: $\frac{dC}{dt} = 8C - 0.005C^2$.	

Let R be the rate of change of C. i.e. $R=\frac{dC}{dt}$

To maximise R, let $\frac{dR}{dC}=0$. Note R is in terms of C.

$$\frac{dR}{dc}=\frac{d\left(\frac{dC}{dt}\right)}{dC}=8-0.01C=0 \quad\Rightarrow\quad C=800.$$

$$\frac{d^2R}{dC^2}=\frac{d^2\left(\frac{dC}{dt}\right)}{dC^2}=-0.01<0$$

$\Rightarrow$ The number of crabs when growth rate is maximum is 800.

Method 2:

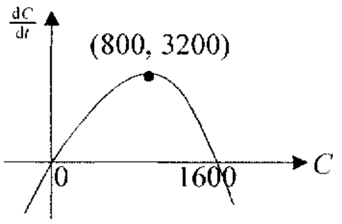
$$\frac{dC}{dt}=-0.005\left(C^2-1600C\right)$$

$$=-0.005\left(C-800\right)^2+3200\le 3200$$

$\frac{dC}{dt}=3200$ is maximum when C = 800.

Method 3:

$$\frac{dC}{dt}=C\left(8-0.005C\right)=0 \Rightarrow C=0 \text{ or } 1600$$



From the graph, $\frac{dC}{dt}=3200$ is maximum when C = 800.

b(ii)

Method 1: using Partial Fraction

$$\frac{dC}{dt}=8C-0.005C^2=0.005C\left(1600-C\right)$$

$$\Rightarrow \frac{1}{C\left(1600-C\right)}\frac{dC}{dt}=0.005$$

Integrate wrt t:

$$\int\frac{1}{C\left(1600-C\right)}\,dC=\int 0.005\,dt$$

$$\Rightarrow \frac{1}{1600}\left[\ln\left|C\right|-\ln\left|1600-C\right|\right]=0.005t+d$$

$\Rightarrow \ln\left|\dfrac{C}{1600-C}\right| = 8t + 1600d$

$\Rightarrow \dfrac{C}{1600-C} = \pm e^{8t+1600d}$

$\Rightarrow \dfrac{C}{1600-C} = Be^{8t}, B = \pm e^{1600d}$

When $t = 0$, $C = 800$, $B = 1$

$\therefore C = \dfrac{1600e^{8t}}{1+e^{8t}}$ (or $C = \dfrac{1600}{1+e^{-8t}}$)

Method 2: Complete the square and use MF26

$\dfrac{dC}{dt} = -0.005\left[(C-800)^2 - 800^2\right]$

$\dfrac{1}{(C-800)^2 - 800^2}\dfrac{dC}{dt} = -0.005$

$\int \dfrac{1}{(C-800)^2-800^2}\,dC = \int -0.005\,dt$

$\dfrac{1}{2(800)}\ln\left|\dfrac{(C-800)-800}{C-800+800}\right| = -0.005t + d$

$\left|\dfrac{(C-800)-800}{C-800+800}\right| = e^{-8t+1600d}$

$\dfrac{C-1600}{C} = \pm e^{-8t+1600d}$

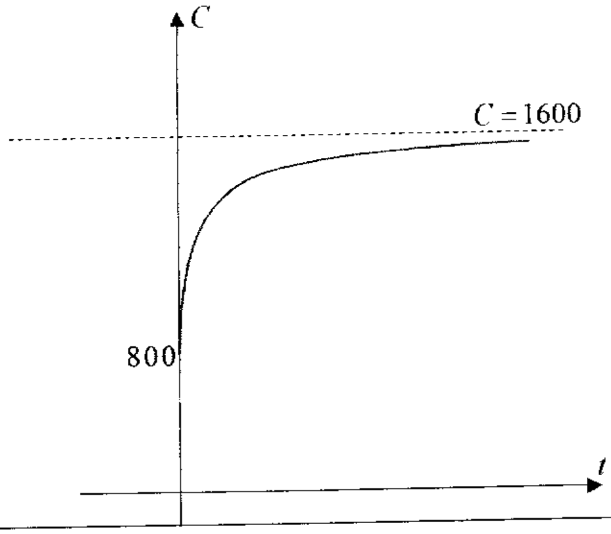
$\dfrac{C-1600}{C} = Be^{-8t}, B = \pm e^{1600d}$

When $t = 0$, $C = 800$.

$B = -1$

$1 - \dfrac{1600}{C} = -e^{-8t}$

$\dfrac{1600}{C} = 1 + e^{-8t} \Rightarrow C = \dfrac{1600}{1+e^{-8t}}$



11 Coordinates axes $Oxyz$ are set up with the origin O at the base of an airport control tower. The x-axis is due East, the y-axis due North and the z-axis vertical. The units of distances are kilometres. An airplane A takes off from the point X. For the first 4 minutes, the position vector of A at time t minutes after take-off, is given by

$$\mathbf{r} = (2+t)\mathbf{i} + (1+2t)\mathbf{j} + 3t\mathbf{k}, \quad 0 \le t \le 4.$$

(i) State the coordinates of X. [1]

(ii) Find the acute angle the flight path makes with the horizontal. [2]

(iii) The airplane enters a cloud at a height of 5 km. Find the coordinates of the point where it enters the cloud. [2]

A second airplane B takes off from the point $(-2,-1,0)$ at the same time as the first airplane A and is traveling at a constant speed in a straight line for the first 4 minutes. Two minutes after take-off, B is at the point $(1,5,\alpha)$.

(iv) Find in terms of α, the position vector of B after t minutes where $0 \le t \le 4$. Explain if it is possible for the two airplanes to collide in the first 4 minutes. [4]

(v) At $t = 4$, a third airplane C was spotted to be equidistant from the first two airplanes. At the same instant, two buildings on the ground D and E are such that A and B are equidistant from both D and E, i.e. $AD = BD$ and $AE = BE$. Find the Cartesian equation of the plane in terms of α in which C, D and E lie. [4]

[Solution]	
(i) $\overrightarrow{OA} = \begin{pmatrix}2\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\2\\3\end{pmatrix}, \quad 0 \le t \le 4$ When $t = 0$, coordinates of $X = (2, 1, 0)$	
(ii) Let θ be the required angle. Normal to the horizontal plane is $\begin{pmatrix}0\\0\\1\end{pmatrix}$ $\sin\theta = \dfrac{\left\lvert\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{\sqrt{1^2+2^2+3^2}\sqrt{1^2}} = \dfrac{3}{\sqrt{14}}$ $\theta = 53.3^\circ$ (1 d.p.) **Alternative Method,** $\cos(90^\circ - \theta) = \dfrac{\left\lvert\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{\sqrt{1^2+2^2+3^2}\sqrt{1^2}} = \dfrac{3}{\sqrt{14}}$ $90^\circ - \theta = 36.7^\circ$ (1 d.p.) $\theta = 53.3^\circ$ (1 d.p.)	

(iii) $\overrightarrow{OA}=\begin{pmatrix}2\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\3\end{pmatrix},\quad 0\le t\le 4$ Since z-axis is vertical, at height 5 km, $3t=5$ $\Rightarrow\ t=\frac{5}{3}$ Position vector of the point when it enters the cloud $=\begin{pmatrix}2\\1\\0\end{pmatrix}+\frac{5}{3}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}\frac{11}{3}\\\frac{13}{3}\\5\end{pmatrix}$ Coordinates of the point $=\left(\frac{11}{3},\frac{13}{3},5\right)$	
(iv) $\begin{pmatrix}1\\5\\\alpha\end{pmatrix}-\begin{pmatrix}-2\\-1\\0\end{pmatrix}=\begin{pmatrix}3\\6\\\alpha\end{pmatrix}=2\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix}$ Position vector of the air plane B after t minutes, $\overrightarrow{OB}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+t\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix},\quad 0\le t\le 4$ **Alternative method,** Let $\overrightarrow{OB}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+t\boldsymbol{d}$ where $0\le t\le 4$ When $t=2$, $\begin{pmatrix}1\\5\\\alpha\end{pmatrix}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+2\boldsymbol{d}\quad\Rightarrow\ \boldsymbol{d}=\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix}$ $\therefore\ \overrightarrow{OB}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+t\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix}$ If the two airplanes collide, $\overrightarrow{OA}=\overrightarrow{OB}$ for some $0\le t\le 4$	

$$\begin{pmatrix}2\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+t\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix}$$ $$t\begin{pmatrix}\frac{1}{2}\\1\\\frac{\alpha}{2}-3\end{pmatrix}=\begin{pmatrix}4\\2\\0\end{pmatrix}$$ There is no solution to the above equation. Hence the two airplanes will not collide.	
(v) At $t=4$, $\overrightarrow{OA}=\begin{pmatrix}2\\1\\0\end{pmatrix}+4\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}6\\9\\12\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}+4\begin{pmatrix}\frac{3}{2}\\3\\\frac{\alpha}{2}\end{pmatrix}=\begin{pmatrix}4\\11\\2\alpha\end{pmatrix}$. The plane contains all the points that each has equal distance to points A and B is the plane that contains the perpendicular bisector of AB and perpendicular to $\overrightarrow{AB}$. Let M be the Mid-point of AB, $$\overrightarrow{OM}=\frac{1}{2}\left[\begin{pmatrix}6\\9\\12\end{pmatrix}+\begin{pmatrix}4\\11\\2\alpha\end{pmatrix}\right]=\begin{pmatrix}5\\10\\6+\alpha\end{pmatrix}$$ $$\overrightarrow{AB}=\begin{pmatrix}4\\11\\2\alpha\end{pmatrix}-\begin{pmatrix}6\\9\\12\end{pmatrix}=\begin{pmatrix}-2\\2\\2\alpha-12\end{pmatrix}=2\begin{pmatrix}-1\\1\\\alpha-6\end{pmatrix}$$ Normal to the plane = $\begin{pmatrix}-1\\1\\\alpha-6\end{pmatrix}$ Vector equation of the plane: $$r\cdot\begin{pmatrix}-1\\1\\\alpha-6\end{pmatrix}=\begin{pmatrix}5\\10\\6+\alpha\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\\alpha-6\end{pmatrix}$$ $$=-5+10+\alpha^2-36$$ $$=\alpha^2-31$$ Cartesian equation of the plane: $$-x+y+(\alpha-6)z=\alpha^2-31$$ **Alternative method 1**: Let coordinates of point C be (x, y, z) and M be the Mid-point of AB.	

$\overrightarrow{AB} \perp \overrightarrow{CM} \Rightarrow \overrightarrow{AB} \cdot \overrightarrow{CM} = 0$ $\Rightarrow \begin{pmatrix} -1 \\ 1 \\ \alpha-6 \end{pmatrix} \cdot \begin{pmatrix} x-5 \\ y-10 \\ z-\alpha-6 \end{pmatrix} = 0$ $\Rightarrow -x+5+y-10+(\alpha-6)z-(\alpha^2-36)=0$ $\Rightarrow -x+y+(\alpha-6)z=\alpha^2-31$ **Alternative method 2**: At $t=4$, $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 9 \\ 12 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 11 \\ 2\alpha \end{pmatrix}$. Let coordinates of point C be (x, y, z) and $AC = BC$. $\Rightarrow \sqrt{(6-x)^2+(9-y)^2+(12-z)^2} = \sqrt{(4-x)^2+(11-y)^2+(2\alpha-z)^2}$ $\Rightarrow 124-4x+4y+(4\alpha-24)z=4\alpha^2$ $\Rightarrow -x+y+(\alpha-6)z=\alpha^2-31$	

2021 TJC Prelim Paper 2

Section A: Pure Mathematics [40 marks]

1 The functions f and g are defined by

$$f: x \mapsto 2-(x-2)^2 \quad \text{for} \quad x \le 0, x \in \mathbb{R},$$

and
$$g: x \mapsto \begin{cases} 4 & \text{for} \quad x \ge 3, x \in \mathbb{R} \\ x+1 & \text{for} \quad x < 3, x \in \mathbb{R} \end{cases}$$

(i) Define f^{-1} in a similar form. [3]

(ii) Explain why g does not have an inverse. [2]

(iii) Find the range of values of α such that $fg(\alpha)$ exists. [2]

2 It is given that $f(r) = \frac{1}{r!}$ where r is a positive integer.

(i) Show that $f(r) - f(r+1) = \frac{r}{(r+1)!}$. [1]

(ii) The sum of the first n terms of the series $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots$ is given by S_n. Find S_n and explain why the series converges. [4]

(iii) By considering **(ii)** and using the standard series from the List of Formulae (MF26), find the exact value of $\sum_{r=2}^{\infty} \frac{r+2}{r!}$. [4]

3 **(a)** In the triangle ABC, $AC = BC = 1$ and angle $ACB = \left(\frac{1}{2}\pi + \theta\right)$ radians. Given that θ is sufficiently small angle, show that

$$AB \approx \sqrt{2} + \frac{\sqrt{2}}{2}\theta - \frac{\sqrt{2}}{8}\theta^2$$ [5]

(b) Let $f(x) = \ln(2-x)$.

(i) Write down the first three non-zero terms of the Maclaurin series for $f(x)$, where $-2 \le x < 2$. [3]

Denote the answer to part **(i)** by $g(x)$.

(ii) Find the range of values of x for which the percentage error between the value of $g(x)$ and the value of $f(x)$ is within 10%. [2]

4 Relative to the origin O, the position vectors A and B are **a** and **b** respectively, where **a** and **b** are non-parallel vectors. The point C is the mid-point of OA and the point D is on the line segment AB produced such that $AD = 3AB$. The line CD meets OB at a point E.

(i) Find the position vector of E in terms of **b**. [4]

(ii) The point F divides the line segment OD in the ratio 1:2. Show that A, E and F are collinear, and find AE: AF. [4]

Given that the vector **a** is a unit vector perpendicular to **a** + 2**b** and angle AOB is $\frac{5\pi}{6}$.

(iii) Show that $|\mathbf{b}| = \frac{1}{\sqrt{3}}$. [2]

(iv) Find the exact area of triangle ACE. [4]

Section B: Statistics [60 marks]

5 For the events A, B and C it is given that, $\mathrm{P}(A) = p$, $\mathrm{P}(B) = 0.4$, $\mathrm{P}(A \cap C') = 0.2$ and $\mathrm{P}(A \cap B \cap C) = \mathrm{P}(A' \cap B' \cap C') = 0$.

It is also given that events A and B are independent, find the range of values of $\mathrm{P}(A \cap B)$. [5]

6 Correlation and Regression [8]

7 A manufacturer claims that the mean lifetime of the Alpha light bulbs he produces is at least 1200 hours. A random sample of 80 Alpha light bulbs is taken and the mean lifetime is found to be 1198.6 hours.

Assuming that the lifetime, T hours, of the Alpha light bulbs is normally distributed with variance 35 hours^2, test the manufacturer's claim at the 2.5% level of significance. You should state your hypotheses and define any symbols that you use. [4]

It is given that the mean lifetime of the Alpha light bulbs is 1200 hours. The manufacturer produces a new type of light bulbs known as Beta light bulbs. He claims a Beta light bulb has twice the mean lifetime of an Alpha light bulb. A random sample of 50 Beta light bulbs is taken, where the mean and standard deviation of the lifetime of this sample are k hours and 9.8 hours respectively. A test at the 5% significance level indicates that the manufacturer's claim is not valid for this sample of Beta light bulbs. Find the range of values of k, giving your answer correct to 2 decimal places. [5]

8 In a Junior College, it is known that the probability that a student is left-handed is 0.125. There are 25 students in a class. The number of students in a randomly chosen class who are left-handed is denoted by L.

(i) State, in context of this question, one assumption needed to model L by a binomial distribution. Hence find the probability that there are at most 3 students who are left-handed in a randomly chosen class. [2]

(ii) Find the probability that the number of students who are right-handed in a randomly chosen class is more than 85% of the class. [2]

(iii) A group of students is randomly chosen from a class. Find the probability that the fifth student chosen is the third student who is left-handed. [3]

(iv) There are n classes in the college. Find the minimum number of classes in the college such that the probability of having not more than 4 classes with at most 3 students who are left-handed in each class is less than 0.005. [3]

9 A random variable X has probability distribution given by

$$P(X=x)=\begin{cases} kx(6-x), & x=1,2,3,4,5 \\ 0, & \text{otherwise.} \end{cases}$$

(i) Show that $k=\frac{1}{35}$. [1]

(ii) Find the mean and variance of X. [3]

40 independent observations of X are taken.

(iii) Find the probability that the sum of the 40 observations of X exceeds 110. [3]

(iv) Find the probability that the maximum value observed is 5. [3]

10 A deck of twenty cards comprises four sets of coloured cards, namely yellow, red, blue and green cards. Each set is made up of five different numbered cards, from "1" to "5".

(i) Four cards are randomly dealt to every player, and there are a total of five players. In how many different ways can the cards be dealt so that a particular player has only red cards? [2]

(ii) Ten cards consisting of a set of yellow cards and a set of blue cards are arranged in a circle. Find the probability that all cards numbered "1" are next to each other and separated from the cards that are numbered "2". [3]

(iii) Three cards are drawn at random from the deck without replacement.

(a) Find the probability that the three cards have the same number. [2]

(b) Find the probability that at least two cards are of the same colour. [2]

(c) Find the probability that three cards are of green colour given that at least two cards are of the same colour. [3]

11 A group of ornithologists in a nature reserve is conducting a research on three endemic species of birds. They will tag and measure the beak lengths of the birds they have caught. After a large number of birds were tagged and measured, they found the beak lengths of the three species of birds have independent normal distributions. The table below gives the mean and standard deviation of the beak length of each species.

Species	**Mean**	**Standard Deviation**
A	40 mm	5 mm
B	60 mm	5 mm
C	40 mm	σ mm

Let A, B and C be the random variables, in mm, denoting the beak lengths of birds of species A, B and C respectively.

(i) The probability of a randomly selected bird of species A having a beak length at least l mm is at least 0.9. Find the range of values of l, correct to 1 decimal place. [2]

(ii) Find the value of k such that $\mathrm{P}(A \le k) = \mathrm{P}(B \ge k)$. [2]

(iii) One ornithologist randomly selected 3 birds of species A and 1 bird of species B. Find the probability that the difference between the total beak lengths of the birds of species A and twice the beak length of the bird of species B is at least 2 mm. [4]

It is found that 73.9% of birds of species C have beak lengths greater than 33.6 mm.

(iv) Find the value of σ. [2]

(v) Another ornithologist randomly selected a sample of 4 birds of species A and n birds of species C. After doing the measurements, he reported that the probability of exactly 3 birds of species A and 5 birds of species C from the sample, each has a beak length more than 33.6 mm is at least 0.0015. Find the maximum number of birds of species C in the sample. [4]

2021 TJC Prelim Paper 2 Suggested Solutions

Section A: Pure Mathematics [40 marks]

1 The functions f and g are defined by

$$\mathrm{f} : x \mapsto 2-(x-2)^2 \quad \text{for} \quad x \le 0, x \in \mathbb{R},$$

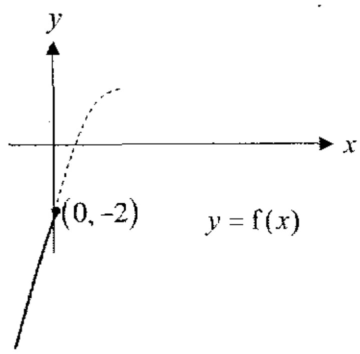
and

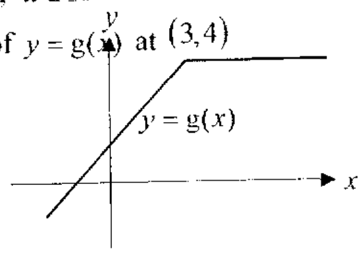
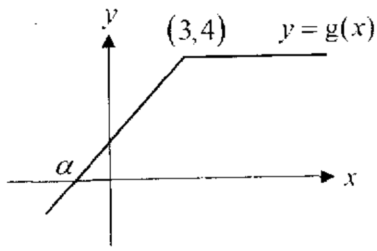
$$\mathrm{g} : x \mapsto \begin{cases} 4 & \text{for} \quad x \ge 3, x \in \mathbb{R} \\ x+1 & \text{for} \quad x < 3, x \in \mathbb{R} \end{cases}$$

(i) Define f^{-1} in a similar form. [3]

(ii) Explain why g does not have an inverse. [2]

(iii) Find the range of values of α such that $\mathrm{fg}(\alpha)$ exists. [2]

(i)	Let $y=\mathrm{f}(x)=2-(x-2)^2$, $x \le 0$ Then $(x-2)^2 = 2-y$ $\Rightarrow$ $x = 2+\sqrt{2-y}$ or $x=2-\sqrt{2-y}$ (reject since $x \le 0$) $\Rightarrow$ $\mathrm{f}^{-1}(x)=2-\sqrt{2-x}$  $D_{\mathrm{f}^{-1}} = R_{\mathrm{f}} = (-\infty, 2]$ Thus $\mathrm{f}^{-1} : x \mapsto 2-\sqrt{2-x}$ for $x \le -2$, $x \in \mathbb{R}$

(ii)	$g : x \mapsto \begin{cases} 4 & \text{for } x \geq 3, \ x \in \mathbb{R} \\ x+1 & \text{for } x < 3, \ x \in \mathbb{R} \end{cases}$ The line $y = 4$ cuts the graph of $y = g(x)$ at infinitely many points. $\Rightarrow$ g is not a 1-1 function $\Rightarrow$ g does not have an inverse  **Alternative solution** $g(3) = 4 = g(5)$ $\Rightarrow$ g is not a 1-1 function $\Rightarrow$ g does not have an inverse
(iii)	(iv) $fg(\alpha)$ not exist $\Rightarrow \quad g(\alpha) \in D_f = (-\infty, 0]$ $\Rightarrow \quad g(\alpha) \leq 0$ $\Rightarrow \quad \alpha + 1 \leq 0$ $\therefore \alpha \leq -1$ 

2 It is given that $f(r)=\frac{1}{r!}$ where r is a positive integer.

(i) Show that $f(r)-f(r+1)=\frac{r}{(r+1)!}$. [1]

(ii) The sum of the first n terms of the series $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots$ is given by S_n. Find S_n and explain why the series converges. [4]

(iii) By considering **(ii)** and using the standard series from the List of Formulae (MF26), find the exact value of $\sum_{r=2}^{\infty}\frac{r+2}{r!}$. [4]

(i)	$f(r)-f(r+1)$ $=\frac{1}{r!}-\frac{1}{(r+1)!}$ $=\frac{(r+1)-1}{(r+1)!}=\frac{r}{(r+1)!}$
(ii)	$S_n=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots \text{to } n \text{ terms}=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots\frac{n}{(n+1)!}$ $=\sum_{r=1}^{n}\frac{r}{(r+1)!}$ $=\sum_{r=1}^{n}\left(f(r)-f(r+1)\right)$ $f(1) - f(2)$ $+ f(2) - f(3)$ $+ f(3) - f(4)$ $+ \cdots$ $+ f(n-1) - f(n)$ $+ f(n) - f(n+1)$ $= f(1) - f(n+1)$ $=1-\frac{1}{(n+1)!}$

	As $n \to \infty, \frac{1}{(n+1)!} \to 0 \therefore S_n \to 1$ Therefore, the series converges. [B1]
(ii)	Consider $\sum_{r=2}^{n} \frac{r+2}{r!} \overset{\text{replace } r \text{ by } (r+1)}{=} \sum_{r+1=2}^{r+1=n} \frac{r+3}{(r+1)!}$ [B1] $\sum_{r=1}^{r=n-1} \frac{r+3}{(r+1)!} = \sum_{r=1}^{n} \frac{r}{(r+1)!} + 3\left(\frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n)!}\right)$ $\sum_{r=2}^{\infty} \frac{r+2}{r!} = \lim_{n\to\infty} \sum_{r=2}^{n} \frac{r+2}{r!}$ $= 1 + 3\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots - \left(1 + \frac{1}{1!}\right)\right)$ $= 1 + 3(e-2) = 3e-5$

3 **(a)** In the triangle ABC, $AC = BC = 1$ and angle $ACB = \left(\frac{1}{2}\pi + \theta\right)$ radians. Given that θ is sufficiently small angle, show that

$$AB \approx \sqrt{2} + \frac{\sqrt{2}}{2}\theta - \frac{\sqrt{2}}{8}\theta^2$$ [5]

(b) Let $f(x) = \ln(2-x)$.

(i) Write down the first three non-zero terms of the Maclaurin series for $f(x)$, where $-2 \le x < 2$. [3]

Denote the answer to part **(i)** by $g(x)$.

(ii) Find the range of values of x for which the percentage error between the value of $g(x)$ and the value of $f(x)$ is within 10%. [2]

(a)	Using cosine rule, $AB^2 = AC^2 + BC^2 - 2(AC)(BC)\cos ACB$ $= 1 + 1 - 2(1)(1)\cos\left(\frac{1}{2}\pi + \theta\right)$ $= 2 - 2\left[\cos\left(\frac{1}{2}\pi\right)\cos\theta - \sin\left(\frac{1}{2}\pi\right)\sin\theta\right]$ $= 2 + 2\sin\theta$
	When θ is sufficiently small, $AB \approx (2+2\theta)^{\frac{1}{2}}$ $= \sqrt{2}(1+\theta)^{\frac{1}{2}}$ $= \sqrt{2}\left[1 + \frac{1}{2}\theta + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}\theta^2 + \dots\right]$ $\approx \sqrt{2}\left(1 + \frac{1}{2}\theta - \frac{1}{8}\theta^2\right)$
(b) (i)	$f(x) = \ln(2-x)$ $= \ln 2 + \ln\left(1 - \frac{1}{2}x\right)$ $\approx \ln 2 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{24}x^3$
(b) (ii)	$g(x) = \ln 2 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{24}x^3$ $\left\lvert\frac{g(x)-f(x)}{f(x)} \times 100\%\right\rvert < 10\%$ $\Rightarrow \; -0.1 < \frac{g(x)-f(x)}{f(x)} < 0.1$ Using GC(graphically), $-1.73 < x < 0.738$ (correct to 3 s.f.)

4 Relative to the origin O, the position vectors A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. The point C is the mid-point of OA and the point D is on the line segment AB produced such that $AD = 3AB$. The line CD meets OB at a point E.

(i) Find the position vector of E in terms of $\mathbf{b}$. [4]

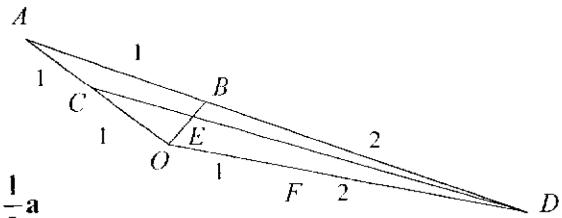
(ii) The point F divides the line segment OD in the ratio 1:2. Show that A, E and F are collinear, and find AE: AF. [4]

Given that the vector $\mathbf{a}$ is a unit vector perpendicular to $\mathbf{a} + 2\mathbf{b}$ and angle AOB is $\frac{5\pi}{6}$.

(iii) Show that $|\mathbf{b}| = \frac{1}{\sqrt{3}}$. [2]

(iv) Find the exact area of triangle ACE [4]

(i)



$\overrightarrow{OC} = \frac{1}{2}\mathbf{a}$

$\overrightarrow{CD} = \overrightarrow{CA} + \overrightarrow{AD}$

$= \frac{1}{2}\mathbf{a} + 3(\mathbf{b} - \mathbf{a})$

$= -\frac{5}{2}\mathbf{a} + 3\mathbf{b}$

$\overrightarrow{OE} = \lambda\mathbf{b}$

Equation of line CD: $\mathbf{r} = \frac{1}{2}\mathbf{a} + \mu\left(-\frac{5}{2}\mathbf{a} + 3\mathbf{b}\right)$

Since E lies on line CD, $\lambda\mathbf{b} = \frac{1}{2}\mathbf{a} + \mu\left(-\frac{5}{2}\mathbf{a} + 3\mathbf{b}\right)$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors,

Equating: $0 = \frac{1}{2} - \frac{5}{2}\mu \Rightarrow \mu = \frac{1}{5}$

$\lambda = 3\mu \Rightarrow \lambda = \frac{3}{5}$

Hence $\overrightarrow{OE} = \frac{3}{5}\mathbf{b}$

(ii)	$\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}$ $=\mathbf{a}+3(\mathbf{b}-\mathbf{a})$ $=-2\mathbf{a}+3\mathbf{b}$ $\overrightarrow{OF}=\frac{1}{3}\overrightarrow{OD}=\frac{1}{3}(-2\mathbf{a}+3\mathbf{b})$ $\overrightarrow{AE}=\overrightarrow{OE}-\overrightarrow{OA}=\frac{3}{5}\mathbf{b}-\mathbf{a}$ $\overrightarrow{AF}=\overrightarrow{OF}-\overrightarrow{OA}=\frac{1}{3}(-2\mathbf{a}+3\mathbf{b})-\mathbf{a}$ $=\mathbf{b}-\frac{5}{3}\mathbf{a}$ $=\frac{5}{3}\left(\frac{3}{5}\mathbf{b}-\mathbf{a}\right)=\frac{5}{3}\overrightarrow{AE}$ Hence A, E, F are collinear. $AE:AF=3:5$
(iii)	$\mathbf{a}.\mathbf{b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}\lvert\mathbf{b}\rvert$ $\mathbf{a}\cdot(\mathbf{a}+2\mathbf{b})=0$ $\lvert\mathbf{a}\rvert^2+2\mathbf{a}\cdot\mathbf{b}=0$ $1+2\left(-\frac{\sqrt{3}}{2}\right)\lvert\mathbf{b}\rvert=0$ $\lvert\mathbf{b}\rvert=\frac{1}{\sqrt{3}}$ (shown)
(iv)	Area of triange $ACE=\frac{1}{2}\left\lvert\overrightarrow{AC}\times\overrightarrow{AE}\right\rvert$ $=\frac{1}{2}\left\lvert-\frac{1}{2}\mathbf{a}\times\left(\frac{3}{5}\mathbf{b}-\mathbf{a}\right)\right\rvert$ $=\frac{3}{20}\lvert-\mathbf{a}\times\mathbf{b}\rvert$ $=\frac{3}{20}\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\frac{5\pi}{6}$ $=\frac{3}{20}\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{2}\right)=\frac{3}{40\sqrt{3}}=\frac{\sqrt{3}}{40}$ units^2

5 For the events A, B and C it is given that, $P(A)=p$, $P(B)=0.4$, $P(A\cap C)=0.2$ and $P(A\cap B\cap C)=P(A'\cap B'\cap C')=0$.

It is also given that events A and B are independent, find the range of values of $P(A\cap B)$. [5]

Since A and B are independent events, we have

$$P(A\cap B)=P(A)\cdot P(B)=0.4p$$

$\because 0\le p\le 1 \Rightarrow 0\le 0.4p\le 0.4$

Also, $p\ge 0.2+0.4p \Rightarrow 0.6p\ge 0.2$

$\Rightarrow p\ge \frac{1}{3}$

$\Rightarrow 0.4p\ge \frac{1}{3}(0.4)=\frac{2}{15}$

$\therefore\ \frac{2}{15}\le P(A\cap B)\le \frac{2}{5}$

Since A and B are independent events, we have

$$P(A\cap B)=P(A)\cdot P(B)=0.4p$$

Maximum $P(A\cap B)$ occurs when $p=1$

Therefore maximum $P(A\cap B)=0.4$

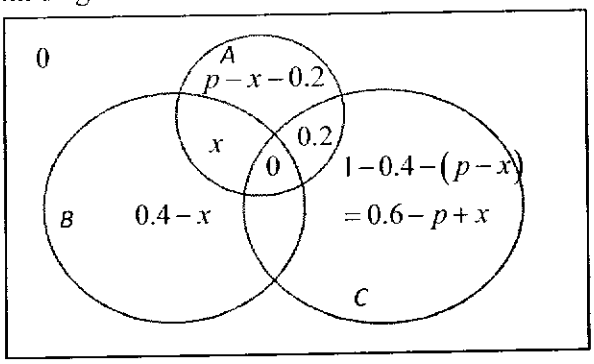
Minimum $P(A\cap B)$ occurs when $p-0.4p=0.2\Rightarrow p=\frac{1}{3}$

Therefore minimum $P(A\cap B)=0.4\left(\frac{1}{3}\right)=\frac{2}{15}$

$\therefore\ \frac{2}{15}\le P(A\cap B)\le \frac{2}{5}$

Let $P(A\cap B)=x$

The Venn diagram is as follow:



Since A and B are independent events, we have

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$$

$$\Rightarrow \quad x = (p)(0.4) = 0.4p$$

$$\Rightarrow \quad p = 2.5x$$

From Venn Diagram, we have

$0 \le 0.4 - x \le 1$ and $0 \le 0.6 - p + x \le 1$ and $0 \le x \le 1$ and $0 \le p - x - 0.2 \le 1$

$\Rightarrow -0.6 \le x \le 0.4$ and $0 \le 0.6 - 2.5x + x \le 1$

and $0 \le x \le 1$ and $0 \le 2.5x - x - 0.2 \le 1$

$\Rightarrow -\frac{3}{5} \le x \le \frac{2}{5}$ and $-\frac{4}{15} \le x \le \frac{2}{5}$ and $0 \le x \le 1$ and $\frac{2}{15} \le x \le \frac{12}{15}$

$$\Rightarrow \quad \frac{2}{15} \le x \le \frac{2}{5}$$

6 It is generally accepted that a person's diet and cardiorespiratory fitness affects his cholesterol levels. The results of a study on the relationship between the cholesterol levels, C mmol/L, and cardiorespiratory fitness, F, measured in suitable units, on 8 individuals with similar diets are given in the following table.

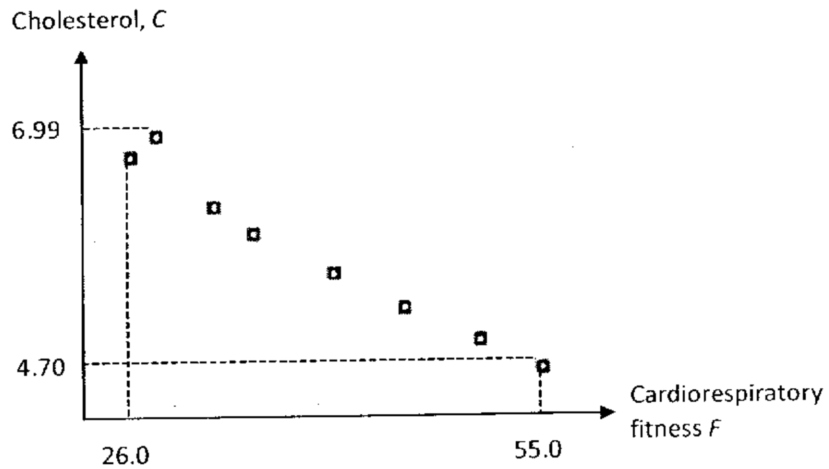
Cardiorespiratory Fitness (F units)	55.0	50.7	45.3	40.2	34.7	31.9	27.9	26.0
Cholesterol (C mmol/L)	4.70	4.98	5.30	5.64	6.04	6.30	6.99	6.79

(i) Draw a scatter diagram of these data. Suppose that the relationship between F and C is modelled by an equation of the form $\ln C = aF + b$, where a and b are constants. Use your diagram to explain whether a is positive or negative. [3]

(ii) Find the product moment correlation coefficient between ln C and F, and the constants a and b for the model in part **(i)**. [3]

(iii) Bronz is a fitness instructor. His cardiorespiratory fitness is 52.0 units. Estimate Bronz's cholesterol level using the model in **(i)** and the values of a and b in part **(ii)**. Comment on the reliability of the estimate. [2]

(i)



$\ln C = aF + b \Rightarrow C = e^{aF+b}$

From the scatter diagram, as F increases, C decreases at a decreasing rate, hence $a < 0$.

(ii) Product moment correlation $r = -0.992$ (3 sf)

Model is $\ln C = -0.013371\,F + 2.2772$ (5 sf)

$\ln C = -0.0134\,F + 2.28$ (3 sf)

(iii) $\ln C = -0.013371\,(52.0) + 2.2772$

$C = e^{1.581908} = 4.86$

Estimate of Bronz's cholesterol level is 4.86 mmol/L

Since $r = -0.992$ is close to -1 and Bronz's cardiorespiratory fitness level, F = 52.0 lies within the data range of $(26.0, 55.0)$, the estimate is reliable.

7 A manufacturer claims that the mean lifetime of the Alpha light bulbs he produces is at least 1200 hours. A random sample of 80 Alpha light bulbs is taken and the mean lifetime is found to be 1198.6 hours.
Assuming that the lifetime, T hours, of the Alpha light bulbs is normally distributed with variance 35 hours^2, test the manufacturer's claim at the 2.5% level of significance. You should state your hypotheses and define any symbols that you use. [4]

It is given that the mean lifetime of the Alpha light bulbs is 1200 hours. The manufacturer produces a new type of light bulbs known as Beta light bulbs. He claims a Beta light bulb has twice the mean lifetime of an Alpha light bulb. A random sample of 50 Beta light bulbs is taken, where the mean and standard deviation of the lifetime of this sample are k hours and 9.8 hours respectively. A test at the 5% significance level indicates that the manufacturer's claim is not valid for this sample of Beta light bulbs. Find the range of values of k, giving your answer correct to 2 decimal places. [5]

	Let T be the lifetime of a randomly chosen Alpha light bulb in hours and μ be the population mean lifetime. $H_0 : \mu = 1200$ $H_1 : \mu < 1200$ Level of significance: 2.5% Under H_0, $\overline{T} \sim N\left(1200, \frac{35}{80}\right)$ and test statistics $Z = \frac{\overline{T} - 1200}{\sqrt{35/80}} \sim N(0,1)$. Using GC with $\overline{t} = 1198.6$, p-value $= 0.0171$ [B1] Since p-value < 0.025, we reject H_0. There is sufficient evidence at the 2.5% level of significance to conclude the mean lifetime of the Alpha light bulbs is less than 1200 hours, i.e. that the manufacturer's claim is not valid.
	Let X be the lifetime of a randomly chosen Beta light bulb in hours. Unbiased estimate of the population variance, $s^2 = \frac{50}{49}(9.8^2) = 98$ $H_0 : \mu = 2400$ $H_1 : \mu \neq 2400$ Level of significance: 5%

	Under H_0, since $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim N\left(2400, \frac{98}{50}\right)$ approximately and test statistics $Z = \frac{\overline{X} - 2400}{\sqrt{98/50}} \sim N(0,1)$ approximately. $z_{cal} = \frac{k - 1200}{\sqrt{98/50}}$ For manufacturer's claim to be not valid at the 5% level of significance, z_{cal} is within the critical region, i.e. $\frac{k - 2400}{\sqrt{98/50}} \le -1.9599$ or $\frac{k - 1200}{\sqrt{98/50}} \ge 1.9599$ Considering 2-tailed critical region $\therefore$ $k \le 2397.26$ or $k \ge 2402.74$

8 In a Junior College, it is known that the probability that a student is left-handed is 0.125. There are 25 students in a class. The number of students in a randomly chosen class who are left-handed is denoted by L.

(i) State, in context of this question, one assumption needed to model L by a binomial distribution. Hence find the probability that there are less than 3 students who are left-handed in a randomly chosen class. [2]

(ii) Find the probability that the number of students who are right-handed in a randomly chosen class is more than 85% of the class. [2]

(iii) A group of students is randomly chosen from a class. Find the probability that the fifth student chosen is the third student who is left-handed. [3]

(iv) There are n classes in the college. Find the minimum number of classes in the college such that the probability of having not more than 4 classes with less than 3 students who are left-handed in each class is less than 0.005. [3]

(i)	Whether a student is left-handed or not is independent of whether any other students is left-handed. The probability of a student is left-handed is the same throughout the sample of 25 students.
	$L \sim \mathrm{B}(25, 0.125)$ $\mathrm{P}(L<3) = \mathrm{P}(L \le 2) = 0.379609...$ $=0.380$ (3 s.f.)
(ii)	Let R be the number of right handed, out of 25 students $R \sim \mathrm{B}(25, 0.875)$ $\mathrm{P}(R > 0.85 \times 25)$ $= \mathrm{P}(R > 21.25) = 1 - \mathrm{P}(R \le 21)$ $= 0.618$ (3 s.f.)
(iii)	Let H be the number of left handed out of 4 students $H \sim \mathrm{B}(4, 0.125)$ $= P(H=2) \times 0.125$ $= 0.00897$
(iv)	Let C be the number of classes, out of n, with less than 3 left handed. $C \sim \mathrm{B}(n, 0.37961)$ $\mathrm{P}(C \le 4) < 0.005$ When $\mathrm{n} = 28$, $\mathrm{P}(C \le 4) = 0.0592 > 0.005$ When $\mathrm{n} = 29$, $\mathrm{P}(C \le 4) = 0.00421 < 0.005$ Therefore minimum number of classes is 29.

9 A random variable X has probability distribution given by

$$P(X=x)=\begin{cases} kx(6-x), & x=1,2,3,4,5 \\ 0, & \text{otherwise.} \end{cases}$$

(i) Show that $k=\frac{1}{35}$. [1]

(ii) Find the mean and variance of X. [3]

40 independent observations of X are taken.

(iii) Find the probability that the sum of the 40 observations of X exceeds 110. [3]

(iv) Find the probability that the maximum value observed is 5. [3]

(i)	$\sum P(X=x)=1$ $5k+8k+9k+8k+5k=1$ $35k=1 \Rightarrow k=\frac{1}{35}$
(ii)	$E(X)=3$ (by symmetry) $\text{Var}(X)=E(X^2)-[E(X)]^2$ $=1^2(5k)+2^2(8k)+3^2(9k)+4^2(8k)+5^2(5k)-3^2$ $=k(371)-3^2$ $=\frac{1}{35}(371)-3^2=1.6$
(iii)	Let $S=X_1+X_2+\ldots+X_{30}$. Since $n=40$ is large, by Central Limit Theorem, $S \sim N(40\times3, 40\times1.6)$ i.e. $S \sim N(120,64)$ approximately. $P(S>110)=0.894$
(iv)	P(max observed = 5) = $1-P(\text{max observed } \le 4)$ **[M1]** $=1-P(X_1\le4)P(X_2\le4)\ldots P(X_{40}\le4)$ $=1-(1-P(X=5))^{40}$ $=1-\left(1-\frac{1}{7}\right)^{40}$ $=0.998$ (to 3 s.f)

10 A deck of twenty cards comprises four sets of coloured cards, namely yellow, red, blue and green cards. Each set is made up of five different numbered cards, from "1" to "5".

(i) Four cards are randomly dealt to every player, and there are a total of five players. In how many different ways can the cards be dealt so that a particular player has only red cards? [2]

(ii) Ten cards consisting of a set of yellow cards and a set of blue cards are arranged in a circle. Find the probability that all cards numbered "1" are next to each other and separated from the cards that are numbered "2". [3]

(iii) Three cards are drawn at random from the deck without replacement.

(a) Find the probability that the three cards have the same number. [2]

(b) Find the probability that at least two cards are of the same colour. [2]

(c) Find the probability that three cards are of green colour given that at least two cards are of the same colour. [3]

(i)	Number of ways $=\binom{5}{4}\times\binom{16}{4}\times\binom{12}{4}\times\binom{8}{4}\times\binom{4}{4}=315315000$	
(ii)	**Mtd1** Number of ways to form ' 11 ' where is not 2 $=\binom{6}{2}2!2!=60$ Required number of ways $=60\times(7-1)!=43200$ **Mtd2** Case1: '2's are separated Number of ways $=2!(6-1)!\binom{6}{3}3!=28800$ Case2: '2's are together Number of ways $=2!2!(6-1)!\binom{6}{2}2!=14400$ Total number of ways = 28800 + 14400 = 43200 Required probability $=\frac{43200}{(10-1)!}=\frac{5}{42}$	

(iii) **(a)**	P(3 cards have the same number) $= \dfrac{\binom{4}{3}\times\binom{5}{1}}{\binom{20}{3}} = \dfrac{20}{1140} = \dfrac{1}{57}$ OR $\dfrac{20}{20}\times\dfrac{3}{19}\times\dfrac{2}{18}$
(b)	1 - P(all different colours) $= 1 - \dfrac{\binom{4}{3}\binom{5}{1}\binom{5}{1}\binom{5}{1}}{\binom{20}{3}} = \dfrac{32}{57}$ OR $1 - \dfrac{20}{20}\times\dfrac{15}{19}\times\dfrac{10}{18}$ OR Case 1: 3 cards same colour $\dfrac{\binom{4}{1}\times\binom{5}{3}}{\binom{20}{3}} = \dfrac{40}{1140} = \dfrac{2}{57}$ Case 2: 2 cards same colour $\dfrac{\binom{4}{2}\times\binom{5}{2}\times\binom{5}{1}\times 2}{\binom{20}{3}} = \dfrac{600}{1140} = \dfrac{10}{19}$ Required probability $= \dfrac{2}{57} + \dfrac{10}{19} = \dfrac{32}{57}$
(c)	P(3 cards are green | at least 2 cards are of the same colour) $= \dfrac{\text{P(3 cards are green and at least 2 cards are of the same colour)}}{\text{P(at least 2 cards are of the same colour)}}$ $= \dfrac{\text{P(3 cards are green)}}{\text{P(at least 2 cards are of the same colour)}}$ $= \dfrac{\dfrac{\binom{5}{3}}{\binom{20}{3}}}{\dfrac{32}{57}} = \dfrac{\dfrac{10}{1140}}{\dfrac{32}{57}} = \dfrac{1}{64}$

	OR $\dfrac{\frac{5}{20}\times\frac{4}{19}\times\frac{3}{18}}{\frac{32}{57}}=\dfrac{1}{64}$

11 A group of ornithologists in a nature reserve is conducting a research on three endemic species of birds. They will tag and measure the beak lengths of the birds they have caught. After a large number of birds were tagged and measured, they found the beak lengths of the three species of birds have independent normal distributions. The table below gives the mean and standard deviation of the beak length of each species.

Species	**Mean**	**Standard Deviation**
A	40 mm	5 mm
B	60 mm	5 mm
C	40 mm	σ mm

Let A, B and C be the random variables, in mm, denoting the beak lengths of birds of species A, B and C respectively.

(i) The probability of a randomly selected bird of species A having a beak length at least l mm is at least 0.9. Find the range of values of l, correct to 1 decimal place. [2]

(ii) Find the value of k such that $\mathrm{P}(A \le k) = \mathrm{P}(B \ge k)$. [2]

(iii) One ornithologist randomly selected 3 birds of species A and 1 bird of species B. Find the probability that the difference between the total beak lengths of the birds of Species A and twice the beak length of the bird of species B is at least 2 mm. [4]

It is found that 73.9% of birds of species C have beak lengths greater than 33.6 mm.

(iv) Find the value of σ. [2]

(v) Another ornithologist randomly selected a sample of 4 birds of species A and n birds of species C. After doing the measurements, he reported that the probability of exactly 3 birds of species A and 5 birds of species C from the sample, each has a beak length more than 33.6 mm is at least 0.0015. Find the maximum number of birds of species C in the sample. [4]

(i)	$A \sim N(40, 5^2)$ $P(A \geq l) \geq 0.9$ $\Rightarrow$ Using a GC, $P(A \geq 33.592) = 0.9$ $\therefore 0 \leq l \leq 33.6$ (3 sf)
(ii)	By using symmetry (graphical, both distributions have same standard deviation), $k = 50$ Or $P\left(Z \leq \frac{k-40}{5}\right) = P\left(Z \geq \frac{k-60}{5}\right)$ $-\left(\frac{k-40}{5}\right) = \frac{k-60}{5}$ $\Rightarrow -k + 40 = k - 60$ $\Rightarrow 2k = 100 \therefore k = 50$
(iii)	$B \sim N(60, 5^2)$ $A_1 + A_2 + A_3 - 2B \sim N\left(3 \times 40 - 2 \times 60, 3(5^2) + 2^2(5^2)\right)$ i.e. $A_1 + A_2 + A_3 - 2B \sim N(0, 175)$ Required probability $= P(\|A_1 + A_2 + A_3 - 2B\| \geq 2)$ $= 2P(A_1 + A_2 + A_3 - 2B \geq 2)$ $= 0.880$ (3 s.f.)
(iv)	$P(C > 33.6) = 0.739$ $P\left(Z > \frac{33.6-40}{\sigma}\right) = 0.739$ $\frac{33.6-40}{\sigma} = -0.64026..$ $\therefore \sigma = 10.0$ (3 s.f.)
(v)	$\binom{4}{3} P(A > 33.6)^3 P(A \leq 33.6)^1 \binom{n}{5} P(C > 33.6)^5 P(C \leq 33.6)^{n-5} \geq 0.0015$ From GC, $5 \leq n \leq 13$ Therefore, maximum number of birds of species C is 13