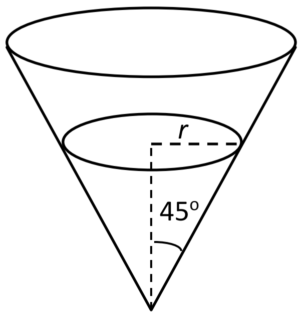
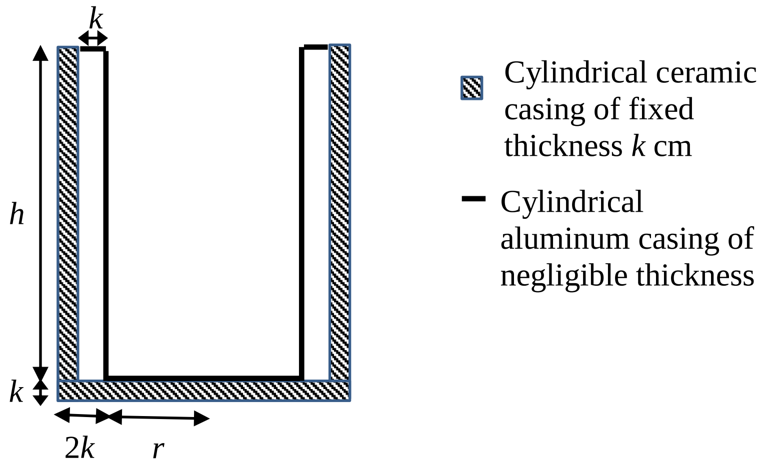
H2 Mathematics 2017 Prelim Paper 1 Question
Answer all questions [100 marks].

1	Water is leaking at a rate of 2 cm^3 per minute from a container in the form of a cone, with its axis vertical and vertex downwards. The semi-vertical angle of the cone is 45^o (see diagram). At time *t* minutes, the radius of the water surface is *r* cm. Find the rate of change of the depth of water when the depth of water in the container is 0.3cm. [4] [The volume of a cone of base radius *r* and height *h* is given by $V=\frac{1}{3}\pi r^2 h$.] 
2	Without using a calculator, solve the inequality $$\frac{x}{x-1}\le\frac{4}{x+2}.$$ [5]
3	**Do not use a calculator in answering this question.** Showing your working, find the complex numbers *z* and *w* which satisfy the simultaneous equations $$4\mathrm{i}z-3w=1+5\mathrm{i} \quad \text{and}$$ $$2z+(1+\mathrm{i})w=2+6\mathrm{i}.$$ [5]
4	**(a)** The points *A* and *B* relative to the origin *O* have position vectors $3\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $-3\mathbf{i}+2\mathbf{j}$ respectively. **(i)** Find the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. [2] **(ii)** Hence or otherwise, find the shortest distance from *B* to line *OA*. [2] **(b)** The points *C*, *D* and *E* relative to the origin *O* have non-zero and non-parallel position vectors **c**, **d** and **e** respectively. Given that $(\mathbf{c}\times\mathbf{d})\bullet\mathbf{e}=0$, state with reason(s) the relationship between *O*, *C*, *D* and *E*. [2]
5	**(i)** Prove by the method of differences that

	$$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{k}{2(n+1)(n+2)},$$ where k is a constant to be determined. [5]
	(ii) Explain why $\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}$ is a convergent series, and state its value. [2]
	(iii) Using your answer in part **(i)**, show that $\sum_{r=1}^{n}\frac{1}{(r+2)^3}<\frac{1}{4}$. [2]
6	A curve C has equation $y=\frac{ax+b}{cx+1}$, where a, b and c are positive real constants and $b>\frac{a}{c}$.
	(i) Sketch C, stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. [3]
	The curve C is transformed by a scaling parallel to y-axis by factor $\frac{1}{2}$ and followed by a translation of 2 units in the positive x-direction.
	(ii) Find the equation of the new curve in the form of $y=\mathrm{f}(x)$. [2]
	It is given that the new curve $y=\mathrm{f}(x)$ passes through the points with coordinates $\left(3,\frac{3}{2}\right)$ and $(6,1)$, and that $y=\frac{3}{4}$ is one of the asymptotes of the new curve $y=\mathrm{f}(x)$.
	(iii) Find the values of a, b and c. [5]
7	**(i)** Given that $\mathrm{f}(x)=\tan\left(\frac{1}{2}x+\frac{1}{4}\pi\right)$, show that $\mathrm{f}'(x)=\frac{1}{2}\left[1+(\mathrm{f}(x))^2\right]$, and find $\mathrm{f}(0)$, $\mathrm{f}'(0)$, $\mathrm{f}''(0)$ and $\mathrm{f}'''(0)$. Hence write down the first four non-zero terms in the Maclaurin series for $\mathrm{f}(x)$. [7]
	(ii) The first three non-zero terms in the Maclaurin series for $\mathrm{f}(x)$ are equal to the first three non-zero terms in the series expansion of $\frac{\cos(ax)}{1+bx}$. By using appropriate expansions from the List of Formulae (MF26), find the possible value(s) for the constants a and b. [5]

8	10 pirates live on a pirate ship and they are ranked based on their seniority. **(a)** One day, the pirates found a treasure chest that consists of some gold coins. The rule which the pirates adhered by to divide all the gold coins are based on their seniority and is as follows: The most senior pirate will get 3 gold coins more than the 2^{nd} most senior pirate. The 2^{nd} most senior pirate will also get 3 gold coins more than the 3^{rd} most senior pirate and so on. Thus, the most junior pirate will get the least number of gold coins. **(i)** If the treasure chest contains 305 gold coins, find the number of gold coins the most senior pirate will get. [3] **(ii)** Find the least number of gold coins the treasure chest must contain if all pirates get some (at least one) gold coins each. [2] **(b)** The pirates need to take turns, one at a time, to be on the lookout for their ship. Each day (24 hours) is divided into 10 shifts rotated among the 10 pirates. The 1^{st} lookout shift starts from 10pm daily and it starts with the most junior pirate to the most senior pirate. The length of their shift is also based on their seniority. The length of shift for the most senior pirate is 10% less than that of the 2^{nd} most senior pirate. The length of shift for the 2^{nd} most senior pirate is 10% less than that of the 3^{rd} most senior pirate and so on. Thus, the most junior pirate has the longest shift. **(i)** Show that the length of shift for the most junior pirate is 3.6848 hours, correct to 4 decimal places. [2] **(ii)** Calculate the length of shift for the 6^{th} most junior pirate. Find the start time of his shift, giving your answer to the nearest minute. [4]
9	A curve C has parametric equations $$x = \sqrt{2}\cos\frac{t}{2}, \quad y = \sqrt{2}\sin t, \quad \text{for } -2\pi \le t \le 2\pi .$$ **(i)** Find $\frac{dy}{dx}$ and verify that curve C has a stationary point at P with parameter $\frac{\pi}{2}$. Hence find the equation of the normal to the curve at point P. [3] **(ii)** Sketch C, indicating clearly all turning points and axial intercepts in exact form. [4]

	(iii) Find the exact area bounded by the curve C. (You may first consider the area bounded by the curve C and the positive x-axis in the first quadrant.) [6]
10	The plane p_1 has equation $\mathbf{r}=\begin{pmatrix}-1\\1\\16\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}+\mu\begin{pmatrix}2\\1\\1\end{pmatrix}$, where λ and μ are real parameters. The point A has position vector $5\mathbf{i}-6\mathbf{j}+7\mathbf{k}$. **(i)** Find a cartesian equation of p_1. [3] **(ii)** Find the position vector of the foot of perpendicular from A to p_1. [4] The plane p_2 has equation $\mathbf{r}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix}=52$. The plane p_3 is obtained by reflecting p_2 about p_1. By considering the relationship between A and p_2, or otherwise, find a cartesian equation of p_3. [6]
11	A company intends to manufacture a cylindrical double-walled ceramic vacuum flask which can hold a fixed V cm^3 of liquid when filled to the brim. The cylindrical vacuum flask is made up of an inner cylindrical aluminum casing (of negligible thickness) with height h cm and radius r cm and an outer cylindrical ceramic casing of fixed thickness k cm. There is a fixed k cm gap between the sides of the inner casing and outer casing where air has been removed to form a vacuum. The diagram below shows the view of the vacuum flask if it is dissected vertically through the centre.



Let the volume of the outer ceramic casing be C cm^3.

(i) Show that the volume of the ceramic casing can be expressed as

$$C = k\left(\frac{2V}{r} + \frac{3kV}{r^2} + \pi\left(r+2k\right)^2\right) \text{ .} \qquad [4]$$

(ii) Let r_1 be the value of r which gives the minimum value of C. Show that r_1 satisfies the equation $\pi r^4 + 2\pi k r^3 - rV - 3kV = 0$. [3]

For the rest of the question, it is given that $k = \frac{1}{4}$ and $V = 250$.

(iii) Find the minimum volume of the ceramic casing, proving that it is a minimum. [3]

(iv) Sketch the graph showing the volume of the ceramic casing as the radius of the aluminum casing varies. [2]

– End Of Paper –

ANNEX B

MJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Differentiation & Applications	7.07
2	Equations and Inequalities	$-2 < x < 1$
3	Complex numbers	$w = -3+5\mathrm{i}$ and $z = 5+2\mathrm{i}$
4	Vectors	(a)(i) 2.35 radian (a)(ii) $h = 2.58$
5	Sigma Notation and Method of Difference	(i) $k=1$ (ii) $\frac{1}{4}$
6	Graphs and Transformation	(ii) $y = \frac{1}{2}\left[\frac{a(x-2)+b}{c(x-2)+1}\right]$ (iii) $a=3,\ b=6$ and $c=2$
7	Maclaurin series	(i) $\mathrm{f}(0)=1;\ \mathrm{f}'(0)=1;\ \mathrm{f}''(0)=1;\ \mathrm{f}'''(0)=2;$ $\mathrm{f}(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 \cdots$ (ii) $a = \pm 1$; $b = -1$
8	AP and GP	(a)(i) 44 (a)(ii) 145 (b)(ii) 1.05pm
9	Application of Integration	(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2\cos t}{\sin\frac{t}{2}}$; $x=1$ (iii) $\frac{16}{3}$ units^2
10	Vectors	(i) $-3x + y + 5z = 84$ (ii) $\overrightarrow{OF} = \begin{pmatrix} -1 \\ -4 \\ 17 \end{pmatrix}$ (iii) $-31x + 22y + 5z = 308$
11	Differentiation & Applications	(iii) 49.7 cm^3

H2 Mathematics 2017 Prelim Exam Paper 1 Solution

1	Solution: $\tan 45^\circ = \frac{r}{h} \Rightarrow r = h$ $V = \frac{1}{3}\pi r^2 h$ $V = \frac{1}{3}\pi h^3$ $\frac{\mathrm{d}V}{\mathrm{d}h} = \pi h^2$ When $h = 0.3$, $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V}\frac{\mathrm{d}V}{\mathrm{d}t}$ $= \frac{1}{\pi(0.3)^2}(-2)$ $= -\frac{200}{9\pi} = -7.07 \;\; \text{(3s.f)}$ The depth of water is decreasing at 7.07 cm per minute.
2	Solution: $\frac{x}{x-1} \le \frac{4}{x+2}$ $\frac{x}{x-1} - \frac{4}{x+2} \le 0$ $\frac{x(x+2)-4(x-1)}{(x-1)(x+2)} \le 0$ $\frac{x^2+2x-4x+4}{(x-1)(x+2)} \le 0$ $\frac{x^2-2x+4}{(x-1)(x+2)} \le 0$ $\frac{\left(x-1\right)^2+3}{(x-1)(x+2)} \le 0$ Since $\left(x-1\right)^2+3>0$ for all $x \in \mathbb{R}$, $(x-1)(x+2)<0$ $-2<x<1$
3	Solution: $4\mathrm{i}z - 3w = 1+5\mathrm{i}$ --------(1) $2z + \left(1+\mathrm{i}\right)w = 2+6\mathrm{i}$ --------(2)

$(2)\times 2\mathrm{i}$

$4\mathrm{i}z+2\mathrm{i}(1+\mathrm{i})w=2\mathrm{i}(2+6\mathrm{i})$

$4\mathrm{i}z+2\mathrm{i}w-2w=4\mathrm{i}-12$ --------(3)

$(3)-(1)$:

$$4\mathrm{i}z+2\mathrm{i}w-2w-(4\mathrm{i}z-3w)=(4\mathrm{i}-12)-(1+5\mathrm{i})$$

$$w+2\mathrm{i}w=-13-\mathrm{i}$$

$$(1+2\mathrm{i})w=-13-\mathrm{i}$$

$$w=\left(\frac{-13-\mathrm{i}}{1+2\mathrm{i}}\right)\left(\frac{1-2\mathrm{i}}{1-2\mathrm{i}}\right)$$

$$w=\frac{-13+26\mathrm{i}-\mathrm{i}-2}{(1)^2-(2\mathrm{i})^2}$$

$$w=\frac{-15+25\mathrm{i}}{5}$$

$$w=-3+5\mathrm{i}$$

Substitute $w=-3+5\mathrm{i}$ into (2)

$2z=2+6\mathrm{i}-(1+\mathrm{i})(-3+5\mathrm{i})$

$2z=2+6\mathrm{i}-(-3+5\mathrm{i}-3\mathrm{i}-5)$

$2z=2+6\mathrm{i}-(-8+2\mathrm{i})$

$2z=10+4\mathrm{i}$

$z=5+2\mathrm{i}$

$\therefore w=-3+5\mathrm{i}$ and $z=5+2\mathrm{i}$.

4

Solution:

(a)(i) Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$$\cos\theta=\frac{\begin{pmatrix}3\\-1\\3\end{pmatrix}\bullet\begin{pmatrix}-3\\2\\0\end{pmatrix}}{\left|\begin{pmatrix}3\\-1\\3\end{pmatrix}\right|\left|\begin{pmatrix}-3\\2\\0\end{pmatrix}\right|}$$

$\theta=\cos^{-1}\left(\dfrac{-11}{\sqrt{19}\sqrt{13}}\right)=134.4^\circ$ (1 d.p) = 2.35 radian (3 s.f)

(a)(ii) Let h be the shortest distance from B to line OA.

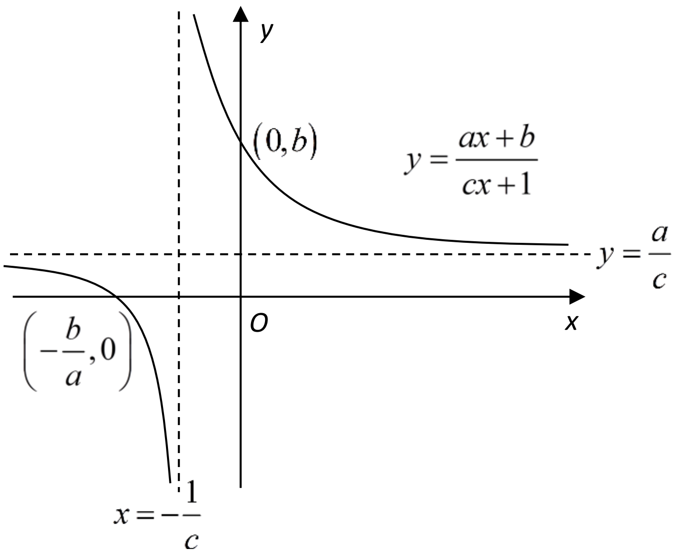
$$\sin 134.42^\circ=\frac{h}{|\mathbf{b}|}$$

$$h=\sqrt{13}\sin 134.42^\circ$$

$$=2.5752$$

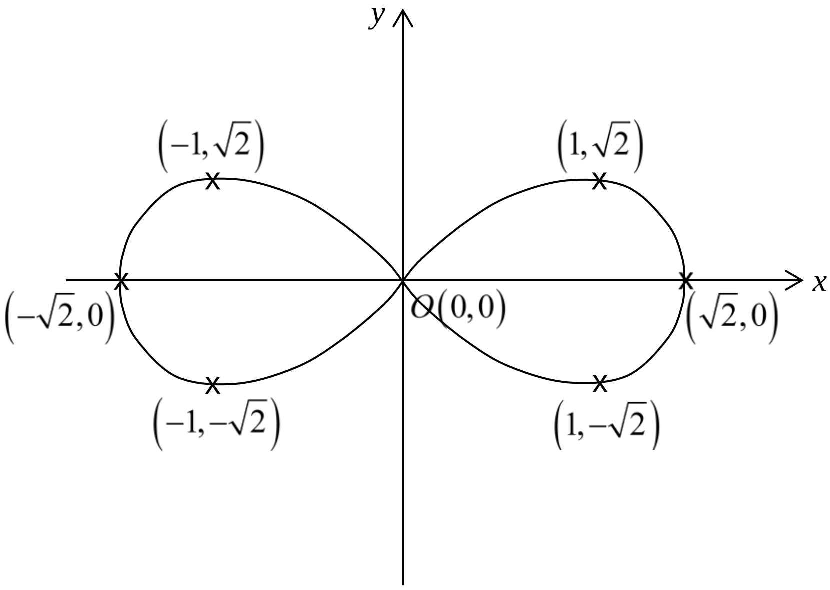
$= 2.58$ units (3 s.f)

	(b) Let $\mathbf{c}\times\mathbf{d}=\mathbf{s}$. 1) $\mathbf{s}\bullet\mathbf{e}=0 \Rightarrow$ $\mathbf{s}$ is perpendicular to **e**. 2) $\mathbf{c}\times\mathbf{d}=\mathbf{s} \Rightarrow$ $\mathbf{s}$ is perpendicular to both **c** and **d**. Since $\mathbf{s}$ is <u>perpendicular to **c**, **d** and **e**</u> and <u>**c**, **d** and **e** passes through common point *O*</u> $\Rightarrow$ points *O*, *C*, *D* and *E* are coplanar.
5	Solution: (i) Let $\dfrac{1}{r(r+1)(r+2)}=\dfrac{A}{r}+\dfrac{B}{r+1}+\dfrac{C}{r+2}$ Using 'cover-up' rule, $A=\dfrac{1}{2},\qquad B=-1,\qquad C=\dfrac{1}{2}$ $\therefore \dfrac{1}{r(r+2)}=\dfrac{1}{2r}-\dfrac{1}{r+1}+\dfrac{1}{2(r+2)}$ $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\sum_{r=1}^{n}\left(\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}\right)$ $=\Big[\dfrac{1}{2}-\dfrac{1}{2}+\dfrac{1}{6}$ $+\dfrac{1}{4}-\dfrac{1}{3}+\dfrac{1}{8}$ $+\dfrac{1}{6}-\dfrac{1}{4}+\dfrac{1}{10}$ $+\dfrac{1}{8}-\dfrac{1}{5}+\dfrac{1}{12}$ $+\ldots$ $+\dfrac{1}{2(n-2)}-\dfrac{1}{n-1}+\dfrac{1}{2n}$ $+\left(\dfrac{1}{2n-1}\right)-\dfrac{1}{n}+\dfrac{1}{2(n+1)}$ $+\dfrac{1}{2n}-\dfrac{1}{n+1}+\dfrac{1}{2(n+2)}\Big]$ $=\dfrac{1}{2}-\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{2(n+1)}-\dfrac{1}{n+1}+\dfrac{1}{2(n+2)}$ $=\dfrac{1}{4}-\dfrac{1}{2(n+1)}+\dfrac{1}{2(n+2)}$ $=\dfrac{1}{4}-\dfrac{1}{2(n+1)(n+2)}$ (proven) $\therefore k=1$

	(ii) As $n \to \infty, \dfrac{1}{2(n+1)(n+2)} \to 0, \displaystyle\sum_{r=1}^{n} \dfrac{1}{r(r+1)(r+2)} \to \dfrac{1}{4}$ $\therefore \displaystyle\sum_{r=1}^{\infty} \dfrac{1}{r(r+1)(r+2)}$ is a convergent series. $\therefore \displaystyle\sum_{r=1}^{\infty} \dfrac{1}{r(r+1)(r+2)} = \dfrac{1}{4}$ (iii) For all $r \geq 1$, $(r+2)^3 > r(r+1)(r+2)$ $\dfrac{1}{(r+2)^3} < \dfrac{1}{r(r+1)(r+2)}$ $\displaystyle\sum_{r=1}^{n} \dfrac{1}{(r+2)^3} < \sum_{r=1}^{n} \dfrac{1}{r(r+1)(r+2)}$ $\displaystyle\sum_{r=1}^{n} \dfrac{1}{(r+2)^3} < \dfrac{1}{4} - \dfrac{1}{2(n+1)(n+2)}$ $\displaystyle\sum_{r=1}^{n} \dfrac{1}{(r+2)^3} < \dfrac{1}{4}$ $\quad \left(\because \dfrac{1}{2(n+1)(n+2)} > 0 \text{ for all } n \geq 1 \right)$
6	Solution: (i)  (ii) Equation of new curve: $y = \dfrac{1}{2}\left[\dfrac{a(x-2)+b}{c(x-2)+1}\right]$

	(iii) Since the new curve $y=\mathrm{f}\left(x\right)$ passes through the points with coordinates $\left(3,\dfrac{3}{2}\right)$ and $\left(6,1\right)$: $\dfrac{3}{2}=\dfrac{1}{2}\left[\dfrac{a\left(3-2\right)+b}{c\left(3-2\right)+1}\right]$ $3=\dfrac{a+b}{c+1}$ $a+b=3c+3$ $a+b-3c=3$ --------(1) $1=\dfrac{1}{2}\left[\dfrac{a\left(6-2\right)+b}{c\left(6-2\right)+1}\right]$ $2=\dfrac{4a+b}{4c+1}$ $4a+b=8c+2$ $4a+b-8c=2$ ---------(2) Since $y=\dfrac{3}{4}$ is one of the asymptotes of $y=\mathrm{f}\left(x\right)$, $\dfrac{3}{4}=\dfrac{1}{2}\left(\dfrac{a}{c}\right)$ $\dfrac{a}{c}=\dfrac{3}{2}$ $2a-3c=0$ --------(3) Solving equations (1), (2) and (3) using GC, $a=3,\ b=6$ and $c=2$.
7	Solution: (i) $\mathrm{f}(x)=\tan\left(\dfrac{1}{2}x+\dfrac{1}{4}\pi\right)$ $\mathrm{f}'(x)=\dfrac{1}{2}\sec^2\left(\dfrac{1}{2}x+\dfrac{1}{4}\pi\right)$ $\quad=\dfrac{1}{2}\left[1+\tan^2\left(\dfrac{1}{2}x+\dfrac{1}{4}\pi\right)\right]$ $\quad=\dfrac{1}{2}\left(1+\left(\mathrm{f}(x)\right)^2\right)$ (shown) $\mathrm{f}''(x)=\mathrm{f}(x)\mathrm{f}'(x)$ $\mathrm{f}'''(x)=\mathrm{f}(x)\mathrm{f}''(x)+\left(\mathrm{f}'(x)\right)^2$ $\mathrm{f}(0)=1,$ $\mathrm{f}'(0)=1,$ $\mathrm{f}''(0)=1,$

	$f'''(0)=2$. $\therefore \quad f(x)=1+x+\frac{1}{2}x^2+\frac{1}{3}x^3\cdots$ (ii) $\frac{\cos(ax)}{1+bx}=\left(1-\frac{(ax)^2}{2!}+...\right)\left(1+(-1)(bx)+\frac{(-1)(-2)}{2!}(bx)^2+...\right)$ $=\left(1-\frac{a^2x^2}{2}+...\right)\left(1-bx+b^2x^2+...\right)$ $\approx 1-bx+b^2x^2-\frac{a^2x^2}{2}$ $=1-bx+\left(b^2-\frac{a^2}{2}\right)x^2$ Comparing coefficients, $x:b=-1$ $x^2:b^2-\frac{a^2}{2}=\frac{1}{2}\Rightarrow\frac{a^2}{2}=\frac{1}{2}\Rightarrow a=\pm 1$
8	Solution: (a)(i) Let a be the number of gold coins the most junior pirate will get. $\frac{10}{2}\left[2a+(10-1)(3)\right]=305$ $a=17$ No of gold coins for most senior pirate $=17+(10-1)(3)$ $=44$ (a)(ii) Least no of gold coins $=\frac{10}{2}\left[2(1)+(10-1)(3)\right]$ $=145$ (b)(i) Let b be the length of shift for the most junior pirate $\frac{b}{1-0.9}\left(1-0.9^{10}\right)=24$ $b=3.6848\text{ hr}\left(\text{to 4 d.p.}\right)$ (shown) (b)(ii) Length of shift for 6th most junior pirate $=3.6848(0.9)^5$ $=2.18\text{ hr}$

	Length of 1st 5 shifts $= \dfrac{3.6848}{1-0.9}\left(1-0.9^{5}\right)$ $=15.090$ $=15$ hrs 5 mins Start time of shift $=1.05$pm
9	(i) $x=\sqrt{2}\cos\dfrac{t}{2} \Rightarrow \dfrac{dx}{dt}=-\dfrac{\sqrt{2}}{2}\sin\dfrac{t}{2}$ $y=\sqrt{2}\sin t \Rightarrow \dfrac{dy}{dt}=\sqrt{2}\cos t$ $\therefore \dfrac{dy}{dx}=-\dfrac{2\cos t}{\sin\frac{t}{2}}$ At $t=\dfrac{\pi}{2}$, $\dfrac{dy}{dx}=-\dfrac{2\cos\frac{\pi}{2}}{\sin\frac{\pi}{4}}=0$ (verified) When $t=\dfrac{\pi}{2}$, $x=\sqrt{2}\cos\left(\dfrac{\pi}{4}\right)=1$ Equation of normal: $x=1$ (ii) 

	(iii) $\text{Area} = 4\int_0^{\sqrt{2}} y\ \mathrm{d}x$ $= 4\int_\pi^0 \sqrt{2}\sin t \cdot \left(-\frac{\sqrt{2}}{2}\sin\frac{t}{2}\right)\mathrm{d}t$ $= 4\int_0^\pi \sin t \cdot \sin\frac{t}{2}\ \mathrm{d}t$ $= 8\int_0^\pi \sin^2\frac{t}{2}\cos\frac{t}{2}\ \mathrm{d}t$ $= 8\left[\frac{2}{3}\sin^3\frac{t}{2}\right]_0^\pi$ $= \frac{16}{3}\ \text{units}^2$ **Alternative Method** $\text{Area} = 4\int_0^{\sqrt{2}} y\ \mathrm{d}x$ $= 4\int_\pi^0 \sqrt{2}\sin t \cdot \left(-\frac{\sqrt{2}}{2}\sin\frac{t}{2}\right)\mathrm{d}t$ $= 4\int_0^\pi \sin t \cdot \sin\frac{t}{2}\ \mathrm{d}t$ $= -2\int_0^\pi \cos\frac{3t}{2} - \cos\frac{t}{2}\ \mathrm{d}t$ $= -2\left[\frac{2}{3}\sin\frac{3t}{2} - 2\sin\frac{t}{2}\right]_0^\pi$ $= \frac{16}{3}\ \text{units}^2$
10	Solutions: (i) $\begin{pmatrix}3\\-1\\2\end{pmatrix}\times\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}-3\\1\\5\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}-3\\1\\5\end{pmatrix}=\begin{pmatrix}-1\\1\\16\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\5\end{pmatrix}=84$ Cartesian equation of p_1 is $-3x+y+5z=84$. (ii) $\overrightarrow{OA}=\begin{pmatrix}5\\-6\\7\end{pmatrix}$ Let the foot of perpendicular from A to p_1 be F.

	$\overrightarrow{OF}=\begin{pmatrix}5\\-6\\7\end{pmatrix}+\beta\begin{pmatrix}-3\\1\\5\end{pmatrix}=\begin{pmatrix}5-3\beta\\-6+\beta\\7+5\beta\end{pmatrix}$ for some $\beta\in\mathbb{R}$ Since F lies on p_1, $\begin{pmatrix}5-3\beta\\-6+\beta\\7+5\beta\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\5\end{pmatrix}=84$ $35\beta+14=84$ $\beta=2$ $\therefore\overrightarrow{OF}=\begin{pmatrix}5-6\\-6+2\\7+10\end{pmatrix}=\begin{pmatrix}-1\\-4\\17\end{pmatrix}$ Note that A lies on p_2 since $\begin{pmatrix}5\\-6\\7\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix}=52$. Let A' be the point of reflection of A about p_1. Note that A' lies on p_3. $\overrightarrow{OF}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$ $\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}=2\begin{pmatrix}-1\\-4\\17\end{pmatrix}-\begin{pmatrix}5\\-6\\7\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}$ $p_1: -3x+y+5z=84$. $p_2: \mathbf{r}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix}=52 \Rightarrow x-2y+5z=52$ By GC, the line of intersection between p_1 and p_2 is $\mathbf{r}=\begin{pmatrix}-44\\-48\\0\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix}$, $\alpha\in\mathbb{R}$ A vector parallel to p_3 is $\overrightarrow{OA'}-\begin{pmatrix}-44\\-48\\0\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}-\begin{pmatrix}-44\\-48\\0\end{pmatrix}=\begin{pmatrix}37\\46\\27\end{pmatrix}$ $\begin{pmatrix}37\\46\\27\end{pmatrix}\times\begin{pmatrix}3\\4\\1\end{pmatrix}=\begin{pmatrix}-62\\44\\10\end{pmatrix}=2\begin{pmatrix}-31\\22\\5\end{pmatrix}$

$$\mathbf{r}\cdot\begin{pmatrix}-31\\22\\5\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}\cdot\begin{pmatrix}-31\\22\\5\end{pmatrix}=308$$

A cartesian equation of p_3 is $-31x+22y+5z=308$

11 Solution:

(i)

$$V=\pi h r^2$$

$$h=\frac{V}{\pi r^2} \quad \text{--------} \quad (*)$$

$$C=\pi(h+k)(r+2k)^2-\pi h(r+k)^2$$

$$=\pi\left(h\left((r+2k)^2-(r+k)^2\right)+k(r+2k)^2\right)$$

$$=\pi\left(h\left(\left(r^2+4rk+4k^2\right)-\left(r^2+2rk+k^2\right)\right)+k(r+2k)^2\right)$$

$$=\pi\left(\frac{V}{\pi r^2}\left(2rk+3k^2\right)+k(r+2k)^2\right) \qquad (\text{from } (*))$$

$$=k\left(V\frac{(2r+3k)}{r^2}+\pi(r+2k)^2\right)$$

$$=k\left(\frac{2V}{r}+\frac{3kV}{r^2}+\pi(r+2k)^2\right) \qquad (\text{shown})$$

(ii) $\dfrac{\mathrm{d}C}{\mathrm{d}r}=k\left(\dfrac{-2V}{r^2}-\dfrac{6kV}{r^3}+2\pi(r+2k)\right)$

When $\dfrac{\mathrm{d}C}{\mathrm{d}r}=0$,

$$k\left(\frac{-2V}{r^2}-\frac{6kV}{r^3}+2\pi(r+2k)\right)=0$$

$$-Vr-3kV+\pi r^3(r+2k)=0$$

$$\pi r^4+2k\pi r^3-Vr-3kV=0 \qquad (\text{Shown})$$

(iii) From GC,

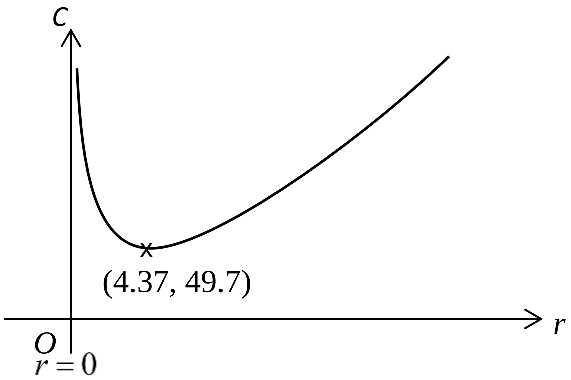
$r_1=4.3736 \quad (\text{since } r>0)$

r	r_1^-	r_1	r_1^+
$\dfrac{\mathrm{d}C}{\mathrm{d}r}$	\	—	/

$\dfrac{\mathrm{d}^2C}{\mathrm{d}r^2}=5.33>0\Rightarrow C$ is a minimum

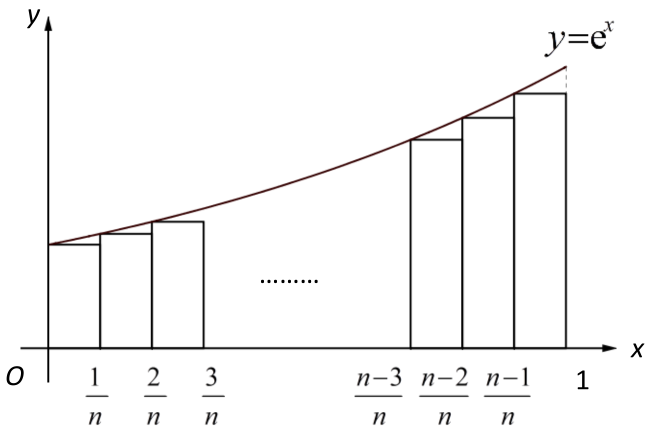
$C=49.7 \quad (3\text{ s.f.})$

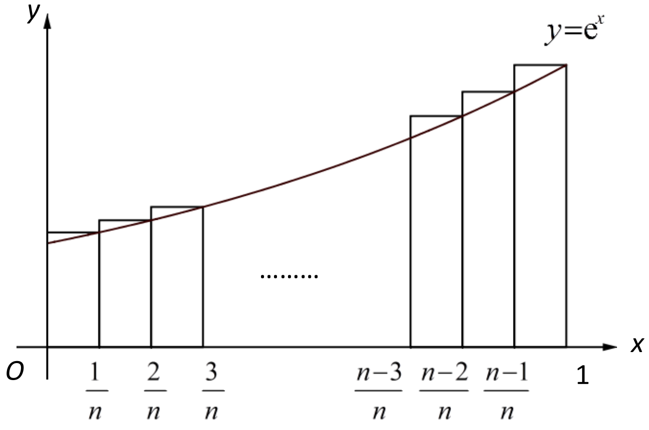
Minimum volume of ceramic casing is 49.7 cm^3.

	Note: cannot use part (iv) graph to solve this part. (iv) 

H2 Mathematics 2017 Prelim Paper 2 Question
Answer all questions [100 marks].

1	The complex number z has modulus 3 and argument $\frac{2\pi}{3}$. **(i)** Find the modulus and argument of $\frac{-2\mathrm{i}}{z^*}$, where z^* is the complex conjugate of z, leaving your answers in the exact form. [3] **(ii)** Hence express $\frac{-2\mathrm{i}}{z^*}$ in the form of $x+\mathrm{i}y$, where x and y are real constants, giving the exact values of x and y in non-trigonometrical form. [2] **(iii)** The complex number w is defined such that $w=1+\mathrm{i}k$, where k is a non-zero real constant. Given that $\frac{-2\mathrm{i}w}{z^*}$ is purely imaginary, find the exact value of k. [2]
2	Two students are investigating the rate of change of the amount of water in a reservoir, x million cubic metres, at time t hour during a rainfall. Student A suggests that x and t are related by the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{2}{(t+1)^3}$. **(i)** Find the general solution of this differential equation. [3] Student B assumes that the amount of water flowing into the reservoir depends only on the rainfall and is at a constant rate of k million cubic metres per hour. The rate at which water flows out from the reservoir is proportional to the square of the amount of water in the reservoir. **(ii)** If the amount of water in the reservoir stabilizes at 0.5 million cubic metres, show that the rate of change of the amount of water in the reservoir can be modelled by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t}=k\left(1-4x^2\right)$. [2] **(iii)** Find x in terms of k and t, given that there are initially 1 million cubic metres of water in the reservoir. [5]
3	The function f is defined by $$\mathrm{f}: x \mapsto \ln\left(x^2-1\right),\ x\in\mathbb{R},\ x>1.$$ **(i)** Find f^{-1} in similar form. [3]

	(ii) Sketch f, f^{-1} and $\mathrm{f}^{-1}\mathrm{f}$ on the same diagram, indicating clearly all asymptotes and axial intercepts. [3] The functions g and h are defined by $$\mathrm{g}: x \mapsto \begin{cases} 4(x-1)^2 & \text{for } 0 \le x < 2, \\ 8-\lvert 2x-8 \rvert & \text{for } 2 \le x < 8, \end{cases}$$ $$\mathrm{h}: x \mapsto 3\sin x, \qquad 0 \le x \le \pi.$$ **(iii)** Sketch the graph of $y = \mathrm{g}(x)$. [3] **(iv)** Prove that the function gh exists and find the range of gh. [2]
4	The graph of $y = \mathrm{e}^x$, for $0 \le x \le 1$, is shown in the diagram below. Rectangles, each of width $\frac{1}{n}$ where *n* is an integer, are drawn under the curve.  **(i)** Show that the total area of all the *n* rectangles, A_n, is $\dfrac{c}{n\left(\mathrm{e}^{\frac{1}{n}}-1\right)}$, where *c* is an exact constant to be found. [3] **(ii)** By considering the Maclaurin Series for $\mathrm{e}^x - 1$, or otherwise, find the value of $\lim_{x \to 0} \frac{1}{x}\left(\mathrm{e}^x - 1\right)$. [3] **(iii)** Hence, without using integration, find the exact value of $\lim_{n \to \infty} A_n$. [2] **(iv)** Give a geometrical interpretation of the value you found in part **(iii)**, and verify your answer in part **(iii)** using integration. [2] Another set of *n* rectangles are drawn, as shown in the diagram below.



The total area of all the n rectangles in the second diagram is denoted by B_n. By considering the concavity of the graph of $y = \mathrm{e}^x$, or otherwise, show that

$$\frac{A_n + B_n}{2} > \int_0^1 \mathrm{e}^x \mathrm{d}x$$

for any positive integer n. [2]

5 Andy needs two passcodes to open a treasure box. Both passcodes consist of three letters and four digits. Each of the three letters can be any of the twenty-six letters of the alphabet A-Z. Each of the four digits can be any of the ten digits 0-9.

(a) The first passcode consists of three letters followed by four digits. It is also known that no letters and digits are repeated. An example of the code is ABC1234.

(i) Find the total number of possible first passcodes. [2]

(ii) An additional hint is given to Andy to break the first passcode. The four digits of the passcode form a number which is odd and greater than 3000. Find the total number of possible first passcodes. [3]

(b) The second passcode has no fixed arrangement for the letters and digits. Given that the letters and digits can be repeated (i.e. 1AA3C34 can be a possible passcode), find the total number of possible second passcodes. [3]

6 The probability function of X is given by

$$\mathrm{P}(X = x) = \begin{cases} (2x-1)\theta & \text{if } x = 1, 2, 3 \\ k & \text{if } x = 4 \\ 0 & \text{otherwise} \end{cases}$$

where $0 < \theta < \frac{1}{9}$.

	(i) Show that $k = 1 - 9\theta$. Find, in terms of θ, the probability distribution of X. [2] **(ii)** Find $\mathrm{E}(X)$ in terms of θ and hence show that $\mathrm{Var}(X) = 26\theta - 196\theta^2$. [3] **(iii)** The random variable Y is related to X by the formula $Y = a + bX$, where a and b are non-zero constants. Given that $\mathrm{Var}(Y) = \frac{1}{3}b^2$, find the value of θ. [3]
7	Coloured lego pieces are packed into boxes of 20 pieces by a particular manufacturer. Each box is made up of randomly chosen coloured lego pieces. The manufacturer produces a large number of lego pieces every day. On average, 15% of lego pieces are red. Explain why binomial distribution is appropriate for modelling the number of red lego pieces in a box. [2] **(i)** Find the probability that a randomly chosen box of lego pieces contains at least 4 red lego pieces. [2] **(ii)** A customer buys 50 randomly chosen boxes containing lego pieces. Find the probability that no more than 19 of these boxes contain at least 4 red lego pieces. [2] It is given instead that the proportion of lego pieces that are red is now *p*. The probability that there is at least one red lego piece but fewer than four red lego pieces in a box, is 0.22198, correct to 5 significant figures. Write down an equation involving *p* and hence find the value of *p*, given that $p > 0.2$. [4]
8	In an assembly line, a machine is programmed to dispense shampoo into empty bottles and the volume of shampoo dispensed into each bottle is a normally distributed continuous random variable *X*. Under ordinary conditions, the expected value of *X* is 325 ml. **(i)** After a routine servicing of the machine, the assembly manager suspects that the machine is dispensing more shampoo than expected. A random sample of 60 bottles is taken and the data is as follows:

Volume of shampoo (correct to nearest ml)	324	325	326	327	328	329	330
Number of bottles	16	20	9	8	4	1	2

Find unbiased estimates of the population mean and variance, giving your answers to 2 decimal places. [2]

Test, at the 5% significance level, whether the assembly manager's suspicion is valid. [4]

Explain what it meant by the phrase 'at 5% significance level' in the context of the question. [1]

(ii) Due to the assembly manager's suspicion, the machine is being recalibrated to dispense 325 ml of shampoo. Another random sample of 50 is taken and a two-tailed test, at the 5% significance level, concluded that the recalibration is done accurately. Given that the volume of shampoo dispensed into each bottle is normally distributed with standard deviation 1.2 ml, find the set of values the mean volume of the 50 bottles can take, giving your answers to 2 decimal places. [4]

9 The consumer price index measures the average price changes in a fixed basket of consumption goods and services commonly purchased by resident households over time. It is commonly used as a measure of consumer price inflation. In the 2013 Singapore household expenditure survey, housing and food made up about half of the average monthly expenditure of an average household.

The table below shows the housing and food price index from 2005 to 2012, where 2005 is the base period, i.e. in 2005, the price index is 100. For example, the food price index of 104.6 in 2007 means that average food prices increased by 4.6% from 2005 to 2007.

Year	2005	2006	2007	2008	2009	2010	2011	2012
Housing Price Index, x	100	100.7	102.3	116.8	123.1	124.3	[redacted]	148
Food Price Index, y	100	101.6	104.6	112.6	115.2	116.8	120.3	123.1

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	(i)	Show that the value of the missing housing price index for 2011 is 136 (nearest integer), given that the regression line of y on x is $y = 54.271 + 0.48363x$, correct to 5 significant figures. [2]
	(ii)	Draw the scatter diagram for these values, labelling the axes clearly. Comment on the suitability of the linear model. [3]
	(iii)	It is required to estimate the housing price index in 2016 where the food price index in 2016 is 134.6. Find the equation of an appropriate regression line for y and $\sqrt{x}$ and use it to find the required estimate. Explain why this estimate might not be reliable. [4]
	(iv)	Find the product moment correlation coefficient between y and $\sqrt{x}$. [1]
	(v)	To simplify recordings and calculations, it would be more convenient to tabulate $\frac{x}{100}$ and $\frac{y}{100}$ instead. Without any further calculations, explain if the product moment correlation coefficient between $\sqrt{\frac{x}{100}}$ and $\frac{y}{100}$ would differ from the value obtained in part **(iv)**. [1]
10	**(i)**	Factory A produces nuts whose mass may be assumed to be normally distributed with mean μ grams and standard deviation σ grams. A random sample of 50 nuts is taken. It is given that the probability that the mean mass is less than 247 grams is 0.018079, correct to 5 significant figures. It is also given that the probability that the total mass exceeds 12600 grams is 0.78397, correct to 5 significant figures. Find the values of μ and σ, giving your answers to the nearest grams. [5]
	(ii)	(For this question, you should state clearly the values of the parameters of any normal distribution you use.) Factory B produces bolts and nuts. The masses, in grams, of bolts and nuts produced are modelled as having independent normal distributions with means and standard deviations as shown in the table:

	Mean Mass (in grams)	Standard Deviation (in grams)
Bolts	745	7.3
Nuts	250	5

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	(a) Find the probability that the mass of a randomly chosen bolt differs from 3 times the mass of a randomly chosen nut by at least 40 grams. [4] **(b)** This factory introduces a new process which is able to reduce the mass of each nut by 10%. Find the probability that the total mass, after the introduction of this process, of 10 randomly chosen nuts is less than 2.24 kg. [3]

− End Of Paper −

ANNEX B

MJC H2 Math JC2 Preliminary Examination Paper 2

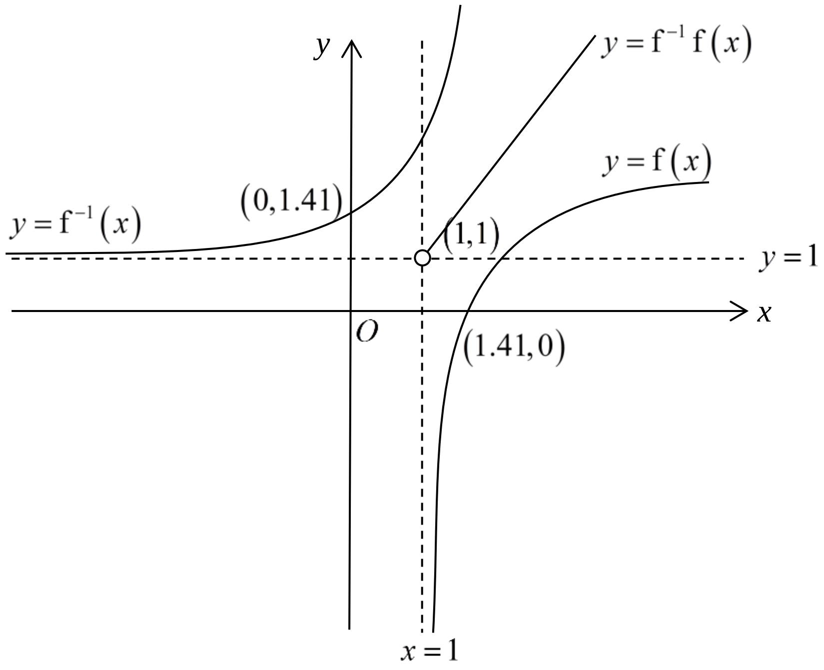
QN	Topic Set	Answers
1	Complex numbers	(i) $\frac{2}{3}$; $\frac{\pi}{6}$ (ii) $\frac{\sqrt{3}}{3}+\frac{1}{3}\mathrm{i}$ (iii) $k=\sqrt{3}$
2	Differential Equations	(i) $x=\frac{1}{t+1}+at+b$, where $b \in \mapsto$ (iii) $x=\frac{1+3\mathrm{e}^{4kt}}{6\mathrm{e}^{4kt}-2}$ or $\frac{1}{3\mathrm{e}^{4kt}-1}+\frac{1}{2}$
3	Functions	(i) $\mathrm{f}^{-1}: x \mapsto \sqrt{1+\mathrm{e}^{x}}$, $x \in \mapsto$ (iv) $\mathrm{R}_{\mathrm{gh}}=[0,6]$
4	Integration techniques	(i) $c=\mathrm{e}-1$ (ii) 1 (iii) $\mathrm{e}-1$
5	P&C, Probability	(a)(i) 78624000 (a)(ii) 27081600 (b) 6151600000
6	DRV	(i) $\begin{array}{|c|c|c|c|c|}\hline x & 1 & 2 & 3 & 4 \\ \hline \mathrm{P}(X=x) & \theta & 3\theta & 5\theta & 1-9\theta \\ \hline\end{array}$ (ii) $\mathrm{E}(X)=4-14\theta$ (iii) $\theta=0.0144$
7	Binomial Distribution	(i) 0.715 (ii) 0.715 (iii) $p=0.250$
8	Hypothesis Testing	(i) $\bar{x}=325.58$; $s^2=2.35$ $p-\text{value}=0.00169<0.05$ (ii) $\{\bar{x} \in \mapsto : 324.67<\bar{x}<325.33\}$
9	Correlation & Linear Regression	(iii) $\sqrt{x}=0.0896y+0.860$; $x=167$ (iv) $r=0.979$
10	Normal Distribution	(i) $\mu=255$; $\sigma=27$ (ii)(a) 0.0214 (ii)(b) $\mathrm{P}(T<2240)=0.241$ (3s.f.)

H2 Mathematics 2017 Prelim Exam Paper 2 Solution

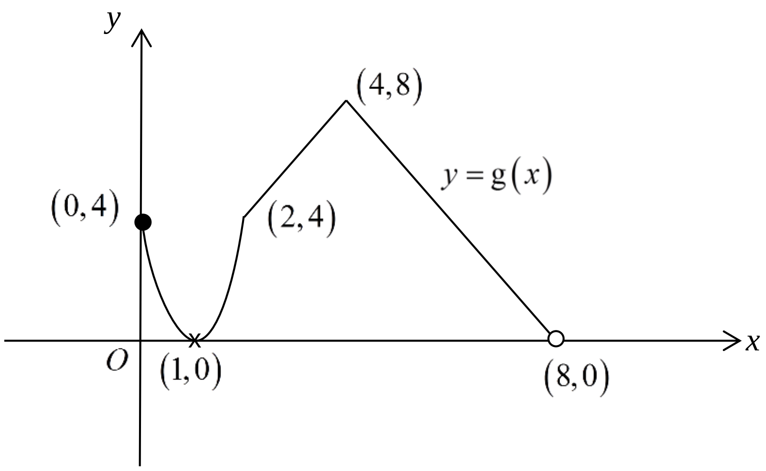
1	Solution: (i) Given $\lvert z\rvert=3,\ \arg(z)=\frac{2\pi}{3}$, $\left\lvert\frac{-2\mathrm{i}}{z*}\right\rvert=\frac{\lvert -2i\rvert}{\lvert z*\rvert}=\frac{2}{3}\quad\left(\because \lvert z\rvert=\lvert z*\rvert\right)$ $\arg\left(\frac{-2\mathrm{i}}{z*}\right)=\arg(-2\mathrm{i})-\arg(z*)$ $=-\frac{\pi}{2}-\left(-\frac{2\pi}{3}\right)\quad\left(\because \arg(z*)=-\arg(z)\right)$ $=\frac{\pi}{6}$ (ii) $\frac{-2\mathrm{i}}{z*}=\frac{2}{3}\left[\cos\left(\frac{\pi}{6}\right)+\mathrm{i}\sin\left(\frac{\pi}{6}\right)\right]$ $=\frac{2}{3}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right)$ $=\frac{\sqrt{3}}{3}+\frac{1}{3}\mathrm{i}$ (iii) $\frac{-2\mathrm{i}w}{z*}=\left(\frac{\sqrt{3}}{3}+\frac{1}{3}\mathrm{i}\right)(1+\mathrm{i}k)$ $=\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{3}k\mathrm{i}+\frac{1}{3}\mathrm{i}-\frac{1}{3}k$ Since $\frac{-2\mathrm{i}w}{z*}$ is purely imaginary, $\frac{\sqrt{3}}{3}-\frac{1}{3}k=0$ $k=\sqrt{3}$
2	Solution: (i)

	$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{2}{(t+1)^3}$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \int 2(t+1)^{-3}\ \mathrm{d}t$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2(t+1)^{-2}}{-2} + a$, where $a \in \mathbb{R}$ $= -(t+1)^{-2} + a$ $x = \int -(t+1)^{-2} + a\ \mathrm{d}t$ $= (t+1)^{-1} + at + b$, where $b \in \mathbb{R}$ $= \frac{1}{t+1} + at + b$, where $b \in \mathbb{R}$ (ii) $\frac{\mathrm{d}x}{\mathrm{d}t} = k - cx^2,\ \ k, c > 0$ When $x = 0.5$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ $k = c(0.5)^2$ $c = 4k$ $\frac{\mathrm{d}x}{\mathrm{d}t} = k - 4kx^2 = k(1-4x^2)$ (shown) (iii)

	$\frac{\mathrm{d}x}{\mathrm{d}t}=k-4kx^2=k\left(1-4x^2\right)$ $\int\frac{1}{1-4x^2}\,\mathrm{d}x=\int k\ \mathrm{d}t,\quad 1-4x^2\neq 0$ $\frac{1}{2(2)}\ln\left\|\frac{1+2x}{1-2x}\right\|=kt+d\ ,\ d\in\mathbb{R}$ $\frac{1}{4}\ln\left\|\frac{1+2x}{1-2x}\right\|=kt+d$ $\ln\left\|\frac{1+2x}{1-2x}\right\|=4kt+4d$ $\frac{1+2x}{1-2x}=\pm\mathrm{e}^{4kt+4d}=A\mathrm{e}^{4kt}$ where $A=\pm\,\mathrm{e}^{4d}$ When $t=0, x=1,$ $\frac{1+2}{1-2}=A$ $A=-3$ $\frac{1+2x}{1-2x}=-3\mathrm{e}^{4kt}$ $1+2x=-3\mathrm{e}^{4kt}+6x\mathrm{e}^{4kt}$ $x\left(2-6\mathrm{e}^{4kt}\right)=-3\mathrm{e}^{4kt}-1$ $x=\frac{-3\mathrm{e}^{4kt}-1}{2-6\mathrm{e}^{4kt}}=\frac{1+3\mathrm{e}^{4kt}}{6\mathrm{e}^{4kt}-2}$ or $\frac{1}{3\mathrm{e}^{4kt}-1}+\frac{1}{2}$
3	Solution: (i) Let $y=\ln\left(x^2-1\right)$. $x=\pm\sqrt{1+\mathrm{e}^y}$ Since $x>1>0$, $\therefore x=\sqrt{1+\mathrm{e}^y}$. $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_\mathrm{f}$ $=\mathbb{R}$ $\mathrm{f}^{-1}:x\mapsto\sqrt{1+\mathrm{e}^x},\ x\in\mathbb{R}$ (ii)



(iii)



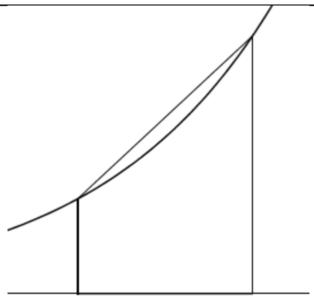
(iv) Since $R_h = [0,3] \subseteq [0,8) = D_g$, therefore the function gh exists.

Restrict D_g to be $[0,3]$

From the graph in **(iii)**, $R_{gh} = [0,6]$.

4	Solution: (i)

	$A_n=\frac{1}{n}\left(e^0+e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n-2}{n}}+e^{\frac{n-1}{n}}\right)$ $=\frac{1}{n}\cdot\frac{e^0\left(1-\left(e^{\frac{1}{n}}\right)^n\right)}{1-e^{\frac{1}{n}}}$ $=\frac{1}{n}\cdot\frac{1-e}{1-e^{\frac{1}{n}}}=\frac{e-1}{n\left(e^{\frac{1}{n}}-1\right)}$ $\therefore c=e-1$ (ii) $e^x-1=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)-1=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$ $\lim_{x\to0}\frac{1}{x}\left(e^x-1\right)=\lim_{x\to0}\left[\frac{1}{x}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)\right]$ $=\lim_{x\to0}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\dots\right)$ $=1$ (iii) $\lim_{n\to\infty}\frac{e-1}{n\left(e^{\frac{1}{n}}-1\right)}=\lim_{x\to0}\frac{e-1}{\frac{1}{x}\left(e^x-1\right)}$ $=e-1$ (iv) e – 1 is the exact area under the graph of $y=e^x$ from x = 0 to x = 1. area $=\int_0^1 e^x dx=e-1$. Since the graph of $y=e^x$ is concave upwards, and $\frac{A_n+B_n}{2}$ is the sum of the area of n trapeziums each of width $\frac{1}{n}$, the area of all trapeziums will be greater than the exact area under the graph, which is $\int_0^1 e^x dx$.



5 Solution:

(a)(i)

Total number of possible passcodes $= 26\times25\times24\times10\times9\times8\times7 = 78624000$

or

$= {}^{26}C_3 \times 3! \times {}^{10}C_4 \times 4! = 78624000$

or

$= {}^{26}P_3 \times {}^{10}P_4 = 78624000$

(a)(ii)

<u>Case 1:</u> 1st digit 4, 6, 8

4, 6, 8 (3 choices)	8 choices	7 choices	1, 3, 5, 7, 9 (5 choices)

Number of possible passcodes

$= 26\times25\times24\times3\times8\times7\times5 = 13104000$

<u>Case 2:</u> 1st digit 3, 5, 7, 9

3, 5, 7, 9 (4 choices)	8 choices	7 choices	4 choices

Number of possible passcodes

$= 26\times25\times24\times4\times8\times7\times4 = 13977600$

Total number of possible passcodes

$= 13104000 + 13977600 = 27081600$

(b)

Total number of possible passcodes

$$= 26^3 \times 10^4 \times \frac{7!}{4!3!} = 6151600000$$

6 Solution:

(i)

x	1	2	3	4

$\mathrm{P}(X=x)$	θ	3θ	5θ	k

Since $\sum_{\text{all } x} \mathrm{P}(X=x)=1$,

$\theta+3\theta+5\theta+k=1$

$\therefore k=1-9\theta$

Probability distribution of X is

x	1	2	3	4
$\mathrm{P}(X=x)$	θ	3θ	5θ	$1-9\theta$

(ii)

$$\begin{aligned}\mathrm{E}(X)&=1(\theta)+2(3\theta)+3(5\theta)+4(1-9\theta)\\&=\theta+6\theta+15\theta+4-36\theta\\&=4-14\theta\end{aligned}$$

$$\begin{aligned}\mathrm{E}(X^2)&=1^2(\theta)+2^2(3\theta)+3^2(5\theta)+4^2(1-9\theta)\\&=\theta+12\theta+45\theta+16-144\theta\\&=16-86\theta\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X)&=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2\\&=16-86\theta-(4-14\theta)^2\\&=16-86\theta-(16-112\theta+196\theta^2)\\&=26\theta-196\theta^2\end{aligned}$$

(iii)

$Y=a+bX$

$\mathrm{Var}(Y)=\mathrm{Var}(a+bX)$

$\mathrm{Var}(Y)=b^2\mathrm{Var}(X)$

$\frac{1}{3}b^2=b^2(26\theta-196\theta^2)$

$196\theta^2-26\theta+\frac{1}{3}=0 \quad (\because b\neq 0)$

Using GC,

$\theta=0.0144$ or $\theta=0.118$ (rejected $\because 0<\theta<\frac{1}{9}$)

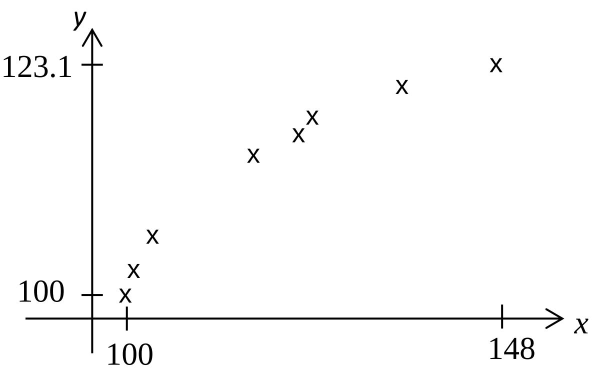
7	Solution:

	A binomial distribution is appropriate as there is a large number of lego pieces with constant probability 0.15 of them being red suggests independence in selection. Moreover, there are only two possible outcomes (red or non red). (i) Let X be the number of lego pieces, out of 20, that are red. $X \sim \mathrm{B}(20, 0.15)$ $\mathrm{P}(X \geq 4) = 1 - \mathrm{P}(X \leq 3)$ $= 0.35227$ $= 0.352 \quad (3\text{s.f.})$ (ii) Let Y be the number of boxes of lego pieces, out of 50, that contain at least 4 red lego pieces. $Y \sim \mathrm{B}(50, 0.35227)$ $\mathrm{P}(Y \leq 19) = 0.71498$ $= 0.715 \quad (3\text{ s.f.})$ (iii) Let A be the number of lego pieces, out of 20, that are red. $A \sim \mathrm{B}(20, p)$ $\mathrm{P}(1 \leq A < 4) = 0.22198$ $\mathrm{P}(A=1) + \mathrm{P}(A=2) + \mathrm{P}(A=3) = 0.22198$ $\binom{20}{1} p(1-p)^{19} + \binom{20}{2} p^2 (1-p)^{18} + \binom{20}{3} p^3 (1-p)^{17} = 0.22198$ $20p(1-p)^{19} + 190p^2(1-p)^{18} + 1140p^3(1-p)^{17} = 0.22198$ Since $0.2 < p < 1$, $p = 0.250$ (3 s.f)
8	Solution: (i) Using GC, Unbiased estimate of population mean is $\bar{x} = 325.58 \ (2\text{ d.p.})$ Unbiased estimate of population variance is $s^2 = 1.5326^2 = 2.35 \ (2\text{ d.p.})$ Let μ denote the population mean volume of shampoo dispensed by the machine. Given $X \sim \mathrm{N}(\mu, \sigma^2) \ \therefore \bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ $\mathrm{H_0}$: $\mu = 325$ $\mathrm{H_1}$: $\mu > 325$ Test statistic: $Z = \frac{\bar{X} - \mu}{S/\sqrt{n}}$ Level of significance: 5%

Alternatively,

Reject $\mathrm{H_0}$ if $z\text{-value} > 1.6449$

	Reject H_0 if $p-\text{value}<0.05$ Under H_0, using GC, $p-\text{value}=0.00160$ (3 s.f) or 0.00169 (3 s.f) Conclusion: Since $p-\text{value}=0.00169<0.05$, we **reject H_0** and conclude that there is **sufficient evidence**, at the 5% significance level, that the mean volume dispensed is more than 325 ml. Thus, the assembly manager's suspicion is valid at 5% level of significance. There is a probability of 0.05 of concluding that the mean volume of shampoo dispensed is more than 325 ml when in fact, it is 325 ml. (ii) H_0: $\mu=325$ H_1: $\mu\neq 325$ Given $X\sim \text{N}\left(\mu,\sigma^2\right)\ \therefore \overline{X}\sim \text{N}\left(\mu,\dfrac{\sigma^2}{n}\right)$ Test statistic: $Z=\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ Level of significance: 5% Since H_0 is not rejected, $-1.9600<z-\text{value}<1.9600$ $-1.9600<\dfrac{\bar{x}-325}{\frac{1.2}{\sqrt{50}}}<1.9600$ $324.67<\bar{x}<325.33 \quad (2\text{ d.p.})$ $\{\bar{x}\in\mathbb{R}:324.67<\bar{x}<325.33\}$
9	Solution: (i) Let k be the missing housing price index for 2011. $\bar{y}=111.775$ and $\bar{x}=\dfrac{815.2+k}{8}$ Since $\bar{y}$ and $\bar{x}$ lies on the regression line, $111.775=54.271+0.48363\left(\dfrac{815.2+k}{8}\right)$ $k=136 \quad (3\text{ s.f.})(\text{shown})$ (ii)



From the scatter diagram, as x increases, y increases at a decreasing rate. Thus the linear model might not be the most appropriate model.

(iii) (Note that there is no clear independent variable.)
From GC, an appropriate regression line would be
$\sqrt{x} = 0.0896y + 0.860 \quad (3 \text{ s.f.})$

When $y = 134.6$, from GC, $x = 167 \quad (3 \text{ s.f.})$.
The estimated housing price index in 2016 is 167.

Since $y = 134.6$ falls outside the data range of y, the linear correlation between y and $\sqrt{x}$ might no longer hold and thus, the estimate is unreliable.

(iv) From GC, $r = 0.979 \quad (3 \text{ s.f.})$.

(v) The product moment correlation coefficient between $\sqrt{\frac{x}{100}}$ and $\frac{y}{100}$ **does not differ** from the value obtained in part **(iv)** as the r-value is **independent of the scale of measurement.**

Note that: $\sqrt{\frac{x}{100}} = \frac{\sqrt{x}}{10}$ means that the values of $\sqrt{x}$ undergo a scaling of 10 units and $\frac{y}{100}$ means that the values of y undergo a scaling of 100 units.

10 (i)
Let X be the mass of a randomly chosen nut in grams.

$X \sim \mathrm{N}\left(\mu, \sigma^2\right)$

$\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{50}\right)$ and $X_1 + \ldots + X_{50} \sim \mathrm{N}\left(50\mu, 50\sigma^2\right)$

Given $\mathrm{P}\left(\overline{X} < 247\right) = 0.018079$ and $\mathrm{P}\left(X_1 + \ldots + X_{50} > 12600\right) = 0.78397$

Standardizing, $Z \sim \mathrm{N}(0,1)$

$$P\left(Z < \frac{247-\mu}{\sigma/\sqrt{50}}\right) = 0.018079$$ $$\frac{247-\mu}{\sigma/\sqrt{50}} = -2.095146$$ $$\mu - 0.2962984\sigma = 247 \ldots(1)$$	$$P\left(Z > \frac{12600-50\mu}{\sqrt{50}\sigma}\right) = 0.78397$$ $$P\left(Z < \frac{12600-50\mu}{\sqrt{50}\sigma}\right) = 0.21603$$ $$\frac{12600-50\mu}{\sqrt{50}\sigma} = -0.7856714$$ $$50\mu - 0.7856714\left(\sqrt{50}\sigma\right) = 12600 \ldots(2)$$

Solving equation (1) and (2), using GC,

$\mu = 255$ (nearest gram)

$\sigma = 27$ (nearest gram)

(ii)(a)

Let Y be the mass of a randomly chosen nut in grams.

$Y \sim \mathrm{N}\left(250, 5^2\right)$

Let W be the mass of a randomly chosen bolt in grams.

$W \sim \mathrm{N}\left(745, 7.3^2\right)$

$W - 3Y \sim \mathrm{N}\left(745 - 3\times 250, 7.3^2 + 3^2 \times 5^2\right)$

i.e. $W - 3Y \sim \mathrm{N}\left(-5, 278.29\right)$

$$\begin{aligned} P\left(\lvert W-3Y\rvert \ge 40\right) &= P\left(W-3Y < -40\right) + P\left(W-3Y > 40\right) \\ &= 0.0214 \text{ (3s.f.)} \end{aligned}$$

(ii)(b)

Let T be total mass of 10 randomly chosen nut, made using new material, in grams.

$T = 0.9Y_1 + 0.9Y_2 + \ldots + 0.9Y_{10} \sim \mathrm{N}\left(10\times 0.9\times 250, 10\times 0.9^2 \times 5^2\right)$

$T \sim \mathrm{N}\left(2250, 202.5\right)$

$P\left(T < 2240\right) = 0.241$ (3s.f.)