BP~303

Class	Index Number

Name : ______________________________

METHODIST GIRLS' SCHOOL

Founded in 1887

MGS

PRELIMINARY EXAMINATION 2024
Secondary 4

Monday
12 August 2024

ADDITIONAL MATHEMATICS
Paper 1

4049/01
2 h 15 min

Candidates answer on the Question Paper.
No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST
Write your class, index number and name in the spaces at the top of this page.
Write in dark blue or black pen
You may use a HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figure, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

/ 90

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)...(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}bc\sin A$$

1 **(i)** Express $y = 4x^2 + 24x + 49$ in the form $a(x+b)^2 + c$. [2]

(ii) Hence, state the maximum value of $\frac{1}{y}$ and the corresponding value of x at which $\frac{1}{y}$ is maximum. [2]

2 By using a suitable substitution, solve the equation $e^{2x} = 4(1 - e^{-2x})$. [3]

3 Show that the equation $x^2 + px + 2p = 2x + 8$ has real roots for all real values of p.

[3]

4 A and B are acute angles such that $\sin(A-B)=\frac{1}{4}$ and $\sin A\cos B=\frac{3}{5}$.

Without using a calculator, find the value of

(i) $\cos A\sin B$, [2]

(ii) $\sin(A+B)$, [1]

(iii) $2\tan A\cot B$. [2]

5 A curve has the equation $y=\dfrac{e^{2x+1}}{x+3}$, where $x \neq -3$.

(i) Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [2]

(ii) Hence, stating your reasons clearly, determine if the graph of $y=\dfrac{e^{2x+1}}{x+3}$ is decreasing or increasing for which $x > 0$. [2]

6 Find the values of k for which the curve $y = 2x^2 + (3k+1)x + 4$ lies entirely above the line $y = 2x - 3k^2 + 5$. [5]

7 A curve is such that $f''(x) = 6x - 10$. The tangent to the curve $y = f(x)$ at the point $(1, -2)$ passes through the point $(0, 13)$. Find the equation of the curve. [6]

8 **(a)** Given that $\log_x p = 9$ and $\log_x q = 6$, find the value of $\log_{\frac{p}{q}} x^2$. [3]

(b) Solve the equation $\lg(3x+1)+\lg(2x-1)=2-\lg 2$ and explain why there is only one solution. [5]

9 Two cubic expressions are defined by $f(x) = x^3 + (a-3)x + 2b$ and $g(x) = 3x^3 + x^2 + 5ax + 4b$ where a and b are constants.

(i) Given that $f(x)$ and $g(x)$ have a common factor $(x + 3)$, show that $a = -4$ and find the value of b. [3]

(ii) Using the values of a and b, factorise $f(x)$ completely. Hence, show that $f(x)$ and $g(x)$ have two common factors. [4]

10 It is given that $y = 4x\sin 2x$.

(i) Show that $\frac{dy}{dx}$ can be expressed in the form $ax\cos 2x + b\sin 2x$, where a and b are constants. [2]

(ii) Given that x is increasing at $\frac{2}{5}$ radians per second, find the rate of change of y with respect to time when $x = \frac{\pi}{2}$. [2]

(iii) Using your answer in part **(i)**, evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos 2x \, \mathrm{d}x$, leaving your answer in terms of π. [4]

(iv) Explain what your answer in part **(iii)** implies about the curve $y = x\cos 2x$. [1]

11 The table shows experimental values of two variables, x and y.

x	2	4	6	8
y	2.25	0.81	0.47	0.33

(i) On the graph paper, plot xy against $\frac{1}{x}$ and draw a straight line graph. [3]

Using your graph,

(ii) estimate the values of x and y for which $x = \frac{3}{y}$, [2]

(iii) express y in terms of x. [4]

12 A particle travels in a straight line, so that t seconds after passing a fixed point O, its velocity, v m/s, is given by $v = 22 + 7t - 2t^2$. The particle comes to instantaneous rest at P. Find

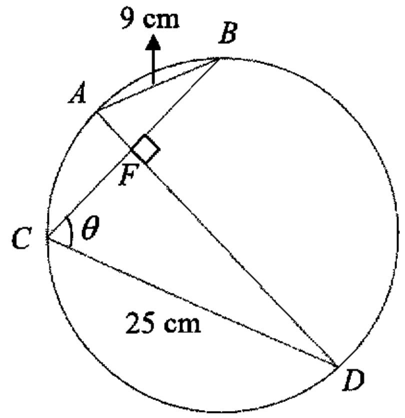
(i) the velocity of the particle when the acceleration is zero, [2]

(ii) the value of t when the particle is at P, [2]

(iii) the distance OP, [2]

(iv) the total distance travelled by the particle in the interval $t = 0$ to $t = 9$. [2]

13 In the diagram, AB and CD are chords of a circle, with $AB = 9$ cm and $CD = 25$ cm. It is given that AD and BC intersect each other at 90° at F and $\angle BCD = \theta$, where θ varies.



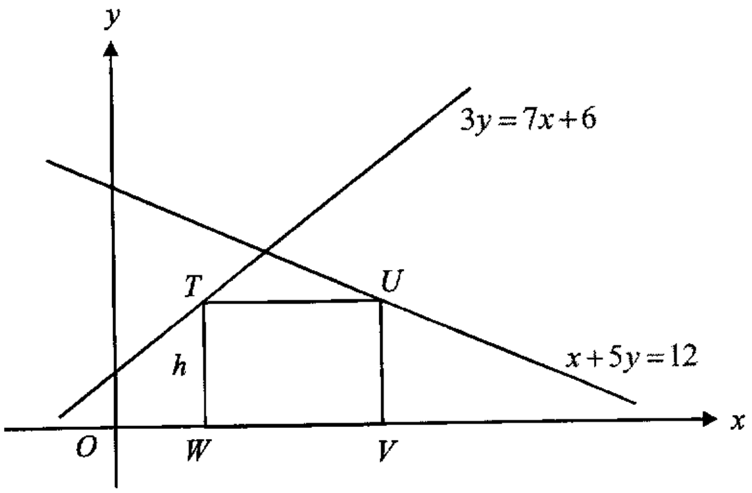
(i) Prove that $AD = 9\cos\theta + 25\sin\theta$. [2]

(ii) Express AD in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [3]

(iii) Find the values of θ when $AD = 26$ cm. [3]

(iv) Calculate the angle θ for which AD is the diameter of the circle. [2]

14 The diagram shows parts of the graphs of $3y = 7x + 6$ and $x + 5y = 12$. T is a point on $3y = 7x + 6$ and U is a point on $x + 5y = 12$. V and W are two points on the x-axis such that $TUVW$ is a rectangle.



It is given that $TW = h$ units.

(i) Find the x-coordinates of T and U in terms of h. [2]

(ii) Hence, show that the area of the rectangle $TUVW$, represented by A square units, is given by $A = \frac{90}{7}h - \frac{38}{7}h^2$. [2]

(iii) Given that h can vary, find the stationary value of A and determine whether this value of A is a maximum or a minimum. [5]

End of Paper

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PAPER 1
SOLUTIONS

1 **(i)** Express $y = 4x^2 + 24x + 49$ in the form $a(x+b)^2 + c$. [2]

$4x^2 + 24x + 49$
$= 4(x^2 + 6x) + 49$
$= 4[(x+3)^2 - 9] + 49$
$= 4(x+3)^2 + 13$

(ii) Hence, state the maximum value of $\frac{1}{y}$ and the corresponding value of x at which $\frac{1}{y}$ is maximum. [2]

Maximum $\frac{1}{y} = \frac{1}{13}$

It occurs when $x = -3$

2 By using a suitable substitution, solve the equation $e^{2x} = 4(1 - e^{-2x})$. [3]

$$e^{2x} = 4(1 - e^{-2x})$$
$$e^{2x} = 4 - \frac{4}{e^{2x}}$$
$$y = 4 - \frac{4}{y}$$
$$y^2 = 4y - 4$$
$$y^2 - 4y + 4 = 0$$
$$(y-2)^2 = 0$$
$$y = 2$$
$$e^{2x} = 2$$
$$x = \frac{\ln 2}{2}$$
$$= 0.347$$

3 Show that the equation $x^2 + px + 2p = 2x + 8$ has real roots for all values of p. [3]

$x^2 + px + 2p = 2x + 8$

$x^2 + (p-2)x + 2p - 8 = 0$

Discriminant

$=(p-2)^2 - 4(1)(2p-8)$

$= p^2 - 4p + 4 - 8p + 32$

$= p^2 - 12p + 36$

$= (p-6)^2$

For all values of p, $(p-6)^2 \geq 0$, hence roots of the equation $x^2 + px + 2p = 2x + 8$ is always real.

4 A and B are acute angles such that $\sin(A-B)=\frac{1}{4}$ and $\sin A\cos B=\frac{3}{5}$.

Without using a calculator, find the value of

(i) $\cos A\sin B$, [2]

$$\sin(A-B)=\frac{1}{4}$$
$$\sin A\cos B-\cos A\sin B=\frac{1}{4}$$
$$\frac{3}{5}-\cos A\sin B=\frac{1}{4}$$
$$\cos A\sin B=\frac{7}{20}$$

(ii) $\sin(A+B)$, [1]

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$
$$\sin(A+B)=\frac{3}{5}+\frac{7}{20}=\frac{19}{20}$$

(iii) $2\tan A\cot B$. [2]

$$2\tan A\cot B$$
$$=2\times\frac{\sin A}{\cos A}\times\frac{\cos B}{\sin B}$$
$$=2\frac{\left(\frac{3}{5}\right)}{\left(\frac{7}{20}\right)}$$
$$=\frac{24}{7}$$

5 A curve has the equation $y=\dfrac{e^{2x+1}}{x+3}$, where $x \neq -3$.

(i) Find an expression for $\dfrac{dy}{dx}$. [2]

$$y=\frac{e^{2x+1}}{x+3}$$
$$\frac{dy}{dx}=\frac{(x+3)\left(2e^{2x+1}\right)-e^{2x+1}}{(x+3)^2}$$
$$=\frac{e^{2x+1}(2x+5)}{(x+3)^2}$$

(ii) Hence, stating your reasons clearly, determine if the graph of $y=\dfrac{e^{2x+1}}{x+3}$ is decreasing or increasing for which $x>0$. [2]

For $x>0$,

$(x+3)^2>0$,

$e^{2x+1}>0$,

$(2x+5)>0$

Thus $\dfrac{dy}{dx}>0$ and y is an increasing function.

6 Find the values of k for which the curve $y = 2x^2 + (3k+1)x + 4$ lies entirely above the line $y = 2x - 3k^2 + 5$. [5]

$$2x^2 + (3k+1)x + 4 > 2x - 3k^2 + 5$$
$$2x^2 + 3kx + x + 4 - 2x + 3k^2 - 5 > 0$$
$$2x^2 + (3k-1)x + 3k^2 - 1 > 0$$
$$b^2 - 4ac < 0$$
$$(3k-1)^2 - 4(2)(3k^2 - 1) < 0$$
$$9k^2 - 6k + 1 - 24k^2 + 8 < 0$$
$$-15k^2 - 6k + 9 < 0$$
$$5k^2 + 2k - 3 > 0$$
$$(5k-3)(k+1) > 0$$
$$k > \frac{3}{5} \quad \text{or} \quad k < -1$$

7 A curve is such that $f''(x)=6x-10$. The tangent to the curve $y=f(x)$ at the point $(1,-2)$ passes through the point $(0,\ 13)$. Find the equation of the curve. [6]

$$f''(x)=6x-10$$
$$f'(x)=\int 6x-10\ \mathrm{d}x$$
$$=3x^2-10x+c$$

gradient of tangent $=\dfrac{-2-13}{1-0}=-15$

at $x=1$, $f'(x)=-15$

$$-15=3(1)-10(1)+c$$
$$-8=c$$

$$f'(x)=3x^2-10x-8$$
$$f(x)=\int 3x^2-10x-8\ \mathrm{d}x$$
$$=x^3-5x^2-8x+d$$

at $(1,-2)$,

$$-2=1-5(1)-8+d$$
$$10=d$$

$$y=x^3-5x^2-8x+10$$

8 **(a)** Given that $\log_x p = 9$ and $\log_x q = 6$, find the value of $\log_{\frac{p}{q}} x^2$. [3]

$$\log_{\frac{p}{q}} x^2$$
$$= \frac{\log_x x^2}{\log_x \frac{p}{q}}$$
$$= \frac{2}{\log_x p - \log_x q}$$
$$= \frac{2}{9-6}$$
$$= \frac{2}{3}$$

(b) Solve the equation $\lg(3x+1) + \lg(2x-1) = 2 - \lg 2$ and explain why there is only one solution. [5]

$$\lg(3x+1) + \lg(2x-1) = 2 - \lg 2$$
$$\lg(3x+1)(2x-1) = \lg \frac{100}{2}$$
$$(3x+1)(2x-1) = 50$$
$$6x^2 - 3x + 2x - 1 - 50 = 0$$
$$6x^2 - x - 51 = 0$$
$$(6x+17)(x-3) = 0$$
$$x = -\frac{17}{6} \quad \text{or} \quad 3$$

$3x+1>0$ and $2x-1>0$

$x > -\frac{1}{3}$ $\qquad$ $x > \frac{1}{2}$

Since $x > \frac{1}{2}$, $x = 3$ is the only solution.

9 Two cubic expressions are defined by $f(x) = x^3 + (a-3)x + 2b$ and $g(x) = 3x^3 + x^2 + 5ax + 4b$ where a and b are constants.

(i) Given that $f(x)$ and $g(x)$ have a common factor $(x+3)$, show that $a = -4$ and find the value of b. [3]

$(-3)^3 + (a-3)(-3) + 2b = 0$

$-27 - 3a + 9 + 2b = 0$

$-3a + 2b = 18$ ---------- (1)

$3(-3)^3 + (-3)^2 + 5a(-3) + 4b = 0$

$-81 + 9 - 15a + 4b = 0$

$-15a + 4b = 72$ ---------- (2)

$(1)\times 2,\ -6a + 4b = 36$ ---------- (3)

(2) - (3), $-9a = 36$

$a = -4$

$b = 3$

(ii) Using the values of a and b, factorise $f(x)$ completely. Hence, show that $f(x)$ and $g(x)$ have two common factors. [4]

$f(x) = x^3 - 7x + 6$

$x^3 - 7x + 6 = (x+3)(x^2 + px + 2)$

compare coeff of x^2, $0 = p + 3$

$p = -3$

$x^3 - 7x + 6 = (x+3)(x^2 - 3x + 2)$

$=(x+3)(x-1)(x-2)$

$g(1) = 3(1)^3 + 1^2 + 5(-4)(1) + 4(3) = -4$

$(x-1)$ is not a factor of $g(x)$

$g(2) = 3(2)^3 + 2^2 + 5(-4)(2) + 4(3) = 0$

$(x-2)$ is a factor of $g(x)$

Thus, $f(x)$ and $g(x)$ have 2 common factors $(x+3)$ and $(x-2)$

10 It is given that $y = 4x\sin 2x$.

(i) Show that $\frac{dy}{dx}$ can be expressed in the form $ax\cos 2x + b\sin 2x$, where a and b are constants. [2]

$$y = 4x\sin 2x$$
$$\frac{dy}{dx} = 4x(\cos 2x)(2) + 4\sin 2x$$
$$\frac{dy}{dx} = 8x\cos 2x + 4\sin 2x$$

(ii) Given that x is increasing at $\frac{2}{5}$ radians per second, find the rate of change of y with respect to time when $x = \frac{\pi}{2}$. [2]

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
$$= (8 \times \left(\frac{\pi}{2}\right)\cos\pi + 4\sin\pi) \times \frac{2}{5}$$
$$= -\frac{8\pi}{5} \text{ radians/sec}$$

(iii) Using your answer in part **(i)**, evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos 2x \, dx$, leaving your answer in terms of π. [4]

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 8x\cos 2x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\sin 2x \, dx = \left[4x\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos 2x \, dx + \frac{1}{2}\left[\frac{-\cos 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \left[\frac{1}{2}x\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos 2x \, dx = \left[\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos 2x \, dx = \left(\frac{\pi}{4}\sin\pi + \frac{1}{4}\cos\pi\right) - \left(\frac{\pi}{8}\sin\frac{\pi}{2} + \frac{1}{4}\cos\frac{\pi}{2}\right)$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos 2x \, dx = -\frac{1}{4} - \frac{\pi}{8}$$

(iv) Explain what your answer in part **(iii)** implies about the curve $y = x\cos 2x$. [1]

It is the area enclosed by the curve $y = x\cos 2x$ below the x-axis between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

11 The table shows experimental values of two variables, x and y.

x	2	4	6	8
y	2.25	0.81	0.47	0.33

(i) On the graph paper, plot xy against $\frac{1}{x}$ and draw a straight line graph. [3]

$\frac{1}{x}$	0.5	0.25	0.17	0.125
xy	4.5	3.24	2.82	2.64

Table of values

Plot xy against $\frac{1}{x}$

Line cuts vertical axis

Using your graph,

(ii) estimate the value of x and y for which $x = \frac{3}{y}$, [2]

Draw horizontal line $xy = 3$

x from 4.8 to 5.2

y from 0.5 to 0.7

(iii) express y in terms of x. [4]

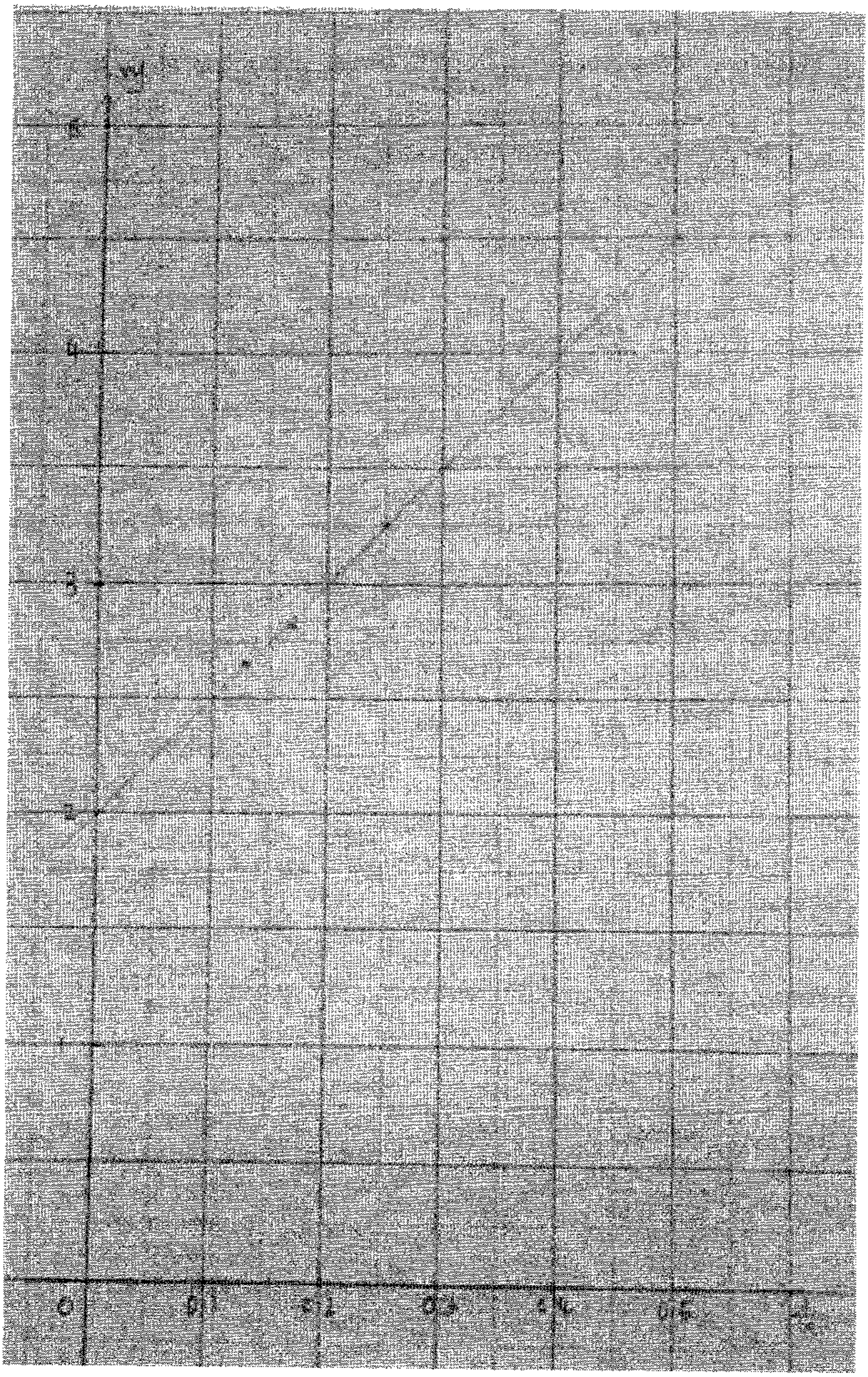
Finding gradient

Gradient from 4.8 to 5.2

Vertical intercept from 1.9 to 2.1

Use $Y = \mathrm{m}X + \mathrm{c}$

$y = \frac{5}{x^2} + \frac{2}{x}$ (oe)



12 A particle travels in a straight line, so that t seconds after passing a fixed point O, its velocity, v m/s, is given by $v = 22 + 7t - 2t^2$. The particle comes to instantaneous rest at P. Find

(i) the velocity of the particle when the acceleration is zero, [2]

$$a = 7 - 4t$$

$$\text{at } a = 0,\ t = \frac{7}{4}$$

$$v = 22 + 7\left(\frac{7}{4}\right) - 2\left(\frac{7}{4}\right)^2 = 28\frac{1}{8} \text{ m/s}$$

(ii) the value of t when the particle is at P, [2]

$$\text{at } v = 0,\ 22 + 7t - 2t^2 = 0$$
$$(11 - 2t)(2 + t) = 0$$
$$t = 5.5 \text{ or } t = -2 \text{ (NA)}$$

(iii) the distance OP, [2]

$$s = \int 22 + 7t - 2t^2 \text{ d}t$$

$$s = 22t + \frac{7}{2}t^2 - \frac{2}{3}t^3 + c$$

at $t = 0, s = 0,$

$c = 0$

$$s = 22t + \frac{7}{2}t^2 - \frac{2}{3}t^3$$

$$\text{at } t = 5.5,\ s = 22(5.5) + \frac{7}{2}(5.5)^2 - \frac{2}{3}(5.5)^3 = 115\frac{23}{24} \text{ m}$$

Alt Mtd :

$$s = \int_0^{5.5} (22 + 7t - 2t^2)\ dt$$

$$= \left[22t + \frac{7}{2}t^2 - \frac{2}{3}t^3\right]_0^{5.5} = 115\frac{23}{24}\ m$$

(iv) the total distance travelled by the particle in the interval $t = 0$ to $t = 9$. [2]

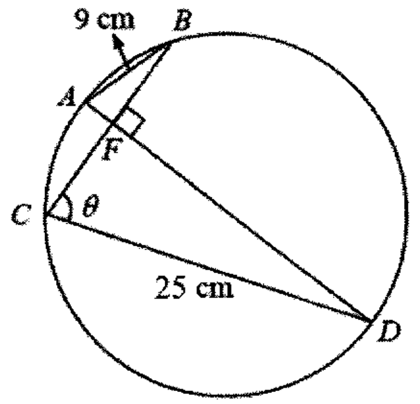
at $t = 0$, $s = 0$

at $t = 5.5$, $s = 115\frac{23}{24}$

at $t = 9$, $s = -\frac{9}{2}$

$$T_D = 115\frac{23}{24} \times 2 + \frac{9}{2} = 236\frac{5}{12} \text{ m}$$

13 In the diagram, AB and CD are chords of a circle, with $AB = 9$ cm and $CD = 25$ cm. It is given that AD and BC intersect each other each at 90° at F and $\angle BCD = \theta$, where θ varies.



(i) Prove that $AD = 9\cos\theta + 25\sin\theta$. [2]

$\angle BCD = \angle DAB = \theta$ (angle in the same segment)
$AD = AF + FD$
$AD = 9\cos\theta + 25\sin\theta$

(ii) Express AD in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [3]

$R = \sqrt{9^2 + 25^2} = \sqrt{706}$

$\tan\alpha = \dfrac{25}{9}$

$\alpha = 70.2011...$

$AD = \sqrt{706}\cos(\theta - 70.2°)$

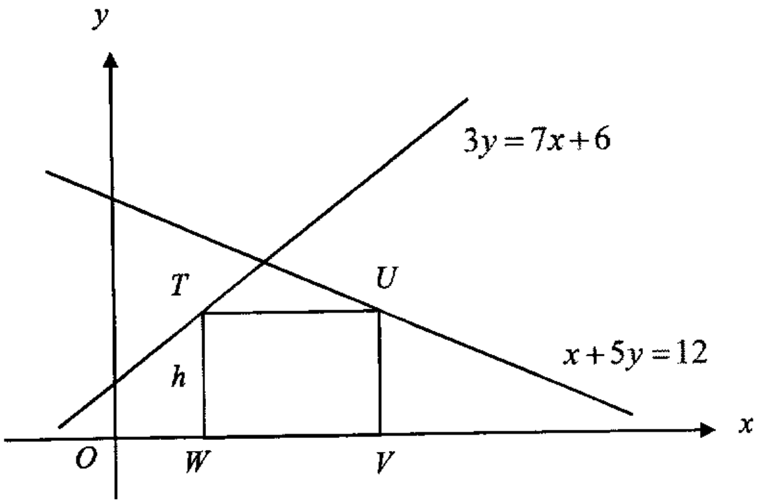
(iii) Find the values of θ when $AD = 26$ cm. [3]

$$\sqrt{706}\cos(\theta - 70.2011°) = 26$$
$$\cos(\theta - 70.2011°) = 0.9785229...$$
$$\theta - 70.2011° = -11.896...,11.896...$$
$$\theta = 58.3°,\ 82.1°$$

(iv) Calculate the angle θ for which AD is the diameter of the circle. [2]

AD is the longest. So $\cos(\theta - 70.2011°) = \cos 0°$
$\theta = 70.2°$

14 The diagram shows parts of the graphs of $3y=7x+6$ and $x+5y=12$. T is a point on $3y=7x+6$ and U is a point on $x+5y=12$. V and W are two points on the x-axis such that $TUVW$ is a rectangle.



It is given that $TW = h$ units.

(i) Find the x-coordinates of T and U in terms of h. [2]

$$3h = 7x_T + 6 \qquad\qquad x_U + 5h = 12$$

$$\frac{3h-6}{7} = x_T \qquad\qquad x_U = 12 - 5h$$

(ii) Hence, show that the area of the rectangle $TUVW$, represented by A square units, is given by $A = \frac{90}{7}h - \frac{38}{7}h^2$. [2]

$$VW = x_2 - x_1 = 12 - 5h - \frac{3h-6}{7}$$

$$VW = \frac{84-35h-(3h-6)}{7} = \frac{90-38h}{7}$$

$$A = \frac{90-38h}{7} \times h = \frac{90}{7}h - \frac{38}{7}h^2$$

(ii) Given that h can vary, find the stationary value of A and determine whether this value of A is a maximum or a minimum. [5]

$$\frac{\mathrm{d}A}{\mathrm{d}h}=\frac{90}{7}-\frac{76}{7}h$$

$$\text{at } \frac{\mathrm{d}A}{\mathrm{d}h}=0,\ \frac{90}{7}=\frac{76}{7}h$$

$$h=1.18$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}h^2}=-\frac{76}{7}<0$$

Thus area is a maximum.

$$A=\frac{90}{7}(1.18421)-\frac{38}{7}(1.18421)^2=7.61278...$$

$$A=7.61$$

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