SRJC Paper 1

1 The complex numbers z and w satisfy the simultaneous equations

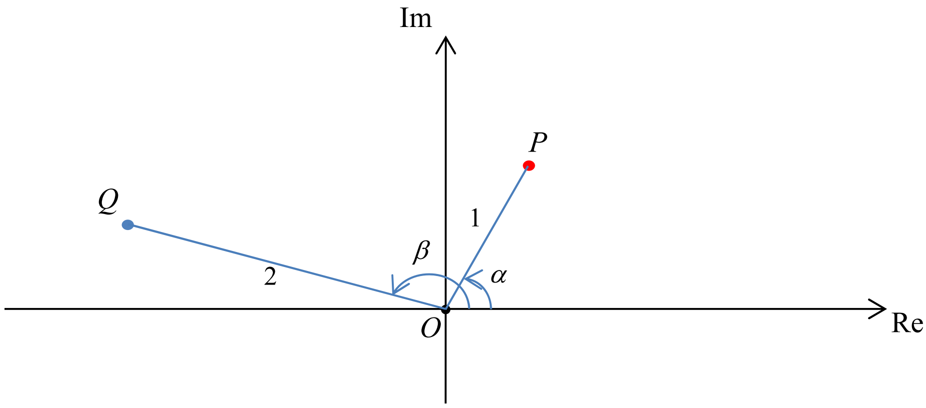
$$\mathrm{i}z + w = 2 + \mathrm{i} \quad \text{and} \quad 2w - (1+\mathrm{i})z = 8 + 4\mathrm{i} \; .$$

Find z and w in the form of $a + \mathrm{i}b$, where a and b are real. [5]

2 Solve the inequality $\dfrac{2x^2 + 2x - 1}{x^2 + 2x} \le 1$.

Hence, solve the inequality $\dfrac{2x^2 + 2|x| - 1}{x^2 + 2|x|} \le 1$. [6]

3 For $\alpha, \beta \in \mathbb{R}$ such that $2\alpha < \beta$, the complex numbers $z_1 = \mathrm{e}^{\mathrm{i}\alpha}$ and $z_2 = 2\mathrm{e}^{\mathrm{i}\beta}$ are represented by the points P and Q respectively in the Argand diagram below.



Find the modulus and argument of the complex numbers given by $\dfrac{\mathrm{i}}{2}z_2$ and $\dfrac{{z_1}^2}{z_2}$. [4]

Copy the given Argand diagram onto your answer script and indicate clearly the following points representing the corresponding complex numbers on your diagram.

(i) A: $\dfrac{\mathrm{i}}{2}z_2$ [1]

(ii) B: $\dfrac{{z_1}^2}{z_2}$ [1]

You are expected to indicate clearly the relevant moduli and arguments for parts **(i)** and **(ii)** on your Argand diagram.

If $\beta = \dfrac{11}{12}\pi$, find the smallest positive integer n such that the point representing the complex number $(z_2)^n$ lies on the negative real axis. [3]

4 The curve C has equation $4y^2 - 8y - x^2 - 4x - 4 = 0$.

(i) Using an algebraic method, find the set of values that y cannot take. [3]

(ii) Showing any necessary working, sketch C and indicate the equations of the asymptotes. [4]

5 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{\pi}{2}\tan\left(\frac{x}{2}\right), \quad x \in \mathbb{R}, -2\pi \le x \le 2\pi.$$

(i) Explain why f^{-1} does not exist. [2]

(ii) The domain of f is restricted to $(-\pi, a)$ such that a is the largest value for which the inverse function f^{-1} exists. State the exact value of a and define f^{-1} in a similar form. [3]

In the rest of the question, the domain of f is $(-\pi, a)$, where a takes the value found in part **(ii)**.

(iii) Sketch, in a single diagram, the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, labelling each graph clearly.Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$ and draw this line on your diagram. [3]

(iv) Verify that $x = \frac{\pi}{2}$ is a root of the equation $x = \mathrm{f}(x)$. Hence, explain why $x = \frac{\pi}{2}$ is also a solution to the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]

6 Referred to the origin O, the two points A and B have position vectors given by **a** and **b**, where **a** and **b** are non-zero vectors. The line l has equation $\mathbf{r} = 2\mathbf{a} + \lambda(\mathbf{a} + 2\mathbf{b})$, where $\lambda \in \mathbb{R}$. The point E is a general point on l and the point D has position vector $2\mathbf{a} - \mathbf{b}$.

Given that vector **a** is a unit vector, vector **b** has a magnitude of $\sqrt{2}$ units and that $\mathbf{a} \cdot \mathbf{b} = 1$,

(i) find the angle between vectors **a** and **b**, and, [2]

(ii) by considering $\overrightarrow{DE} \bullet \overrightarrow{DE}$, find an expression for the square of the distance DE, leaving your answer in terms of λ. [3]

Hence or otherwise, find the exact shortest distance of D to l, and write down the position vector of the foot of the perpendicular from D to l, in the form $p\mathbf{a} + q\mathbf{b}$. [3]

7 **(a)** By considering the Maclaurin expansion for $\cos x$, show that the expansion of $\sec x$ up to and including the term in x^4 is given by $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$. Hence show that the expansion for $\ln(\sec x)$ up to and including the term in x^4 is given by $\left[\frac{1}{2}x^2 + Ax^4\right]$ where A is an unknown constant to be determined. [4]

(b) The variables x and y satisfy the conditions **(A)** and **(B)** as follows:

$$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y \quad \text{---}\mathbf{(A)}$$

$$y = 0 \text{ when } x = 0 \quad \text{---}\mathbf{(B)}$$

(i) Obtain the Maclaurin expansion of y, up to and including the term in x^3. [4]

(ii) Verify that both conditions **(A)** and **(B)** hold for the curve $\ln(1+y) = \tan^{-1} x$.[2]

(iii) Hence, without using a graphing calculator, find an approximation for

$$\int_0^{\frac{1}{2}} \left(\mathrm{e}^{\tan^{-1} x} - 1\right) \mathrm{d}x .$$ [2]

8 **(a)** The fifth, ninth and eleventh terms of a geometric progression are also the seventh, twenty-fifth and forty-ninth terms of an arithmetic progression with a non-zero common difference respectively.

Show that $3R^6 - 7R^4 + 4 = 0$, where R is the common ratio of the geometric progression and determine if the geometric progression is convergent. [4]

(b) A semicircle with radius 12 cm is cut into 8 sectors whose areas follow a geometric progression. The first sector, which is the largest, has an area of A cm^2. The second sector has an area of Ar cm^2, the third sector has an area of Ar^2 cm^2, and so on, where r is a positive constant. Given also that the total area of the odd-numbered sectors is 10π cm^2 more than that of the even-numbered sectors, find the values of A and r. [5]

(c) The production levels of a particular coal mine in any year is 4% less than in the previous year. Show that the total production of the coal mine can never exceed 25 times the production in the first year. [2]

9

(a) Using the substitution $u = 2x+3$, find $\int \frac{x}{(2x+3)^3}\,dx$ in the for $-\frac{Px+Q}{R(2x+3)^2}+c$

where P, Q and R are positive integers to be determined. [3]

Hence find $\int \frac{x\ln(4x+3)}{(2x+3)^3}\,dx$. [3]

(b) Find $\int \sin 4x \cos 6x\,dx$. [2]

Hence or otherwise, find $\int e^x \sin 4e^x \cos 6e^x\,dx$. [1]

10 A particle is moving along a curve, C, such that its position at time t seconds after it is set into motion is given by the parametric equations

$$x = t + e^{-2t},\ y = t - e^{-2t}.$$

(i) State the coordinates of the initial position of the particle. [1]

(ii) Explain what would happen to the path of the particle after a long time. [1]

At the time of 2 seconds after the particle was set into motion, an external force struck the particle resulting in the particle moving in a straight line along the normal to the path at the point of collision.

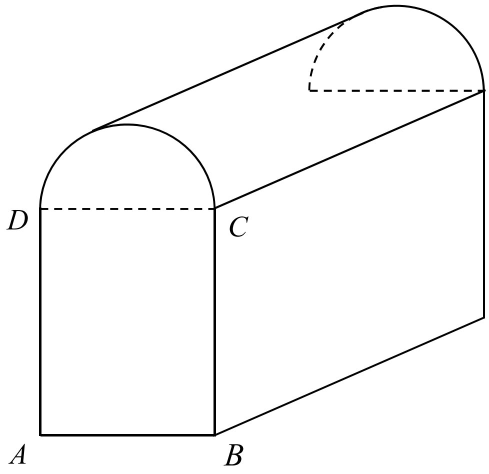
(iii) Find an equation for the normal to the curve C at the point $t = 2$, leaving your answer correct to 3 decimal places. [3]

After T seconds, where $T > 2$, the particle reaches point A, which lies on the x-axis, and stops moving.

(iv) Find the coordinates of the point A. Hence, give a sketch of the path traced by the particle, indicating the coordinates of any axial intercepts. [4]

(v) Find the total area bounded by the path of the particle in the first T seconds and the positive x-axis. [4]

11



A heavy wooden chest has a cross-sectional area made up of a rectangle and a semi-circle as shown in the diagram above. The wooden chest is constructed such that the perimeter of the cross-sectional area is 100 cm. It is given that the wooden chest is $2(a+b)$ cm long and the lengths of AB and BC are $2a$ cm and $2b$ cm respectively, where $a < 70$.

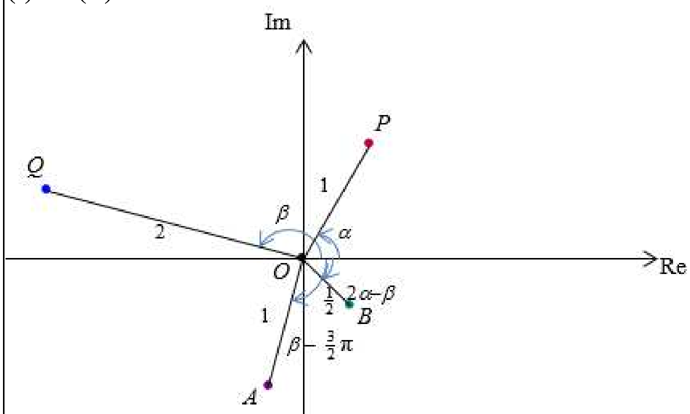
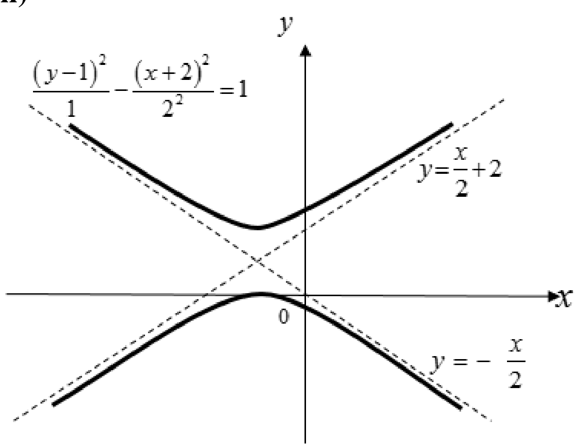
(i) Express b in terms of a. [1]

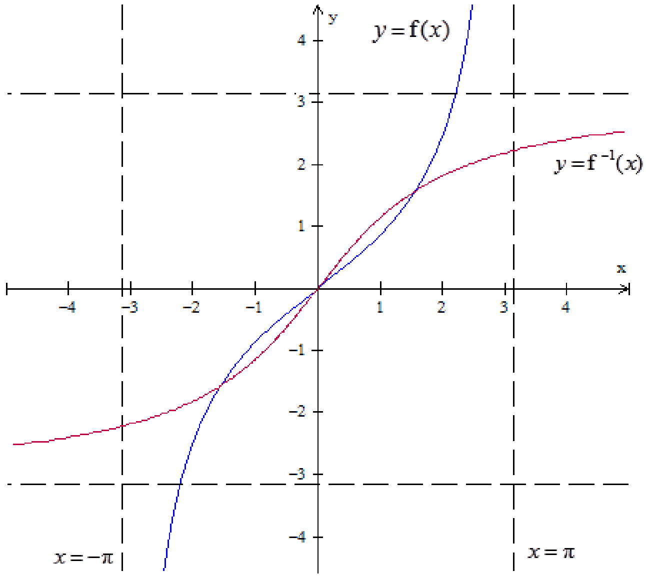
(ii) Show that the cross-sectional area of the wooden chest is given by $S = 100a - \frac{a^2}{2}(\pi + 4)$ and find the volume of the chest in terms of a and π. [4]

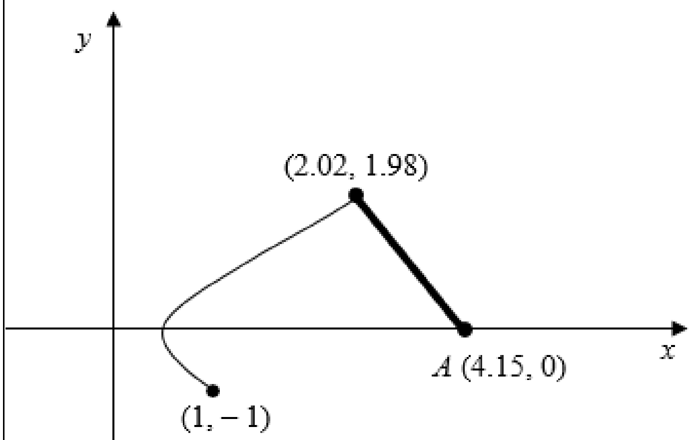
(iii) As a varies, find the value of a such that the volume of this wooden chest is greatest and find this volume correct to 2 decimal places. [5]

ANNEX B

SRJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Complex numbers	$z=-1+\mathrm{i}$ and $w=3+2\mathrm{i}$
2	Equations and Inequalities	$-2<x\le -1$ or $0<x\le 1$, $-1\le x\le 1,\ x\ne 0$
3	Complex numbers	$\left\lvert \frac{\mathrm{i}}{2}z_2 \right\rvert = 1$, $\arg(\frac{\mathrm{i}}{2}z_2) = \beta-\frac{3\pi}{2}$ $\left\lvert \frac{z_1^{\ 2}}{z_2} \right\rvert = \frac{1}{2}$, $\arg\left(\frac{z_1^{\ 2}}{z_2}\right) = 2\alpha-\beta$ **(i) & (ii)**  Smallest n required = 12
4	Graphs and Transformation	**(i)** $0<y<2$ **(ii)** 

5	Functions	**(ii)** $a=\pi$, $f^{-1}: x \mapsto 2\tan^{-1}\left(\frac{2x}{\pi}\right)$, $x \in R$. **(iii)**  The line required is $y=x$. **(iv)** Since the graphs of $y=f(x)$ and $y=f^{-1}(x)$ intersect along the line $y = x$, and since $x=\frac{\pi}{2}$ is a root of the equation $x=f(x)$, thus, the graphs of $y=f(x)$ and $y=f^{-1}(x)$ must also intersect at the point $x=\frac{\pi}{2}$.
6	Vectors	**(i)** $\theta = 45°$ **(ii)** $13\lambda^2+10\lambda+2$ Exact shortest distance from D to l is $\frac{1}{\sqrt{13}}$ units $\overrightarrow{OF}=\frac{21}{13}\mathbf{a}-\frac{10}{13}\mathbf{b}$
7	Maclaurin series	**(a)** $\frac{1}{2}x^2+\frac{1}{12}x^4$, $A=\frac{1}{12}$ **(b) (i)** $y=x+\frac{x^2}{2}-\frac{x^3}{6}+...$ **(iii)** $\frac{55}{384}$
8	AP and GP	**(a)** $r = \pm\sqrt{2}$ so $\lvert r\rvert > 1$ Hence, the geometric progression is not convergent.

		(b) $r = 0.75610$, $A = 61.8$
9	Integration techniques	**(a)** $P = 4$, $Q = 3$ and $R = 8$ $\int \frac{x\ln(4x+3)}{(2x+3)^3}\,dx = -\frac{(4x+3)\ln(4x+3)+2(2x+3)}{8(2x+3)^2}+C$ **(b)** $-\frac{1}{20}\cos 10x+\frac{1}{4}\cos 2x+C$, $-\frac{1}{20}\cos 10e^x+\frac{1}{4}\cos 2e^x+C$
10	Differentiation & Applications	**(i)** $(1,-1)$ **(ii)** The path of the particle **approaches the line $y = x$** **(iii)** $y = -0.929x + 3.857$ **(iv)** A (4.15, 0)  **(v)** 3.56 units^2
11	Differentiation & Applications	**(i)** $b = \frac{100-a(\pi+2)}{4}$ **(ii)** $5000a - 75\pi a - \frac{a^3}{4}(\pi^2+2\pi-8)$ **(iii)** $a = 12.7$, greatest volume = 29671.95 cm^3

H2 Mathematics 2017 Prelim Exam Paper 1 Question
Answer all questions [100 marks].

1

$\mathrm{i}z + w = 2 + \mathrm{i} -----(1)$

$2w - 1 - \mathrm{i}z = \dfrac{20}{2-\mathrm{i}} ---(2)$

Let $w = 2 + \mathrm{i} - \mathrm{i}z ---(3)$

Substitute eq (3) into eq (2)

$2(2 + \mathrm{i} - \mathrm{i}z) - z - \mathrm{i}z = 8 + 4\mathrm{i}$

$4 + 2\mathrm{i} - 3\mathrm{i}z - z = 8 + 4\mathrm{i} ----(5)$

Let $z = a + b\mathrm{i}$

Substitute $z = a + b\mathrm{i}$ into eq(5)

$4 + 2\mathrm{i} - 3\mathrm{i}(a + b\mathrm{i}) - (a + b\mathrm{i}) = 8 + 4\mathrm{i}$

$4 + 2\mathrm{i} - 3a\mathrm{i} + 3b - a - b\mathrm{i} = 8 + 4\mathrm{i}$

Comparing real and imaginary parts:

$4 + 3b - a = 8(\text{real parts}) ---(6)$

$2 - 3a - b = 4(\text{imaginary parts}) ---(7)$

Eq(6)$\times 3$ $-$ eq(7)

$10 + 10b = 20$

$10b = 10$

$b = 1$

Since $b = 1$, $4 + 3(1) - a = 8 \Rightarrow a = -1$

$\therefore z = -1 + \mathrm{i}$

Substituting $z = -1 + \mathrm{i}$ into eq(3) to solve for w

$w = 2 + \mathrm{i} + \mathrm{i} + 1 = 3 + 2\mathrm{i}$

Answer: $z = -1 + \mathrm{i}$ and $w = 3 + 2\mathrm{i}$

2

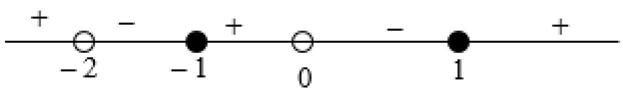
$\dfrac{2x^2 + 2x - 1}{x^2 + 2x} \le 1$

$\dfrac{2x^2 + 2x - 1}{x^2 + 2x} - 1 \le 0$

$\dfrac{2x^2 + 2x - 1 - x^2 - 2x}{x^2 + 2x} \le 0$

$\Rightarrow \dfrac{x^2 - 1}{x(x+2)} \le 0$

$\Rightarrow \dfrac{(x+1)(x-1)}{x(x+2)} \le 0$



Thus, $-2 < x \le -1$ or $0 < x \le 1$

Replacing x with $|x|$,

$-2 < |x| \leq -1$ or $0 < |x| \leq 1$

$-2 < |x| \leq -1 \ \Rightarrow$ no solution

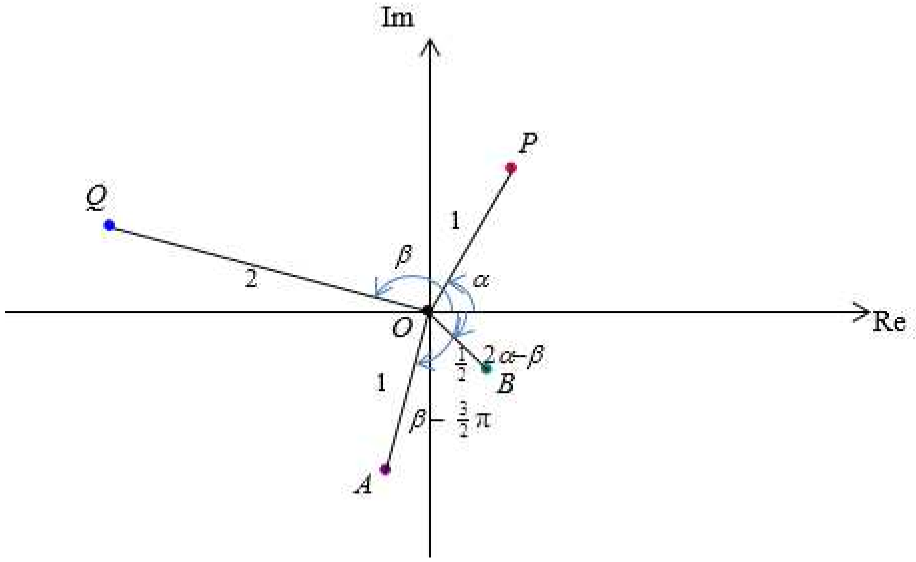
For $0 < |x| \leq 1$,

$0 < |x|$ and $|x| \leq 1$

$x \in \square, x \neq 0$ and $-1 \leq x \leq 1$

Thus, range of values: $-1 \leq x \leq 1,\ x \neq 0$

3



$$\frac{\mathrm{i}}{2} z_2 = \left(\frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)\left(2\mathrm{e}^{\mathrm{i}\beta}\right) = \mathrm{e}^{\mathrm{i}\left(\beta + \frac{\pi}{2}\right)}$$

Modulus = 1

Argument = $\beta + \frac{\pi}{2} - 2\pi = \beta - \frac{3\pi}{2}$

(i) Point A correctly plotted

$$\frac{{z_1}^2}{z_2} = \frac{\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\alpha}}{2\mathrm{e}^{\mathrm{i}\beta}} = \frac{1}{2}\mathrm{e}^{\mathrm{i}(2\alpha - \beta)}$$

Modulus = $\frac{1}{2}$

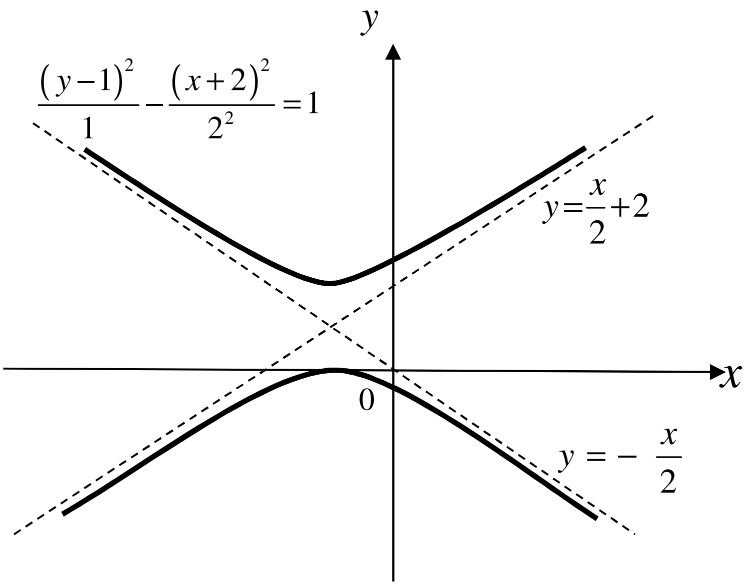
Argument = $2\alpha - \beta$

(ii) Point B correctly plotted

(b) $(z_2)^n = 2^n \mathrm{e}^{\mathrm{i}\frac{11\pi}{12}n}$

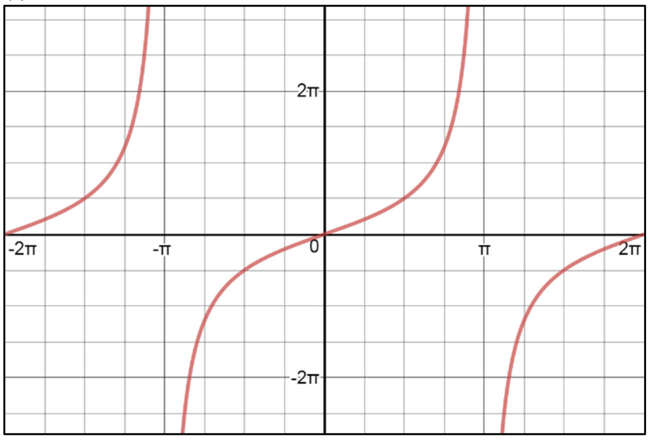
Since the point lies on the negative real axis, $\arg(z_2)^n = (2k + 1)\pi$ for $k \in \mathbb{Z}$.

$\therefore \ \frac{11}{12} n\pi = (2k + 1)\pi$

		$\Rightarrow\ n=\frac{12}{11}(2k+1)$ $\Rightarrow$ Smallest n required = 12
4	**(i)**	$-x^2-4x+\left(4y^2-8y-4\right)=0$ For values that y cannot take, there are no real solutions for x and discriminant<0. Therefore, $(-4)^2-4(-1)\left(4y^2-8y-4\right)<0$ $16+16y^2-32y-16<0$ $y^2-2y<0$ $y(y-2)<0$ $\therefore 0<y<2$ Set of values that y cannot take is $\{y\in ℝ: 0<y<2\}$.
	(ii)	$4y^2-8y-x^2-4x-4=0$ $4\left[(y-1)^2-1\right]-\left[(x+2)^2-4\right]-4=0$ $4(y-1)^2-4-(x+2)^2=0$ $\frac{(y-1)^2}{1}-\frac{(x+2)^2}{2^2}=1$ 

5

(i)



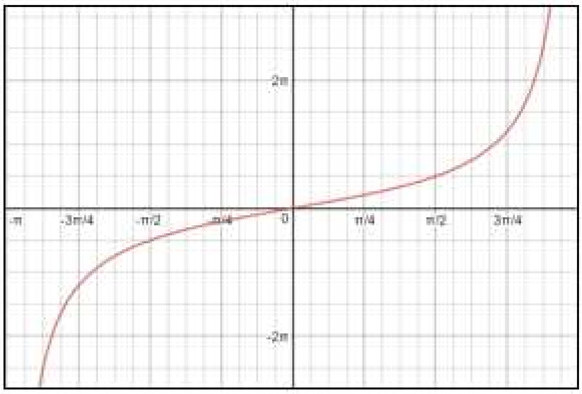
$y = \mathrm{f}(x)$

The horizontal line $y = 1$ cuts the graph of $y = \mathrm{f}(x)$ at **2 points**. Thus, $\mathrm{f}(x)$ is not a one-one function and the inverse of $\mathrm{f}(x)$ does not exist for the domain $[-2\pi, 2\pi]$.

OR

Any horizontal line $y = k$ ($k \in$ □) cuts the graph at more than one point. Thus, $\mathrm{f}(x)$ is not a one-one function and the inverse of $\mathrm{f}(x)$ does not exist for the domain $[-2\pi, 2\pi]$.

(ii)



$a = \pi$

To make x the subject of y

$$y = \frac{\pi}{2}\tan\left(\frac{x}{2}\right)$$

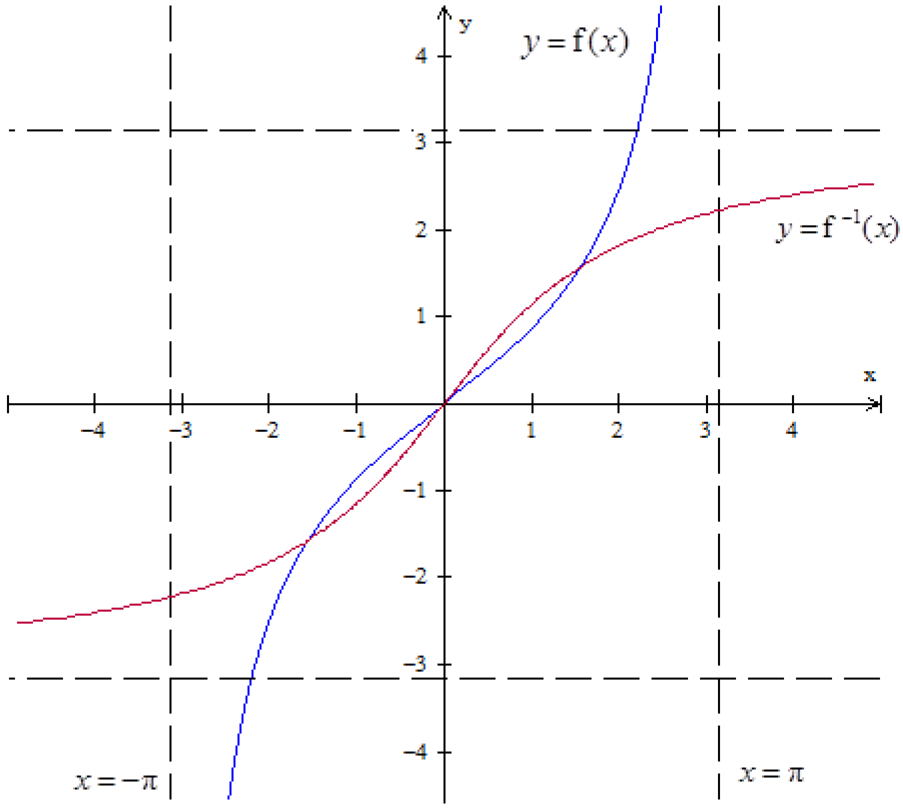
$$\frac{2y}{\pi} = \tan\left(\frac{x}{2}\right)$$

$$\tan^{-1}\left(\frac{2y}{\pi}\right) = \frac{x}{2}$$

$$\Rightarrow x = 2\tan^{-1}\left(\frac{2y}{\pi}\right)$$

$$\mathrm{f}^{-1} : x \mapsto 2\tan^{-1}\left(\frac{2x}{\pi}\right), \quad x \in R.$$

(iii)



The line required is $y = x$.

(iv)

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\tan\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

Thus, $x = \frac{\pi}{2}$ is a root of the equation $x = f(x)$.

Since the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect along the line $y = x$, and since $x = \frac{\pi}{2}$ is a root of the equation $x = f(x)$, thus, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ must also intersect at the point $x = \frac{\pi}{2}$.

6	(i) $\mathbf{a}.\mathbf{b} = \|a\|\|b\|\cos\theta \Rightarrow \|1\|\|\sqrt{2}\|\cos\theta$ $\mathbf{a}.\mathbf{b} = 1 \quad \therefore \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°$ (by inspection) (ii) $\overline{DE} = \overline{OE} - \overline{OD} = 2\mathbf{a} + \lambda(\mathbf{a}+2\mathbf{b}) - (2\mathbf{a} - \mathbf{b}) = \mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b}),\ \lambda \in R$ To find the square of the distance DE $DE^2 = [\mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})].[\mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})]$ $= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}+2\mathbf{b}).(\mathbf{a}+2\mathbf{b}) + 2\lambda\mathbf{b}.(\mathbf{a}+2\mathbf{b})$ $= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}.\mathbf{a} + 4\mathbf{a}.\mathbf{b} + 4\mathbf{b}.\mathbf{b}) + 2\lambda(\mathbf{b}.\mathbf{a} + 2\mathbf{b}.\mathbf{b})$ $= 2 + \lambda^2(1 + 4(1) + 4(2)) + 2\lambda(1 + 2(2))$ as $\mathbf{a}.\mathbf{a} = 1$, $\mathbf{b}.\mathbf{b} = 2$ and $\mathbf{a}.\mathbf{b} = 1$ $= 2 + 13\lambda^2 + 10\lambda$ $= 13\lambda^2 + 10\lambda + 2$ (iii) **<u>Method One:</u>** $DE^2 = 13\left[\lambda^2 + \frac{10}{13}\lambda\right] + 2$ $= 13\left(\lambda + \frac{10}{26}\right)^2 + 2 - \frac{25}{13} = 13\left(\lambda + \frac{5}{13}\right)^2 + \frac{1}{13}$ $DE = \sqrt{13\left(\lambda + \frac{5}{13}\right)^2 + \frac{1}{13}}$ The perpendicular distance from E to l occurs when D is closest to l, that is when DE is minimum or $\lambda = -\frac{5}{13}$. Exact shortest distance from D to l is $\frac{1}{\sqrt{13}}$ units.

	Method Two: DE is minimum when DE^2 is minimum: $\frac{\mathrm{d}}{\mathrm{d}x}(DE^2) = 26\lambda + 10$ To find stationary point: When $\frac{\mathrm{d}}{\mathrm{d}x}(DE^2) = 0,\ 26\lambda + 10 = 0$ $\therefore \lambda = -\frac{5}{13}$. Since DE^2 is quadratic and coefficient of $\lambda^2 > 0$, DE^2 is minimum at $\lambda = -\frac{5}{13}$ $\therefore$ perpendicular distance from D to l occur when $\lambda = -\frac{5}{13}$. $DE^2 = 13\lambda^2 + 10\lambda + 2 = 13\left(-\frac{5}{13}\right)^2 + 10\left(-\frac{5}{13}\right) + 2 = \frac{1}{13}$ Exact shortest distance from D to l is $\frac{1}{\sqrt{13}}$ units. (iv) Let F be the foot of the perpendicular from D to l. $\overrightarrow{OF} = 2\mathbf{a} - \frac{5}{13}(\mathbf{a} + 2\mathbf{b}) = \frac{21}{13}\mathbf{a} - \frac{10}{13}\mathbf{b}$
7	**(a)** $\sec x = \frac{1}{\cos x}$ $= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots\right)^{-1}$ $= 1 + (-1)\left[-\frac{1}{2}x^2 + \frac{1}{24}x^4\right] + \frac{(-1)(-2)}{2!}\left[-\frac{1}{2}x^2 + \frac{1}{24}x^4\right]^2 + \ldots$ $= 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{4}x^4 + \ldots$ $= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$ (up to x^4) (shown) $\ln(\sec x) \approx \ln\left[1 + \frac{1}{2}x^2 + \frac{5}{24}x^4\right]$ $= \left[\frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots\right] - \frac{1}{2}\left[\frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots\right]^2$

$$=\frac{1}{2}x^2+\frac{5}{24}x^4 \;-\frac{1}{2}\left(\frac{1}{4}x^4\right)+\ldots$$

$$=\frac{1}{2}x^2+\frac{1}{12}x^4$$

Thus $A=\frac{1}{12}$

(b)(i) $\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y$

$$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left(1+x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(1-2x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}$$

At $x = 0$, $y = 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=1,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1,\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-1$$

Thus, $y=x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots$

i.e. $y=x+\frac{x^2}{2}-\frac{x^3}{6}+\ldots$

(ii) $\ln\left(1+y\right)=\tan^{-1}x \Rightarrow \frac{1}{1+y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+x^2}$

$\therefore \left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y$ so condition **(A)** is satisfied.

At $x = 0$,

$\ln\left(1+y\right)=\tan^{-1}0=0 \Rightarrow 1+y=\mathrm{e}^0$

$\therefore y=0$

(iii) $\int_0^{\frac{1}{2}}\left(\mathrm{e}^{\tan^{-1}x}-1\right)\,\mathrm{d}x \approx \int_0^{\frac{1}{2}}\left(x+\frac{x^2}{2}-\frac{x^3}{6}\right)\,\mathrm{d}x=\frac{55}{384}$

8 **(a)**Let a denote the first term of the geometric progression.

Likewise, let b and d denote the first term and common difference of the arithmetic progression.

$\therefore$ $ar^4 = b + 6d$ …Eq(1)

$ar^8 = b + 24d$ …Eq(2)

$ar^{10} = b + 48d$ …Eq(3)

Eq(2) – Eq(1): $ar^8 - ar^4 = 18d$ …Eq(4)

Eq(3) – Eq(2): $ar^{10} - ar^8 = 24d$ …Eq(5)

Eq(5)/Eq(4): $\frac{ar^8\left(r^2-1\right)}{ar^4\left(r^4-1\right)}=\frac{24d}{18d}$

$$\frac{r^4}{r^2+1}=\frac{4}{3}$$

$3r^4 = 4r^2 + 4$ (Shown)

	From GC, $r = \pm\sqrt{2}$ so $\lvert r\rvert > 1$ Hence, the geometric progression is not convergent. **(b)** Let a be the 1st term and r be the common ratio of the G.P. $S_8 = \dfrac{A\left(1-r^8\right)}{1-r} = 72\pi$ ----- (1) $S_{odd} - S_{even} = 10\pi$ $\Rightarrow \dfrac{A\left(1-(r^2)^4\right)}{1-r^2} - \dfrac{Ar\left(1-(r^2)^4\right)}{1-r^2} = 10\pi$ $\dfrac{A\left(1-r^8\right)}{(1-r)(1+r)}\left[1-r\right] = 10\pi$ ----- (2) (1) ÷ (2): $\dfrac{1-r}{1+r} = \dfrac{10}{72}$ $72 - 72r = 10 + 10r$ $82r = 62$ $r = 0.75610$ Substituting into equation (1), A = 61.8 (to 3 s.f.) Let the production level in the first year be a. Total production of the coal mine = $\dfrac{a}{1-0.96}$ = $25a$ Thus, the total production of the coal mine can never exceed 25 times the production in the first year.
9	(a) Given $u = 2x+3 \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2$ $\displaystyle\int \frac{x}{(2x+3)^3}\,\mathrm{d}x = \int \frac{\frac{1}{2}(u-3)}{u^3}\cdot\frac{1}{2}\,\mathrm{d}u$ $\displaystyle= \frac{1}{4}\int\left[u^{-2} - 3u^{-3}\right]\mathrm{d}u$ $\displaystyle= \frac{1}{4}\left[-u^{-1} + \frac{3}{2}u^{-2}\right] + C$ $\displaystyle= -\frac{1}{4(2x+3)} + \frac{3}{8(2x+3)^2} + C$ $\displaystyle= \frac{-2(2x+3)+3}{8(2x+3)^2} + C$

	$=-\dfrac{4x+3}{8(2x+3)^2}+C$ $P = 4$, $Q = 3$ and $R = 8$ $\int \dfrac{\ln(4x+3)^x}{(2x+3)^3}\,\mathrm{d}x$ $=\int \dfrac{x}{(2x+3)^3}\cdot\ln(4x+3)\,\mathrm{d}x$ Let $\dfrac{\mathrm{d}v}{\mathrm{d}x}=\dfrac{x}{(2x+3)^3}, u=\ln(4x+3)$ $=-\dfrac{(4x+3)\ln(4x+3)}{8(2x+3)^2}-\int -\dfrac{(4x+3)}{8(2x+3)^2}\cdot\dfrac{4}{(4x+3)}\mathrm{d}x + C$ $=-\dfrac{(4x+3)\ln(4x+3)}{8(2x+3)^2}+\dfrac{1}{2}\int(2x+3)^{-2}\,\mathrm{d}x + C$ $=-\dfrac{(4x+3)\ln(4x+3)}{8(2x+3)^2}+\dfrac{1}{2}(2x+3)^{-1}\left(-\dfrac{1}{2}\right)+C$ $=-\dfrac{(4x+3)\ln(4x+3)}{8(2x+3)^2}-\dfrac{1}{4(2x+3)}+C$ $=-\dfrac{(4x+3)\ln(4x+3)+2(2x+3)}{8(2x+3)^2}+C$ (b) $\int \sin 4x\cos 6x\,\mathrm{d}x$ $=\dfrac{1}{2}\int \sin 10x+\sin(-2x)\,\mathrm{d}x$ $=\dfrac{1}{2}\int \sin 10x-\sin 2x\,\mathrm{d}x$ $=\dfrac{1}{2}\left[-\dfrac{1}{10}\cos 10x+\dfrac{1}{2}\cos 2x\right]+C$ $=-\dfrac{1}{20}\cos 10x+\dfrac{1}{4}\cos 2x+C$ $\int e^x\sin 4e^x\cos 6e^x\,\mathrm{d}x$ $=-\dfrac{1}{20}\cos 10e^x+\dfrac{1}{4}\cos 2e^x+C$
10	**(i)** At the original position, $t = 0$ $x=0+\mathrm{e}^0=1$ and $y=0-\mathrm{e}^0=-1$ Thus the coordinates are $(1,-1)$. **(ii)** As t tends to infinity, $\mathrm{e}^{-2t}\to 0$ so $x\to t$ and $y\to t$ Thus, the path of the particle **approaches the line $y = x$**

(iii) $\frac{dy}{dt}=1+2e^{-2t}$ and $\frac{dx}{dt}=1-2e^{-2t}$

$\frac{dy}{dx}=\frac{1+2e^{-2t}}{1-2e^{-2t}}$

At $t = 2$, $x=2+e^{-4}=2.01832$, $y=2-e^{-4}=1.98168$ and $\frac{dy}{dx}=\frac{1+2e^{-4}}{1-2e^{-4}}$

Gradient of normal = $\frac{2e^{-4}-1}{1+2e^{-4}}=-0.92933$

Thus, an equation for C_2 is $y-1.98168=-0.92933(x-2.01832)$

i.e. $y=-0.92933x+3.85737$

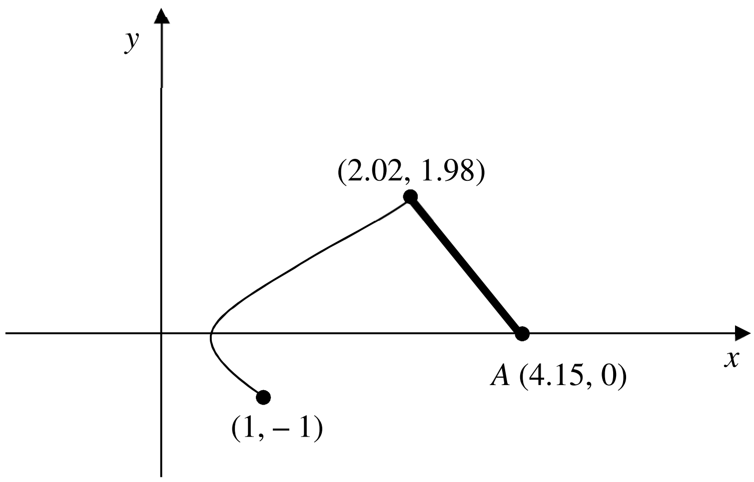
i.e. $y=-0.929x+3.857$ (correct to 3 d.p.)

(iv) At point A, $y = 0$

$0=-0.929x+3.857 \Rightarrow x=4.15178$

Coordinates of A are (4.15, 0)

Sketch of motion of particle:



(v) Consider the curve C_1 when $y = 0$,

$t=e^{-2t}$ and solving by GC, $t = 0.4263$

Thus, $x = 0.85261$

Required area

$=\int_{0.852}^{2.02} y\ dx+\int_{2.02}^{4.15}(-0.929x+3.857)\ dx$

$=\int_{0.4263}^{2}(t-e^{-2t})(1-2e^{-2t})dt+\int_{2.02}^{4.15}(-0.929x+3.857)\ dx$

= 3.5576 units^2

= 3.56 units^2

11

(i) Perimeter of cross-sectional area $=100=(2a+4b)+\frac{1}{2}(2\pi a)$

$\Rightarrow 100=4b+a(\pi+2)$

$$\Rightarrow b = \frac{100 - a(\pi + 2)}{4}$$

(ii) $S = (2a)(2b) + \frac{1}{2}(\pi a^2)$

$$= 4a\left[\frac{100 - a(\pi+2)}{4}\right] + \frac{\pi}{2}a^2$$

$$= 100a - a^2(\pi + 2) + \frac{\pi}{2}a^2$$

$$= 100a - \frac{a^2}{2}(2\pi + 4 - \pi)$$

$$= 100a - \frac{a^2}{2}(\pi + 4) \quad \text{(shown)}$$

Note that, $a + b = a + \frac{100 - a(\pi+2)}{4}$

$$= \frac{4a + 100 - a(\pi + 2)}{4}$$

$$= \frac{1}{4}\left[100 + a(2 - \pi)\right]$$

$$V = \left[100a - \frac{a^2}{2}(\pi + 4)\right]2(a + b)$$

$$= \left[100a - \frac{a^2}{2}(\pi + 4)\right]\cdot\frac{2}{4}\left[100 + a(2-\pi)\right]$$

$$= \frac{a}{2}\left[100 - \frac{a}{2}(\pi + 4)\right]\cdot\left[100 + a(2 - \pi)\right]$$

$$= 5000a - 75\pi a - \frac{a^3}{4}(\pi^2 + 2\pi - 8)$$

(iii) $\frac{dV}{da} = 5000 - 150\pi a - \frac{3}{4}a^2(\pi^2 + 2\pi - 8)$

When $\frac{dV}{da} = 0$, using the GC, $a = 12.70471$ or $a = 64.36321$

For $a = 12.70471$			
A	a^-	a	a^+
Sign	–	0	+
$\frac{dV}{da}$	/	—	\

For $a = 64.36321$			
a	a^-	a	a^+
sign	–	0	+
$\frac{dV}{da}$	\	—	/

Thus when $a = 12.70471 = 12.7$ (3 s.f.), volume is greatest.
Using the GC, greatest volume is 29671.95154=29671.95 cm^3.

− End Of Paper −

SRJC Paper 2

1 **(i)** Prove that $\dfrac{\sin(A-B)}{\cos A\cos B}=\tan A-\tan B$. [1]

(ii) Hence, by considering a suitable expression of A and B, find

$$\sum_{r=1}^{N}\frac{\sin x}{\cos\left[(r+1)x\right]\cos(rx)}.$$ [3]

(iii) Using your answer to part **(ii)**, find $\displaystyle\sum_{r=1}^{N}\left(\frac{\sqrt{3}}{2\cos\frac{r\pi}{3}\cos\frac{(r+1)\pi}{3}}\right)$, leaving your answer in terms of N. [2]

2 **(i)** Find $\displaystyle\int_2^n \frac{9x}{\left(x^2-1\right)^3}\,\mathrm{d}x$, where $n \geq 2$ and hence evaluate $\displaystyle\int_2^\infty \frac{9x}{\left(x^2-1\right)^3}\,\mathrm{d}x$. [3]

(ii) Sketch the curve $y=\dfrac{9x}{\left(x^2-1\right)^3}$ for $x\geq 0$. [2]

(iii) The region R is bounded by the curve, the line $y=\dfrac{2}{3}$ and the line $x = 5$.

Write down the equation of the curve when it is translated by $\dfrac{2}{3}$ units in the negative y-direction. [1]

Hence or otherwise, find the volume of the solid formed when R is rotated completely about the line $y=\dfrac{2}{3}$, leaving your answer correct to 3 decimal places. [2]

3 **(a)** **(i)** Show that $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta-\dfrac{1}{3}\sin^3\theta\right)=\cos^3\theta$. [1]

(ii) Find the solution to the differential equation $\operatorname{cosec} x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\cos^2 x$ in the form $y=\mathrm{f}(x)$, given that $y=0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{3}+\dfrac{2}{\pi}$ when $x=0$. [4]

(b) Show, by means of the substitution $v=x^2y$, that the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+2y+4x^2y=0$$

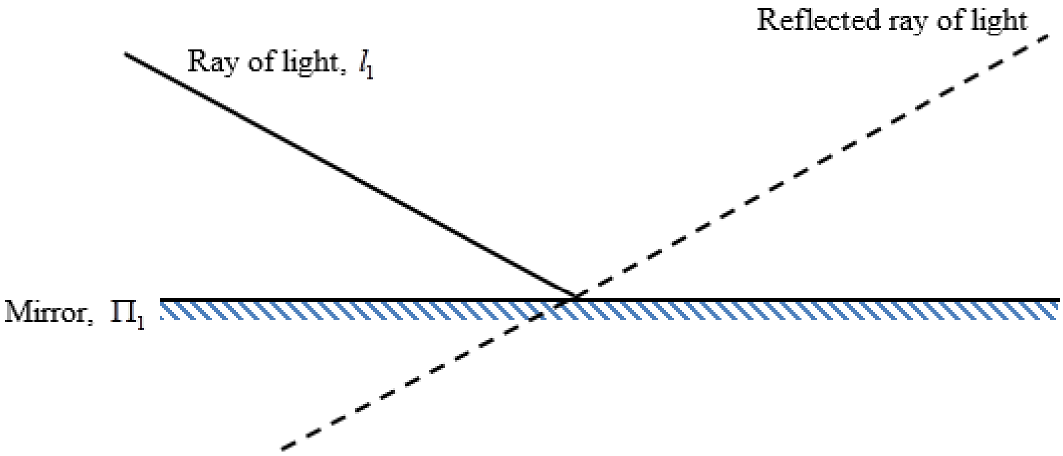
can be reduced to the form

$$\frac{\mathrm{d}v}{\mathrm{d}x}=-4vx .$$

Hence find y in terms of x, given that $y=\dfrac{1}{3}$ when $x=-3$. [6]

4 In the study of light, we may model a ray of light as a straight line.

A ray of light, l_1, is known to be parallel to the vector $2\mathbf{i}+\mathbf{k}$ and passes through the point P with coordinates $(1,1,0)$. The ray of light hits a mirror, and is reflected by the mirror which may be modelled by a plane Π_1 containing the points A, B and C with coordinates $(-1,1,0)$, $(0,0,2)$ and $(0,3,-3)$ respectively. This scenario is depicted in the diagram below:



(i) Show that an equation for plane Π_1 is given by $-x+5y+3z=6$. [3]

(ii) Find the coordinates of the point where the ray of light meets the mirror. [2]

(iii) Determine the position vector of the foot of the perpendicular from the point P to the mirror and hence, find an equation of the line that may be used to model the reflected ray of light. [6]

A second ray of light which is parallel to the mirror may be modelled by the line l_2, with Cartesian equation $\dfrac{x-1}{2}=\dfrac{z-2}{\alpha}, y=\beta$. Given that the distance between l_2 and the mirror is $\dfrac{14}{\sqrt{35}}$ units, find the values of the positive constants α and β. [4]

5 A random variable X has the probability distribution given in the following table.

x	2	3	4	5
$P(X=x)$	0.2	a	b	0.45

Given that $E(\lvert X-4\rvert)=\dfrac{11}{10}$, find the values of a and b. [3]

Two independent observations of X are taken. Find the probability that one of them is 2 and the other is at most 4. [2]

6 In a large consignment of mangoes, 4.5% of the mangoes are damaged.

(i) A total of 21 mangoes are selected at random. Calculate the probability that not more than 3 mangoes are damaged. [2]

(ii) The mangoes are randomly selected and packed into boxes of 21. For shipping purposes, the boxes are packed into cartons, with each carton containing 12 boxes. A box containing more than 3 damaged mangoes is considered low standard. Calculate the probability that, in a randomly selected carton, there are at least 2 boxes which are of low standard. [3]

(iii) Find the probability that a randomly chosen box that is of low standard contains no more than five damaged mangoes. [3]

7 **(a)** Seven boys and five girls formed a group in a school orientation. During one of the game segments, they are required to arrange themselves in a row. Find the exact probability that

(i) the girls are separated from one another, [2]

(ii) there will be exactly one boy between any two girls. [2]

In another game segment, they are required to sit at a round table with twelve identical chairs. Find the exact probability that one particular boy is seated between two particular girls. [2]

(b) The events A and B are such that $\mathrm{P}(A)=\dfrac{7}{10}$, $\mathrm{P}(B)=\dfrac{2}{5}$ and $\mathrm{P}(A\mid B)=\dfrac{13}{20}$.

(i) Find $\mathrm{P}(A\cup B)$, [3]

(ii) State, with a reason, whether the events A and B are independent. [1]

(c) A man plays a game in which he draws balls, with replacement, from a bag containing 3 yellow balls, 2 red balls and 4 black balls. If he draws a black ball, he loses the game and if he draws a red ball he wins the game. If he draws a yellow ball, the ball is replaced and he draws again. He continues drawing until he either wins or loses the game. Find the probability that he wins the game. [2]

8 A company manufactures bottles of iced coffee. Machines A and B are used to fill the bottles with iced coffee.

(i) Machine A is set to fill the bottles with 500 ml of iced coffee. A random sample of 50 filled bottles was taken and the volume of iced coffee (x ml) in each bottle was measured. The following data was obtained

$$\sum x = 24965 \quad \sum (x-\bar{x})^2 = 365$$

Calculate unbiased estimates of the population mean and variance. Test at the 2% level of significance, whether the mean volume of iced coffee per bottle is 500 ml. [6]

(ii) The company claims that Machine B filled the bottles with μ_0 ml of iced coffee. A random sample of 70 filled bottles was taken and the mean is 489.1 ml with standard deviation 4 ml. Find the range of values of μ_0 for which there is sufficient evidence for the company to have overstated the mean volume at the 2% level of significance. [5]

9 An online survey revealed that 34.1% of junior college students spent between 3 to 3.8 hours on their mobile phones daily. Assuming that the amount of time a randomly chosen junior college student spends on mobile phones daily follows a normal distribution with mean 3.4 hours and standard deviation σ hours, show that $\sigma = 0.906$, correct to 3 decimal places. [3]

Find the probability that

(i) four randomly chosen students each spend between 3 to 3.8 hours daily on their mobile phones. [1]

(ii) the total time spent on their mobile phones daily by the three randomly chosen junior college students is less than twice that of another randomly chosen junior college student. [3]

(iii) State an assumption required for your calculations in **(i)** and **(ii)** to be valid. [1]

N samples, each consisting of 50 randomly selected junior college students, are selected. It is expected that 15 of these samples will have a mean daily time spent on mobile phones greater than 3.5 hours.

(iv) Estimate the value of *N*. [4]

10 In a medical study, researchers investigated the effect of varying amounts of calcium intake on the bone density of Singaporean women of age 50 years. A random sample of eighty 50-year-old Singaporean women was taken.

(i) Explain, in the context of this question, the meaning of the phrase 'random sample'. [1]

The daily calcium intake (x mg) of the women was varied and the average percentage increase in bone density (y%) was measured. The data is as shown in the table below.

x (in mg)	700	800	900	1000	1050	1100
y (%)	0.13	0.78	1.38	1.88	2.07	2.10

(ii) Calculate the product moment correlation coefficient and suggest why its value does not necessarily mean that the best model for the relationship between x and y is $y = a + bx$. [2]

(iii) Draw a scatter diagram representing the data above. [2]

The researchers suggest that the change in bone density can instead be modelled by the equation $\ln(P - y) = a + bx$.

The product moment correlation coefficient between x and $\ln(P - y)$ is denoted by r. The following table gives values of r for some possible values of P.

P	3	5	10
r		–0.993803	–0.991142

(iv) Calculate the value of r for $P = 3$, giving your answer correct to 6 decimal places. Use the table and your answer to suggest with reason, which of 3, 5 or 10 is the most appropriate value of P. [2]

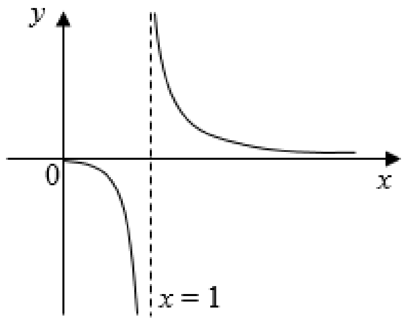
The Healthy Society wants to recommend a daily calcium intake that would ensure an average of 1.8% increase in bone density.

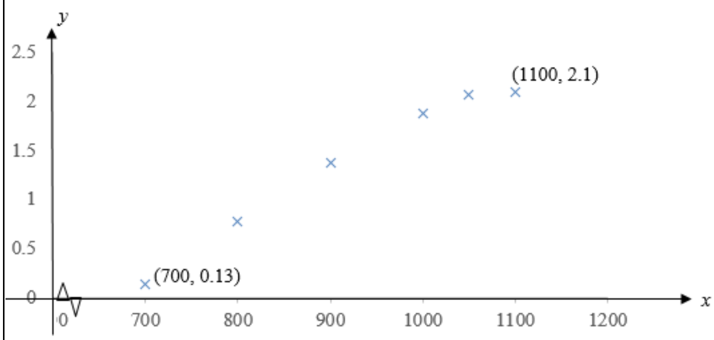
(v) Using the value of P found in part **(iv)**, calculate the values of a and b and use your answer to estimate the daily calcium intake that the Health Society should recommend. Comment on the reliability of the estimate obtained. [4]

(vi) Give an interpretation, in the context of the question, of the meaning of the value of P. [1]

ANNEX B

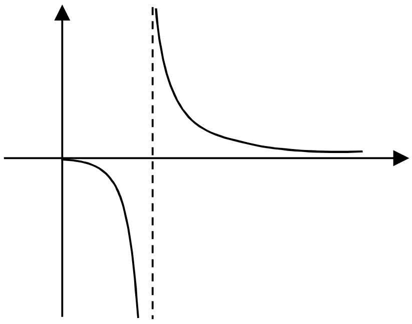
SRJC H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Sigma Notation and Method of Difference	**(ii)** $\tan(N+1)x - \tan x$ **(iii)** $\tan\left[\frac{(N+1)\pi}{3}\right] - \sqrt{3}$
2	Application of Integration	**(i)** $\frac{1}{4} - \frac{9}{4\left(n^2-1\right)^2}$, $\frac{1}{4}$ **(ii)**  **(iii)** $y = \frac{9x}{\left(x^2-1\right)^3} - \frac{2}{3}$, 3.385 units^3
3	Differential Equations	**(a) (ii)** $y = \frac{1}{3}\left(\sin x - \frac{\sin^3 x}{3}\right) + \frac{2}{\pi}x$ **(b)** $y = \frac{3e^{18-2x^2}}{x^2}$
4	Vectors	**(ii)** (5, 1, 2) **(iii)** $\overrightarrow{OF} = \begin{pmatrix} 33/35 \\ 9/7 \\ 6/35 \end{pmatrix}$, $l_1' : \mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 72 \\ -10 \\ 29 \end{pmatrix}, \gamma \in \mathbb{R}$ $\alpha = \frac{2}{3}$, $\beta = 3$
5	DRV	$a = 0.25$ and $b = 0.1$, 0.18
6	Binomial Distribution	**(i)** 0.987 **(ii)** 0.0106 **(iii)** 0.981
7	P&C, Probability	**(a) (i)** $\frac{7}{99}$ **(ii)** $\frac{1}{198}$, $\frac{1}{55}$ **(b)** 0.84 **(c)** $\frac{1}{3}$

8	Hypothesis Testing	**(i)** $\bar{x} = 499.3$, $s^2 \approx 7.45$, p-value = 0.06974 **(ii)** $\mu_0 \geq 490$
9	Normal Distribution	**(i)** 0.0135 **(ii)** 0.0781 **(iii)** Assumption: The amount of time spent by a randomly chosen student on mobile phones is independent of the amount of time spent by another randomly chosen student. **(iv)** $N = 69$
10	Correlation & Linear Regression	**(i)** The phrase 'random sample' means that every 50-year-old Singaporean woman has an **<u>equal probability of being included in the sample</u>**. **(ii)** $r = 0.988$ **(iii)**  **(iv)** $r = -0.995337$ **(v)** $a = 3.24$, $b = -0.00310$ The recommended daily calcium intake is 988 mg. Since the r value is – 0.995 is close to – 1, there is a strong negative linear correlation between $\ln(P - y)$ and x. Also since the value of $y = 1.8$ is within the data range, thus, the estimate obtained is reliable. **(vi)** The value of P is the maximum percentage increase in bone density achievable as the daily calcium intake increases.

H2 Mathematics 2017 Prelim Exam Paper 2 Question
Answer all questions [100 marks].

1	**(i)** $\frac{\sin(A-B)}{\cos A\cos B}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B}=\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}=\tan A-\tan B$ **(ii)** $\sum_{r=1}^{N}\frac{\sin x}{\cos(r+1)x\cos rx}=\frac{\sin(2x-x)}{\cos 2x\cos x}+\frac{\sin(3x-2x)}{\cos 3x\cos 2x}+\frac{\sin(4x-3x)}{\cos 4x\cos 3x}+\ldots+\frac{\sin((N+1)x-Nx)}{\cos(N+1)x\cos Nx}$ $=(\tan 2x-\tan x)$ $+(\tan 3x-\tan 2x)$ $+(\tan 4x-\tan 3x)$ $\vdots$ $+(\tan(N-1)x-\tan(N-2)x$ $+(\tan Nx-\tan(N-1)x$ $+(\tan(N+1)x-\tan Nx$ $=\tan(N+1)x-\tan x$ **(iii)** When $x=\frac{\pi}{3}$, $\sum_{r=1}^{N}\frac{\sin x}{\cos(r+1)x\cos rx}=\sum_{r=1}^{N}\left(\frac{\sqrt{3}}{2\cos\frac{r\pi}{3}\cos\frac{(r+1)\pi}{3}}\right)$ Thus, required sum = $\tan\left[(N+1)\left(\frac{\pi}{3}\right)\right]-\tan\left(\frac{\pi}{3}\right)=\tan\left[\frac{(N+1)\pi}{3}\right]-\sqrt{3}$
2	**(i)** $\int_2^n\frac{9x}{(x^2-1)^3}\,dx=\frac{9}{2}\int_2^n\frac{2x}{(x^2-1)^3}\,dx$ $=\frac{9}{2}\left[-\frac{1}{2}(x^2-1)^{-2}\right]_2^n$ $=\frac{9}{2}\left[-\frac{1}{2(n^2-1)^2}+\frac{1}{18}\right]$ $=\frac{1}{4}-\frac{9}{4(n^2-1)^2}$

	$\lim_{n\to\infty}\left[\int_2^n \frac{9x}{\left(x^2-1\right)^3}\ \mathrm{d}x\right]=\lim_{n\to\infty}\left[\frac{1}{4}-\frac{9}{4\left(n^2-1\right)^2}\right]$ $=\frac{1}{4}$ **(ii)**  **(iii)** The equation of the transformed curve is $y=\frac{9x}{\left(x^2-1\right)^3}-\frac{2}{3}$. Volume of revolution $=\pi\int_2^5\left(\frac{9x}{\left((x^2-1\right)^3}-\frac{2}{3}\right)^2\ \mathrm{d}x\ =3.385\text{ units}^3$ (to 3 d.p.)
3	**(a) (i)** $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)$ $=\cos\theta-\sin^2\theta\cos\theta$ $=\cos\theta\left(1-\sin^2\theta\right)$ $=\cos\theta\left(\cos^2\theta\right)=\cos^3\theta$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\sin x\cos^2 x$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(-\sin x\right)\left(\cos x\right)^2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(\cos x\right)^3}{3}+C$ $=\frac{1}{3}\left(\cos x\cdot\cos^2 x\right)+C$ $=\frac{1}{3}\left(\cos x\cdot\left(1-\sin^2 x\right)\right)+C$ $=\frac{1}{3}\left(\cos x-\cos x\cdot\sin^2 x\right)+C$

$$y=\frac{1}{3}\left(\sin x-\frac{\sin^3 x}{3}\right)+Cx+D$$

When $x = 0$ and $y = 0$, $D = 0$

When $x = 0$ and $\frac{dy}{dx}=\frac{1}{3}+\frac{2}{\pi}$, $C=\frac{2}{\pi}$

$$y=\frac{1}{3}\left(\sin x-\frac{\sin^3 x}{3}\right)+\frac{2}{\pi}x$$

(b) $v=x^2y$ --------------------(1)

$\frac{dv}{dx}=2xy+x^2\frac{dy}{dx}$ -------- (2)

$x\frac{dy}{dx}+2y+4x^2y=0$ ----- (3)

(3) × x, $x^2\frac{dy}{dx}+2xy+4x^2y(x)=0$ -------- (4)

$\frac{dv}{dx}+4x\left(x^2y\right)=0$

$\frac{dv}{dx}+4vx=0$

$\frac{dv}{dx}=-4vx$ (Shown)

$\frac{dv}{dx}=-4vx$

$\int\frac{1}{v}\,dv=-4\int x\,dx$

$\ln|v|=-2x^2+c$

$v=\pm e^{-2x^2+c}$

$v=Ae^{-2x^2}$, where $A=\pm e^c$

$x^2y=Ae^{-2x^2}$

Given that $y=\frac{1}{3}$ when $x=-3$,

$(-3)^2\left(\frac{1}{3}\right)=Ae^{-18}$

$A=3e^{18}$

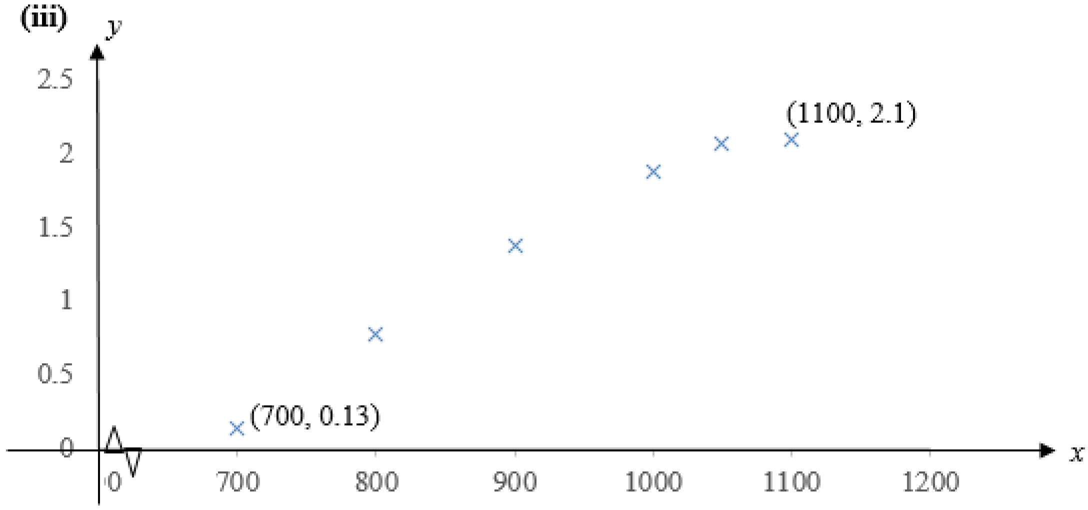
	$y=\dfrac{3e^{18-2x^2}}{x^2}$
4	**(i)** $l_1 : \mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\0\\1\end{pmatrix}, \lambda \in \mathbb{R}$ $\overrightarrow{AB} = \begin{pmatrix}0\\0\\2\end{pmatrix} - \begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\2\end{pmatrix}$; $\overrightarrow{AC} = \begin{pmatrix}0\\3\\-3\end{pmatrix} - \begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\-3\end{pmatrix}$; $\overrightarrow{BC} = \begin{pmatrix}0\\3\\-3\end{pmatrix} - \begin{pmatrix}0\\0\\2\end{pmatrix} = \begin{pmatrix}0\\3\\-5\end{pmatrix}$ A normal to the plane is: $\begin{pmatrix}1\\-1\\2\end{pmatrix} \times \begin{pmatrix}1\\2\\-3\end{pmatrix} = \begin{pmatrix}-1\\5\\3\end{pmatrix}$ $\mathbf{r} \cdot \begin{pmatrix}-1\\5\\3\end{pmatrix} = \begin{pmatrix}0\\0\\2\end{pmatrix} \cdot \begin{pmatrix}-1\\5\\3\end{pmatrix} = 6$ Thus an equation for Π_1 is $-x+5y+3z=6$. (shown) **(ii)** Let N be the point of intersection between the line and the plane. $\overrightarrow{ON} = \begin{pmatrix}1+2\lambda\\1\\\lambda\end{pmatrix}$ for some $\lambda \in \mathbb{R}$ Since N lies on the plane, $\begin{pmatrix}1+2\lambda\\1\\\lambda\end{pmatrix} \cdot \begin{pmatrix}-1\\5\\3\end{pmatrix} = 6 \Rightarrow \lambda = 2$ Thus, coordinates of N are (5, 1, 2). **(iii)** Let the foot of the perpendicular from P to the plane be denoted by F. $l_{PF} : \mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}-1\\5\\3\end{pmatrix}, \mu \in \mathbb{R}$ Since F lies on l_{PF}, $\overrightarrow{OF} = \begin{pmatrix}1-\mu\\1+5\mu\\3\mu\end{pmatrix}$ for some $\mu \in \mathbb{R}$ Since F lies on the plane, $\begin{pmatrix}1-\mu\\1+5\mu\\3\mu\end{pmatrix} \cdot \begin{pmatrix}-1\\5\\3\end{pmatrix} = 6$ Solving, $\mu = \dfrac{2}{35}$

	$\overrightarrow{OF}=\begin{pmatrix}33/35\\9/7\\6/35\end{pmatrix}$ Let the reflection of point *P* in the mirror be *P'*. By the midpoint theorem, $\overrightarrow{OP'}=2\overrightarrow{OF}-\overrightarrow{OP}=\begin{pmatrix}31/35\\11/7\\12/35\end{pmatrix}$ A direction vector for the reflected line is $\begin{pmatrix}5\\1\\2\end{pmatrix}-\begin{pmatrix}31/35\\11/7\\12/35\end{pmatrix}=\begin{pmatrix}144/35\\-4/7\\58/35\end{pmatrix}=\frac{2}{35}\begin{pmatrix}72\\-10\\29\end{pmatrix}$ Thus, an equation of the reflected line is: $l_1':\mathbf{r}=\begin{pmatrix}5\\1\\2\end{pmatrix}+\gamma\begin{pmatrix}72\\-10\\29\end{pmatrix},\gamma\in\mathbb{R}$ Since l_2 is parallel to Π_1, $\begin{pmatrix}2\\0\\\alpha\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\3\end{pmatrix}=0\Rightarrow\alpha=\frac{2}{3}$ $\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}1\\\beta\\2\end{pmatrix}=\begin{pmatrix}-1\\-\beta\\0\end{pmatrix}$ Since the distance is $\frac{14}{\sqrt{35}}$, $\left\lvert\frac{\begin{pmatrix}-1\\-\beta\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\3\end{pmatrix}}{\sqrt{35}}\right\rvert=\frac{14}{\sqrt{35}}$ $\lvert 1-5\beta\rvert=14$ Solving, $\beta=-\frac{13}{5}$ (rejected) or $\beta=3$
5	$\sum_{\text{all }x}\mathrm{P}(X=x)=1=0.2+a+b+0.45\Rightarrow a+b=0.35\ \dots\ (1)$ $E(\lvert X-4\rvert)=1\frac{1}{10}\Rightarrow\sum_{\text{all }x}\lvert x-4\rvert\mathrm{P}(X=x)=\frac{11}{10}$ $\Rightarrow 2(0.2)+a+0+0.45=\frac{11}{10}$ $\Rightarrow a=0.25$ and $b=0.1$

	P(required) $=P(X_1=2, X_2=2)+2\left[P(X_1=2, X_2=3)+P(X_1=2, X_2=4)\right]$ $=0.2\times0.2+2\left[0.2\times0.25+0.2\times0.1\right]$ $=0.18$
6	**(i)** Let X be the random variable "number of damaged mangoes out of 21 mangoes". $X \sim B(21, 0.045)$ $P(X\le 3)=0.98673=0.987$ (3 s.f.) **(ii)** Let Y be the random variable "number of boxes of mangoes out of 12 boxes which are of low standard". $Y \sim B(12, 1-0.98673) \Rightarrow Y \sim B(12, 0.013268)$ $P(Y\ge 2)=1-P(Y\le 1)$ $=1-0.98936=0.01064=0.0106$ (3 s.f.) **(iii)** P(required) = $P(X\le 5 \mid \text{box is of low standard})$ $=P\left(X\le 5 \mid X>3\right)$ $=\dfrac{P\left(X\le 5\cap X>3\right)}{P\left(X>3\right)}$ $=\dfrac{P\left(X=4\right)+P\left(X=5\right)}{1-P\left(Y\le 3\right)}$ $=\dfrac{0.011219+0.0017975}{1-0.98673}$ $=0.981$
7	**(a)(i)** Required probability = $\dfrac{7!\times{}^{8}C_{5}\times 5!}{12!}$ $=\dfrac{7}{99}$ **(a)(ii)** Required probability = $\dfrac{7!\times 4\times 5!}{12!}$ $=\dfrac{1}{198}$ Required probability = $\dfrac{\left(10-1\right)!\times 2!}{\left(12-1\right)!}$ $=\dfrac{1}{55}$

	(b)(i) $P(A\mid B)=\frac{13}{20}$ $\frac{P(A\cap B)}{P(B)}=\frac{13}{20}$ $P(A\cap B)=\frac{13}{20}\left(\frac{2}{5}\right)=\frac{13}{50}$ (or 0.26) $P(A\cup B)=P(A)+P(B)-P(A\cap B)$ $=\frac{7}{10}+\frac{2}{5}-\frac{13}{50}$ $=\frac{21}{25}$ (or 0.84) **(b)(ii)** Since $P(A\mid B)\neq P(A)$, therefore events *A* and *B* are not independent. **Alternatively,** Since $P(A\cap B)=\frac{13}{50}$ and $P(A)\times P(B)=\frac{7}{10}\times\frac{2}{5}=\frac{7}{25}\neq P(A\cap B)$, therefore events *A* and *B* are not independent. **(c)** Probability of winning the game $=\frac{2}{9}+\frac{2}{9}\left(\frac{3}{9}\right)+\frac{2}{9}\left(\frac{3}{9}\right)^2+...$ $=\frac{2/9}{1-\frac{3}{9}}$ $=\frac{1}{3}$
8	**(i)** Let *X* be the random variable denoting volume of the randomly chosen iced coffee bottle in ml from Machine *A*. $\bar{x}=\frac{24965}{50}=499.3$ Unbiased estimate of population variance $s^2=\frac{1}{n-1}\sum(x-\bar{x})^2=\frac{50}{49}\left(\frac{365}{50}\right)=\frac{365}{49}=7.4489\approx 7.45$ $H_0:\mu=500$ $H_1:\mu\neq 500$ Two tailed Z test at 2% level of significance Under H_0, since the sample size of 50 is large, by Central Limit Theorem

	$\overline{X} \sim \text{N}(500, \frac{7.4489}{50})$ approx. From GC, p-value = 0.06974> 0.02 Conclusion: Since the p-value is more than the level of significance, we do not reject H_0 and conclude that there is insufficient evidence at 2% that the mean volume is not 500ml. **(ii)** Let Y be the random variable denoting the volume of a randomly chosen iced coffee bottle in ml from Machine B. Unbiased estimate for population variance = $\frac{70}{69}\left(4^2\right)=16.232$ $H_0 : \mu = \mu_0$ $H_1 : \mu < \mu_0$ One tailed Z test at 2% level of significance Under H_0, since the sample size of 70 is large, by Central Limit Theorem $\overline{Y} \sim \text{N}\left(\mu_0, \frac{16.232}{70}\right)$ approx. Value of test statistic, $z_{\text{test}} = \frac{489.1-\mu_0}{\sqrt{\frac{16.232}{70}}}$ For H_0 to be rejected, p-value ≤ 0.02 $\frac{489.1-\mu_0}{\sqrt{\frac{16.232}{70}}} \leq -2.053748911$ $\mu_0 \geq 490$ (to 3 s.f.)
9	Let X denote the random variable representing the amount of time a randomly chosen junior college student spends on mobile phones each day. $\therefore X \sim \text{N}(3.4, \sigma^2)$ $\text{P}(3 < X < 3.8) = 0.341$ $\text{P}\left(\frac{3-3.4}{\sigma} < Z < \frac{3.8-3.4}{\sigma}\right) = 0.341$ $\text{P}\left(\frac{-0.4}{\sigma} < Z < \frac{0.4}{\sigma}\right) = 0.341$ $\Rightarrow \text{P}\left(Z < \frac{-0.4}{\sigma}\right) = \frac{1-0.341}{2} = 0.3295$ From GC, $\frac{-0.4}{\sigma} = -0.4412942379$ $\Rightarrow \sigma = 0.90642 = 0.906$ (3 dp) **(i)** Probability required $= (0.341)^4$ $= 0.0135$ (3 sf)

	(ii)	Probability required $= \mathrm{P}(X_1 + X_2 + X_3 < 2X_4)$ $= \mathrm{P}(X_1 + X_2 + X_3 - 2X_4 < 0)$ $X_1 + X_2 + X_3 - 2X_4 \sim \mathrm{N}(3.4\times3 - 2\times3.4,\ 0.90642^2\times3 + 2^2\times0.90642^2)$ i.e. $X_1 + X_2 + X_3 - 2X_4 \sim \mathrm{N}(3.4, 5.75118)$ $\therefore$ From GC, $(X_1 + X_2 + X_3 - 2X_4 < 0) = 0.0781$ (3 sf)
	(iii)	Assumption: The amount of time spent by a randomly chosen student on mobile phones is independent of the amount of time spent by another randomly chosen student.
	(iv)	$\overline{X} \sim \mathrm{N}\left(3.4, \frac{0.90642^2}{50}\right)$ From GC, $\mathrm{P}\left(\overline{X} > 3.5\right) = 0.217663$ Since expected number of samples with mean time exceeding 3.5 hours = 15, then $0.217663 \times N = 15$ $\Rightarrow N = 68.9 \approx 69$
10	**(i)**	The phrase 'random sample' means that every 50-year-old Singaporean woman has an **<u>equal probability of being included in the sample</u>**.
	(ii)	$r = 0.988$ (to 3 s.f.) Although the r-value = 0.988 is close to 1, the value is not 1 so there may be another model with $\lvert r\rvert$ closer to 1. Hence a linear model may not be the best model for the relationship between x and y.
	(iii)	
	(iv)	Using the GC, when $P = 3$, $r = -0.995337$ (to 6 d.p.) When $P = 3$, $\lvert r\rvert$ is closest to 1 and thus, $P = 3$ is the most appropriate value.
	(v)	When $P = 3$, using the GC, $a = 3.2446 = 3.24$ (to 3 s.f.)

	$b = -0.0030988 = -0.00310$ (to 3 s.f.) When $y = 1.8$, and $P = 3$, $\ln(3-1.8) = 3.2446 - 0.0030988x$ $x = 988$ Thus, the recommended daily calcium intake is 988 mg. Since the r value is -0.995 is close to -1, there is a strong negative linear correlation between $\ln(P-y)$ and x. Also since the value of $y = 1.8$ is within the data range, thus, the estimate obtained is reliable. **(vi)** The value of P is the maximum percentage increase in bone density achievable as the daily calcium intake increases.

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