H2 2017 Preliminary Exam Paper 1 Question
Answer all questions [100 marks].

1 Without using a calculator, solve the inequality

$$\frac{3x^2+7x+1}{x+3} < 2x-1. \qquad [4]$$

2 The function p is defined by $\mathrm{p}: x \mapsto \dfrac{1-x^2}{1+x^2},\ x \in \mapsto.$

(i) Find algebraically the range of p, showing your working clearly. [3]

(ii) Show that $\mathrm{p}(x)=\mathrm{p}(-x)$ for all $x \in \mapsto$ [1]

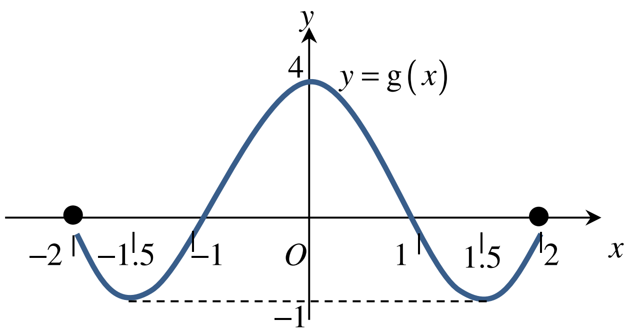
It is given that $\mathrm{q}(x)=\mathrm{p}\left(\dfrac{1}{2}x-4\right),\ x \in \mapsto.$

(iii) State a sequence of transformations that will transform the graph of p on to the graph of q. Hence state the line of symmetry for the graph of q.
[3]

3 The function f is defined by

$$\mathrm{f}: x \mapsto (x-k)^2\ ,\ x<k \text{ where } k>5.$$

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]



The diagram above shows the curve with equation $y=\mathrm{g}(x)$, where $-2 \le x \le 2$. The curve crosses the x–axis at $x=-2,\ x=-1,\ x=1$ and $x=2$, and has turning points at $(-1.5,-1)$, $(0,4)$ and $(1.5,-1)$.

(ii) Explain why the composite function fg exists. [2]

(iii) Find in terms of k,

(a)the value of fg (-1) [1]

(b)the range of fg. [2]

4 It is given that $z=-1-\mathrm{i}\sqrt{3}$.

(i) Given that $\dfrac{(\mathrm{i}z)^n}{z^2}$ is purely imaginary, find the smallest positive integer n. [4]

The complex number w is such that $|wz|=4$ and $\arg\left(\dfrac{w^*}{z^2}\right)=-\dfrac{5\pi}{6}$.

(ii) Find the value of $|w|$ and the exact value of $\arg(w)$ in terms of π. [3]

On an Argand diagram, points A and B represent the complex numbers w and z respectively.

(iii) Referred to the origin O, find the exact value of the angle OAB in terms of π. Hence, or otherwise, find the exact value of $\arg(z-w)$ in terms of π. [2]

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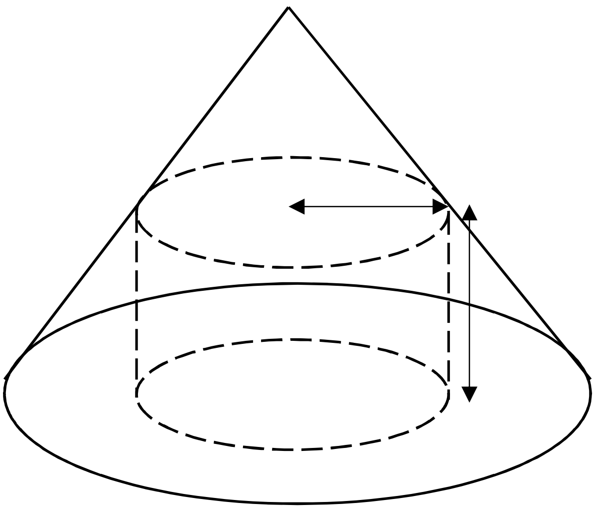


A metal cylinder of radius r cm and height h cm is inscribed in a circular cone paperweight of base radius 4 cm and height 6 cm (see diagram).

It is determined that the volume of the cylinder, V cm^3, should be as large as possible to provide weight to the paperweight. Show that

$$V = \frac{4\pi}{9}\left(36h - 12h^2 + h^3\right).$$ [2]

Hence find the exact maximum value of V. [5]

The metal cylinder is known to expand under heat. An experiment shows that the height of the cylinder is increasing at a rate of 0.04 cm s^{-1} at an instant when $h = 1.5$. Find the rate of change of V at this instant. [2]

6 Timber cladding is the application of timber planks over timber planks to provide the layer intended to control the infiltration of weather elements.

(a) Using method A, 20 rectangular planks are used and the lengths of the planks form an arithmetic progression with common difference d cm. The shortest plank has length 65 cm and the longest plank has length 350 cm.

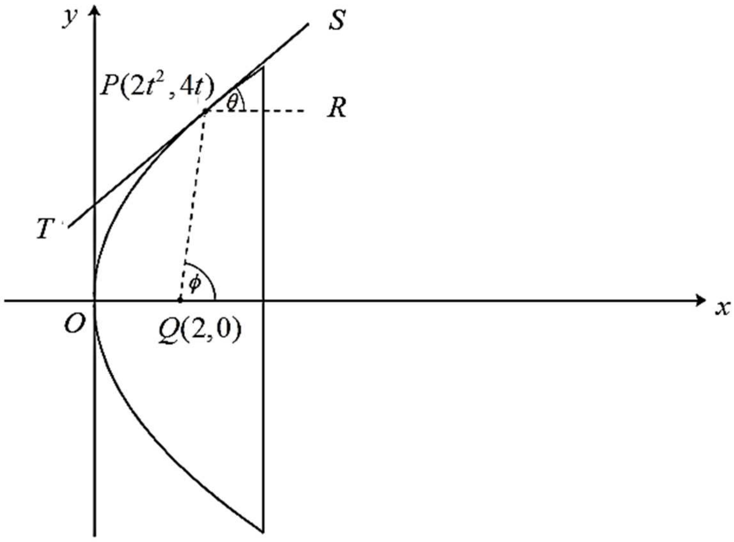
(i) Find the value of d. [2]

(ii) Find the total length of all the planks. [2]

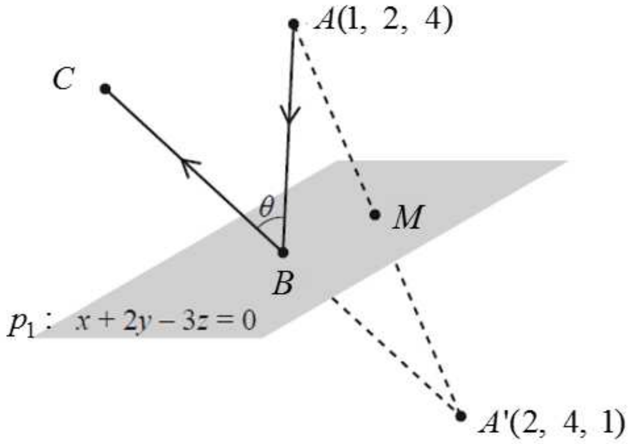
(b) Using method B, a long plank of 2000 cm is sawn off by a machine into n smaller rectangular planks. The length of the first plank is a cm and each successive plank is $\frac{8}{9}$ as long as the preceding plank.

(i) Show that the total length of the planks sawn off can never be greater than k times the length of the first plank, where k is an integer to be determined. [2]

	(ii) Given that $a = 423$, find the greatest possible integral value of n and the corresponding length of the shortest plank. [4]
7	**(i)** Express $\dfrac{1}{r^2-1}$ in partial fractions, and deduce that $$\frac{1}{r(r^2-1)}=\frac{1}{2}\left[\frac{1}{r(r-1)}-\frac{1}{r(r+1)}\right].$$ [2] **(ii)** Hence, find the sum, S_n, of the first n terms of the series $$\frac{1}{2\times3}+\frac{1}{3\times8}+\frac{1}{4\times15}+\dots .$$ [4] **(iii)** Explain why the series converges, and write down the value of the sum to infinity. [2] **(iv)** Find the smallest value of n for which S_n is smaller than the sum to infinity by less than 0.0025. [3]
8	A drug is administered by an intravenous drip. The drug concentration, x, in the blood is measured as a fraction of its maximum level. The drug concentration after t hours is modelled by the differential equation $$\frac{\mathrm{d}x}{\mathrm{d}t}=k\left(1+x-2x^2\right),$$ where $0 \le x < 1$, and k is a positive constant. Initially, $x = 0$. (i) Find an expression for x in terms of k and t. [5] After one hour, the drug concentration reaches 75% of its maximum level. (ii) Show that the exact value of k is $\frac{1}{3}\ln 10$, and find the time taken for the drug concentration to reach 90% of its maximum level. [3] A second model is proposed with the following differential equation $$\frac{\mathrm{d}x}{\mathrm{d}t}=\sin^2\left(\frac{1}{2}t\right),$$ where x is the drug concentration, measured as a fraction of its maximum level, in the blood after t hours. Initially, $x = 0$. **(iii)** Find an expression for x in terms of t. [3] **(iv)** Explain, with the aid of a sketch, why this proposed second model is inappropriate. [2]

9	
	The figure above shows a cross-section of a searchlight whose inner reflective surface is modelled, in suitable units, by the curve $$x = 2t^2, \quad y = 4t, \quad -\sqrt{2} \le t \le \sqrt{2}.$$ The inner reflective surface of the searchlight has the shape produced by rotating the curve about the x-axis. **(i)** Show that the curve has cartesian equation $y^2 = 8x$, and find the volume of revolution of the curve, giving your answer as a multiple of π. [3] $P(2t^2, 4t)$ is a point on the curve with parameter t. *TS* is the tangent to the curve at *P*, and *PR* is the line through *P* parallel to the x-axis. *Q* is the point $(2,0)$. The angles that *PS* and *QP* make with the positive x-direction are θ and ϕ respectively. **(ii)** By considering the gradient of the tangent *TS*, show that $\cot\theta = t$. [2] **(iii)** Find the gradient of the line *QP* in terms of t. Hence show that $\phi = 2\theta$, and show that angle *TPQ* is equal to θ. [5] A lamp bulb is placed at *Q*. **(iv)** Use your answer to part **(iii)** to describe the direction of the reflected light from the bulb. [1] **(v)** Find a cartesian equation of the locus of the mid-point *M* on *PQ* as t varies. [2]
10	Federal Aviation Administration data shows that there were an increase in aviation incidents caused by laser illuminations reported by pilots in 2015 and 2016. A simplified laboratory model is set up to investigate the effects of a laser beam on plexiglass, a common material used to make cockpit windscreen.

The piece of plexiglass is represented by a plane p_1 with equation $x+2y-3z=0$.



Referred to the origin, a laser beam *ABC* is fired from the point A with coordinates $(1,\ 2,\ 4)$, and is reflected at the point *B* on p_1 to form a reflected ray *BC* as shown in the diagram above. It is given that *M* is the midpoint of AA', where the point A' has coordinates $(2,\ 4,\ 1)$.

(i) Show that AA' is perpendicular to p_1. [2]

(ii) By finding the coordinates of *M*, show that M lies in p_1. [2]

The vector equation of the line *AB* is $\mathbf{r}=\begin{pmatrix}1\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix},\ \lambda\in \mapsto.$

(iii) Find the coordinates of *B*. [2]

The acute angle between the incident ray *AB* and the reflected ray *BC* is θ (see diagram).

(iv) Given that $A'BC$ is a straight line, find the value of θ. Hence, or otherwise, write down the acute angle between the line *AB* and p_1. [3]

To reduce the effect of laser illumination on the pilot sitting in the cockpit at point A', a scientist proposes to include a protective film, represented by a plane p_2, such that the perpendicular distance from p_1 to p_2 is 0.5.

(v) State the possible cartesian equations of p_2. [2]

To further investigate the effects of a laser beam on plexiglass, separate laser beams are fired such that the incident ray *AD* is now a variable line which is also fired from the same point *A* and is reflected at the variable point *D* on p_1 to form a reflected ray *DE*.

(vi) Given that AD is perpendicular to the previous ray AB, find the minimum possible distance between B and D. [2]

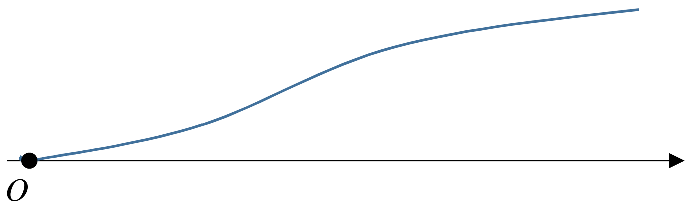
(vii) Find the acute inclination of the reflected ray DE to the z-axis when DE is inclined at $60°$ to the x-axis and $45°$ to the y-axis. [3]

− End Of Paper −

ANNEX B

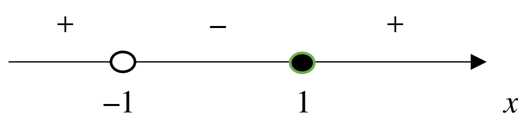
TPJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Equations and Inequalities	$x < -3$
2	Graphs and Transformation	(i) $-1 < y \le 1$ (iii) Translation by 4 units in the positive x-direction, followed by -Stretch of factor 2 parallel to the x-axis. **Alternative Answers:** Stretch of factor 2 parallel to the x-axis, followed by Translation by 8 units in the positive x-direction
3	Functions	(i) $\mathrm{f}^{-1}(x) = -\sqrt{x} + k$ $D_{\mathrm{f}^{-1}} = (0,\ \infty)$ (ii) $R_g = [-1,\ 4]$ $D_f = (-\infty,\ k)$ Since $k > 5$, $R_g \subseteq D_f$. Thus fg exists. (iii)(a) $\mathrm{fg}(-1) = \mathrm{f}(0) = k^2$ (b) $R_{\mathrm{fg}} = \left[(4-k)^2,\ (-1-k)^2\right]$ $= \left[(4-k)^2,\ (1+k)^2\right]$
4	Complex numbers	(i) $\therefore$ smallest positive integer $n = 5$. (ii) $\lvert w\rvert = 2$, $\arg(w) = \frac{13\pi}{6}$ (iii) **Hence Method:** $\arg(z-w) = -\left[\pi - \frac{\pi}{6} - \frac{\pi}{12}\right]$ $= -\left[\frac{5\pi}{6} - \left(\frac{1}{2}\left\{\pi - \frac{5\pi}{6}\right\}\right)\right]$ $= -\frac{3\pi}{4}$ (exact) **Otherwise Method:** $z - w = \left(-1-\sqrt{3}\right) + \left(-1-\sqrt{3}\right)\mathrm{i}$ $\arg(z-w) = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$

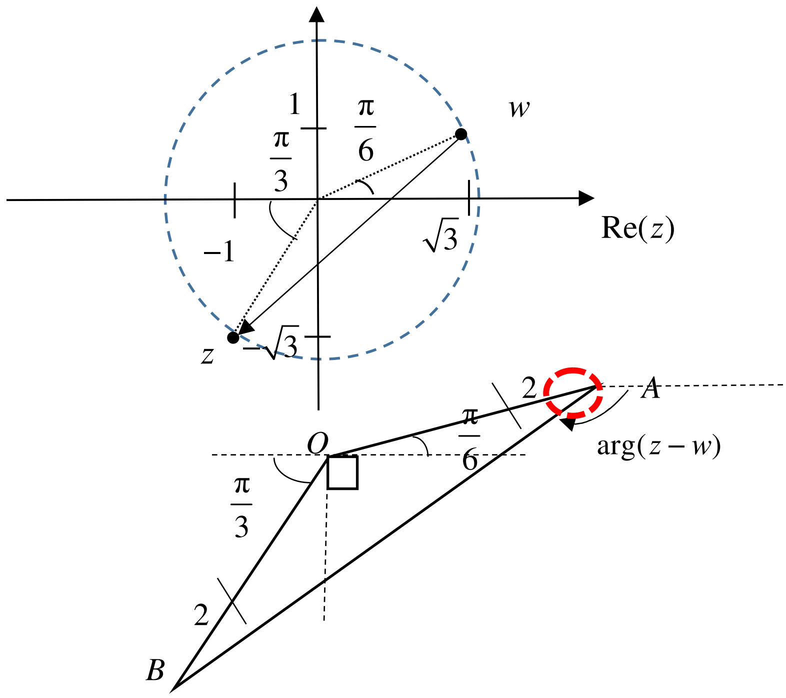
5	Differentiation & Applications	$V=\frac{128\pi}{9}$ $\frac{dV}{dt}=0.12\pi \quad \text{cm}^3\text{s}^{-1}$
6	AP and GP	(a)(i) $d=15$ (ii) $S_{20}=4150$ cm (b)(i) $k=9$ (ii) $n=6$, Length = 235 cm
7	Sigma Notation and Method of Difference	(ii) $\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$ (iii) As $n\to\infty$, $\frac{1}{2(n+1)(n+2)}\to 0.$ $\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\to\frac{1}{4}$ Sum to infinity $=\frac{1}{4}$ (iv)13
8	Differential Equations	(i) $x=\frac{e^{3kt}-1}{e^{3kt}+2}$ (ii)1.45 hours (iii) $x=\frac{1}{2}t-\frac{1}{2}\sin t$ (iv)  The graph shows that as time increases, the drug concentration still continue to increase / the curve shows a strictly increasing function beyond the maximum level of drug concentration.
9	Application of Integration	(i) 64π (iv) The reflected light from the bulb <u>produces a horizontal beam</u> of light/ produces a beam of line parallel to x-axis.

		(v) $y^2 = 4(x-1)$
10	Vectors	(ii) $\left(\frac{3}{2},\ 3,\ \frac{5}{2}\right)$ (iii) $\left(0,\ 3,\ 2\right)$ (iv) $\theta = 80.4°,\ 49.8°$ (v) $x+2y-3z=-\frac{\sqrt{14}}{2}$ or $x+2y-3z=\frac{\sqrt{14}}{2}$ (vi) $BD=\frac{\sqrt{6}}{\cos 49.8°}=3.79$ units (vii) $60°$

H2 Mathematics 2017 Preliminary Exam Paper 1 Solutions

1	$\dfrac{3x^2+7x+1}{x+3}<2x-1$ $\dfrac{3x^2+7x+1}{x+3}-(2x-1)<0$ $\dfrac{3x^2+7x+1-(2x-1)(x+3)}{x+3}<0$ $\dfrac{x^2+2x+4}{x+3}<0$ $\dfrac{(x+1)^2+3}{x+3}<0$ Since $(x+1)^2+3>0$ for all real x, the inequality reduces to: $x+3<0$ $\Rightarrow\ x<-3$
2	Let $y=\dfrac{1-x^2}{1+x^2},\ x\in\mathbb{R}$: $y(1+x^2)=1-x^2$ $(y+1)x^2+(y-1)=0$ Discriminant ≥ 0: $0^2-4(y+1)(y-1)\geq 0$ $-4(y^2-1)\geq 0$ $y^2-1\leq 0$ $y^2\leq 1$ $-1\leq y\leq 1$ Since $y=-1$ is an asymptote, $-1<y\leq 1$ **<u>Alternative Method:</u>** Let $y=\dfrac{1-x^2}{1+x^2},\ x\in\mathbb{R}$: $y(1+x^2)=1-x^2$ $(y+1)x^2+(y-1)=0$ $x^2=\dfrac{1-y}{y+1},\ y\neq -1$ Since $x^2\geq 0\ \forall x\in\mathbb{R}$, $\dfrac{1-y}{y+1}\geq 0$  $\therefore -1<y\leq 1$

2 (ii)	$\mathrm{p}(-x)=\dfrac{1-(-x)^2}{1+(-x)^2}$ $=\dfrac{1-x^2}{1+x^2}$ $=\mathrm{p}(x)$ for all $x\in$ □ (shown)
2(iii)	Graph of $\mathrm{q}(x)=\mathrm{p}\left(\dfrac{1}{2}x-4\right)$, $x\in$ □ is obtained from the graph of $\mathrm{p}(x)$ by: - Translation by 4 units in the positive *x*-direction, followed by Stretch of factor 2 parallel to the *x*-axis.
3(i)	Let $y=(x-k)^2$ $x-k=\pm\sqrt{y}$ $x=-\sqrt{y}+k \quad (\because x<k)$ $\mathrm{f}^{-1}(x)=-\sqrt{x}+k$ $\mathrm{D}_{\mathrm{f}^{-1}}=(0,\ \infty)$
3(ii)	$\mathrm{R_g}=[-1,\ 4]$ $\mathrm{D_f}=(-\infty,\ k)$ Since $k>5$, $\mathrm{R_g}\subseteq\mathrm{D_f}$. Thus fg exists.
3(iii)	$\mathrm{fg}(-1)=\mathrm{f}(0)=k^2$ Using $\mathrm{R_g}=[-1,\ 4]$, and the fact that f is a strictly decreasing function in the given domain, $\mathrm{R_{fg}}=\left[(4-k)^2,\ (-1-k)^2\right]$ $=\left[(4-k)^2,\ (1+k)^2\right]$
4(i)	$\lvert z\rvert=\sqrt{1^2+\sqrt{3}^2}=2 \qquad \arg z=-\left[\pi-\tan^{-1}\left(\dfrac{\sqrt{3}}{1}\right)\right]=-\dfrac{2\pi}{3}$ $z=2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$ $\dfrac{(\mathrm{i}z)^n}{z^2}=\dfrac{\mathrm{e}^{\mathrm{i}\left(\frac{n\pi}{2}\right)}2^n\mathrm{e}^{\mathrm{i}\left(-\frac{2n\pi}{3}\right)}}{2^2\mathrm{e}^{\mathrm{i}\left(-\frac{4\pi}{3}\right)}}$ $=2^{n-2}\mathrm{e}^{\mathrm{i}\left(\frac{n\pi}{2}-\frac{2n\pi}{3}+\frac{4\pi}{3}\right)}$ $=2^{n-2}\mathrm{e}^{\mathrm{i}\left(\frac{(8-n)\pi}{6}\right)}$ $\dfrac{(\mathrm{i}z)^n}{z^2}$ is purely imaginary: $\cos\left(\dfrac{(8-n)\pi}{6}\right)=0$ $\dfrac{(8-n)\pi}{6}=(2k+1)\dfrac{\pi}{2},\ k\in$ □ $n=5-6k,\ k\in$ □ **<u>Note:</u>** You may also have alternative form: $\dfrac{(8-n)\pi}{6}=(2k-1)\dfrac{\pi}{2},\ k\in$ □ $n=11-6k,\ k\in$ □

	$\therefore$ smallest positive integer $n=5$. **Alternative Method:** $n\arg(\mathrm{i}z)-2\arg(z)=n\arg(\mathrm{i})+n\arg(z)-2\arg(z)$ $=\dfrac{n\pi}{2}-\dfrac{2n\pi}{3}+\dfrac{4\pi}{3}$ $=\dfrac{(8-n)\pi}{6}$
4 (ii)	$\lvert wz\rvert=4$ $2\lvert w\rvert=4$ $\lvert w\rvert=2$ $\arg\left(\dfrac{w^*}{z^2}\right)=-\dfrac{5\pi}{6}$ $-\arg(w)-2\arg(z)=-\dfrac{5\pi}{6}$ $\arg(w)=\dfrac{5\pi}{6}-2\left(-\dfrac{2\pi}{3}\right)$ $=\dfrac{13\pi}{6}$ Since $-\pi<\arg(w)\le\pi$, $\arg(w)=\dfrac{\pi}{6}$ (exact).
4(iii)	 $\angle OAB=\dfrac{1}{2}\left\{\pi-\left[\left(\dfrac{\pi}{2}-\dfrac{\pi}{3}\right)+\dfrac{\pi}{2}+\dfrac{\pi}{6}\right]\right\}=\dfrac{\pi}{12}$ **<u>Hence Method:</u>** $\arg(z-w)=-\left[\pi-\dfrac{\pi}{6}-\dfrac{\pi}{12}\right]$ $=-\left[\dfrac{5\pi}{6}-\left(\dfrac{1}{2}\left\{\pi-\dfrac{5\pi}{6}\right\}\right)\right]$ $=-\dfrac{3\pi}{4}$ (exact)

Otherwise Method:

$z-w=\left(-1-\sqrt{3}\right)+\left(-1-\sqrt{3}\right)\mathrm{i}$ $\qquad \arg(z-w)=-\left(\pi-\frac{\pi}{4}\right)=-\frac{3\pi}{4}$

5

Using similar triangles: $\frac{r}{4}=\frac{6-h}{6}$

$r=\frac{2}{3}(6-h)$

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi\left(\frac{2}{3}(6-h)\right)^2 h \\ &= \frac{4\pi}{9}\left(36-12h+h^2\right)h \\ &= \frac{4\pi}{9}\left(36h-12h^2+h^3\right) \quad \text{(shown)} \end{aligned}$$

For maximum V, $\frac{\mathrm{d}V}{\mathrm{d}h}=0$:

$\frac{4\pi}{9}\left(36-24h+3h^2\right)=0$

Using GC: $h=2$ or $h=6$ (Rejected as $h = 6$ is height of cone)

Method 1 (1st derivative sign test)

h	2^-	2	2^+
Sign of $\frac{\mathrm{dV}}{\mathrm{dh}}$	+	0	-
slope	/	—	\

Thus, maximum volume $V=\frac{128\pi}{9}$ when $h = 2$ cm.

Method 2 (2nd derivative test)

$\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=\frac{4\pi}{9}(-24+6h)$

When $h=2$: $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=-\frac{16\pi}{3}<0$

Thus, maximum volume $V=\frac{128\pi}{9}$.

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}V}{\mathrm{d}h}.\frac{\mathrm{d}h}{\mathrm{d}t} \\ &= \frac{4\pi}{9}\left(36-24(1.5)+3(1.5)^2\right)(0.04) \\ &= 0.12\pi \quad \text{cm}^3\text{s}^{-1} \end{aligned}$$

(Accept: 0.377 cm^{3}s^{-1})

6(a)(i)

$u_{20}=a+(n-1)d$

$350=65+19d$

$d=15$

6(a)(ii)

$S_{20}=\frac{20}{2}(65+350)$

$\quad =4150$ cm $\qquad$ (Accept: 41.5 m)

6(b)(i)	$S_\infty = \dfrac{a}{1-\frac{8}{9}}$ $= 9a$ $\therefore$ integer $k = 9$.
6 (i)	Method 1: Number of ways $= \dbinom{14}{3} \times 3! = 2184$ Method 2: Number of ways $= 14 \times 13 \times 12 = 2184$
6(b)(ii)	$S_n \le 2000$ $\dfrac{423\left[1-\left(\frac{8}{9}\right)^n\right]}{1-\frac{8}{9}} \le 2000$ $1-\left(\frac{8}{9}\right)^n \le \dfrac{2000}{3807}$ $\left(\frac{8}{9}\right)^n \ge \dfrac{1807}{3807}$ $n \le \dfrac{\ln\left(\frac{1807}{3807}\right)}{\ln\left(\frac{8}{9}\right)}$ $n \le 6.3267$ $\therefore$ Largest integer $n = 6$. Length of shortest plank is $u_6 = 423\left(\dfrac{8}{9}\right)^{6-1}$ $= 235$ cm (3 s.f.)
7(i)	$\dfrac{1}{r^2-1} = \dfrac{1}{2(r-1)} - \dfrac{1}{2(r+1)}$ $\dfrac{1}{r(r^2-1)} = \dfrac{1}{r}\left[\dfrac{1}{2(r-1)} - \dfrac{1}{2(r+1)}\right]$ $= \dfrac{1}{2}\left[\dfrac{1}{r(r-1)} - \dfrac{1}{r(r+1)}\right]$
7 (ii)	$S_n = \dfrac{1}{2\times 3} + \dfrac{1}{3\times 8} + \dfrac{1}{4\times 15} + \ldots + (n\text{th term})$

$$= \sum_{r=2}^{n+1} \frac{1}{r(r^2-1)}$$

$$= \frac{1}{2}\sum_{r=2}^{n+1}\left[\frac{1}{r(r-1)} - \frac{1}{r(r+1)}\right]$$

$$
\begin{aligned}
= \frac{1}{2}\Bigg[& \frac{1}{2\times 1} - \frac{1}{2\times 3} \\
& + \frac{1}{3\times 2} - \frac{1}{3\times 4} \\
& + \frac{1}{4\times 3} - \frac{1}{4\times 5} \\
& \vdots \\
& \vdots \\
& \vdots \\
& + \frac{1}{(n-1)\times(n-2)} - \frac{1}{(n-1)\times n} \\
& + \frac{1}{(n)\times(n-1)} - \frac{1}{n\times(n+1)} \\
& + \frac{1}{(n+1)\times n} - \frac{1}{(n+1)\times(n+2)}\Bigg]
\end{aligned}
$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$$

$$= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

7 (iii)

As $n \to \infty$, $\dfrac{1}{2(n+1)(n+2)} \to 0.$

$$\frac{1}{4} - \frac{1}{2(n+1)(n+2)} \to \frac{1}{4}$$

Sum to infinity $= \dfrac{1}{4}$

7 (iv)

$$(0<)\frac{1}{4} - S_n < 0.0025$$

$$\Rightarrow (0<)\frac{1}{4} - \left[\frac{1}{4} - \frac{1}{2(n+1)(n+2)}\right] < 0.0025$$

$$\Rightarrow (0<)\frac{1}{2(n+1)(n+2)} < 0.0025$$

$$\Rightarrow (n+1)(n+2) > 200$$

Using G.C.

$n < -15.651$ or $n > 12.651$

Since $n \in \mathbb{Z}^+$,

Smallest value of $n = 13$

8(i)

Method 1: Using Partial Fractions

$$\frac{1}{1+x-2x^2}\frac{dx}{dt}=k$$

$$\frac{1}{1+x-2x^2}=\frac{1}{(1-x)(1+2x)}=\frac{2/3}{2x+1}-\frac{1/3}{x-1}$$

$$\int \frac{1}{1+x-2x^2}\ dx=\int k\ dt$$

$$\frac{2}{3}\int \frac{1}{2x+1}\ dx-\frac{1}{3}\int \frac{1}{x-1}\ dx=\int k\ dt$$

$$\frac{1}{3}\ln|2x+1|-\frac{1}{3}\ln|x-1|=kt+C$$

$$\frac{1}{3}\ln\left|\frac{2x+1}{x-1}\right|=kt+C$$

$$\frac{2x+1}{x-1}=Ae^{3kt},\ A=\pm e^{3C}$$

$$x=\frac{Ae^{3kt}+1}{Ae^{3kt}-2}$$

When $t=0,\ x=0$: $\quad 0=\dfrac{A+1}{A-2}\Rightarrow A=-1$

$$\therefore x=\frac{e^{3kt}-1}{e^{3kt}+2}$$

Method 2: Completing the square

$$\frac{1}{1+x-2x^2}\frac{dx}{dt}=k$$

$$\int \frac{1}{1+x-2x^2}\ dx=\int k\ dt$$

$$\int \frac{1}{-2\left(x-\frac{1}{4}\right)^2+\frac{9}{8}}\ dx=\int k\ dt$$

$$\frac{1}{2}\int \frac{1}{\left(\frac{3}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}\ dx=\int k\ dt$$

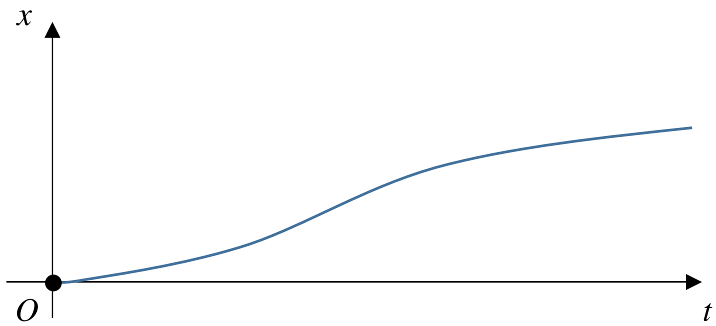
$$\frac{1}{2}\left(\frac{1}{2\left(\frac{3}{4}\right)}\right)\ln\left|\frac{\frac{3}{4}+x-\frac{1}{4}}{\frac{3}{4}-\left(x-\frac{1}{4}\right)}\right|=kt+C$$

$$\frac{1}{3}\ln\left|\frac{\frac{1}{2}+x}{1-x}\right|=kt+C$$

$$\frac{1}{3}\ln\left|\frac{2x+1}{2(1-x)}\right|=kt+C$$

$$\frac{2x+1}{2(1-x)}=Ae^{3kt},\ A=\pm e^{3C}$$

$$x=\frac{2Ae^{3kt}-1}{2\left(Ae^{3kt}+1\right)}$$

	When $t=0,\ x=0$: $\quad 0=\dfrac{2A-1}{2(A+1)} \Rightarrow A=\dfrac{1}{2}$ $\therefore x=\dfrac{e^{3kt}-1}{e^{3kt}+2}$
8 (ii)	When $t=1,\ x=\dfrac{3}{4}$: $\quad \therefore \dfrac{3}{4}=\dfrac{e^{3k}-1}{e^{3k}+2} \Rightarrow e^{3k}=10$ $\Rightarrow k=\dfrac{1}{3}\ln 10$ (shown) $\therefore x=\dfrac{10^t-1}{10^t+2}$ When $x=\dfrac{9}{10}$: $\quad \therefore \dfrac{9}{10}=\dfrac{10^t-1}{10^t+2} \Rightarrow 10^t=28$ $\Rightarrow t=\dfrac{\ln 28}{\ln 10}$ = 1.45 hours (3 s.f.) **<u>Also Accept:</u>** 86.8 mins (3 s.f.)
8 (iii)	$\dfrac{dx}{dt}=\sin^2\left(\dfrac{1}{2}t\right)$ $=\dfrac{1}{2}-\dfrac{1}{2}\cos t$ $x=\displaystyle\int \dfrac{1}{2}-\dfrac{1}{2}\cos t \ dt$ $=\dfrac{1}{2}t-\dfrac{1}{2}\sin t+C$ When $t=0,\ x=0$: $\quad C=0$ $\therefore x=\dfrac{1}{2}t-\dfrac{1}{2}\sin t$
8(iv)	 The graph shows that as time increases, the drug concentration still continue to increase / the curve shows a strictly increasing function beyond the maximum level of drug concentration.
9(i)	$y^2=(4t)^2=16t^2$ $=8(2t^2)$ $=8x$ (shown)

	$\text{Volume} = \pi\int_0^4 8x\,\mathrm{d}x$ $= \pi\left[4x^2\right]_0^4$ $= 64\pi$
9(ii)	$\frac{\mathrm{d}x}{\mathrm{d}t} = 4t,\ \frac{\mathrm{d}y}{\mathrm{d}t} = 4$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{t}$ Gradient of tangent $TS = \tan\theta$ $\therefore \tan\theta = \frac{1}{t}$ $\cot\theta = t$ (shown
9 (iii)	Gradient of line $QP = \frac{4t-0}{2t^2-2}$ $= \frac{2t}{t^2-1}$ $= \frac{2/\tan\theta}{1/\tan^2\theta - 1}$ $= \frac{2\tan\theta}{1-\tan^2\theta}$ $= \tan 2\theta$ $\tan\phi = \tan 2\theta \Rightarrow \phi = 2\theta$ (shown) $\angle QPR = 180° - \phi$ (interior angles) $= 180° - 2\theta$ (by earlier results) $\angle TPQ + (180° - 2\theta) + \theta = 180°$ $\therefore \angle TPQ = \theta$ (shown)
9 (iv)	The reflected light from the bulb produces a horizontal beam of light/ produces a beam of line parallel to x-axis
9 (v)	Midpoint $M = \left(\frac{2+2t^2}{2}, \frac{4t+0}{2}\right)$ $= \left(1+t^2,\ 2t\right)$ $\begin{cases} x = 1+t^2 \\ y = 2t \Rightarrow t = \frac{y}{2} \end{cases}$ Locus of midpoint M is: $x = 1 + \frac{y^2}{4}$ $y^2 = 4(x-1)$

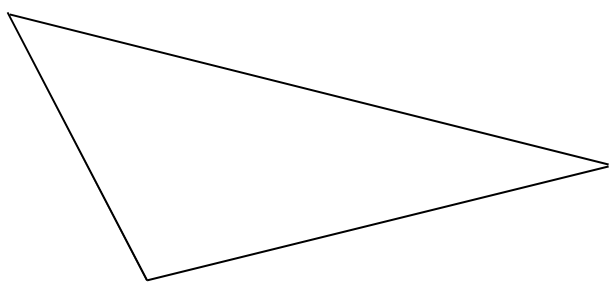
10(i)	$\overrightarrow{AA'} = \begin{pmatrix} 2-1 \\ 4-2 \\ 1-4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ Since $\overrightarrow{AA'} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \underset{\sim}{n}_1$, $\overrightarrow{AA'}$ is parallel to the normal of p_1, and thus $\overrightarrow{AA'}$ is perpendicular to p_1. **<u>Alternative Method:</u>** Since $\overrightarrow{A'A} = \begin{pmatrix} 1-2 \\ 2-4 \\ 4-1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = -\underset{\sim}{n}_1$, $\overrightarrow{A'A}$ is parallel to the normal of p_1, and thus $\overrightarrow{A'A}$ is perpendicular to p_1
10 (ii)	Since M is the midpoint of A and A': $\overrightarrow{OM} = \frac{1}{2}\left[\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right] = \begin{pmatrix} 3/2 \\ 3 \\ 5/2 \end{pmatrix}$ Coordinates of M are $\left(\frac{3}{2}, 3, \frac{5}{2}\right)$. **<u>Note:</u>** Question asks for coordinates form. Since $\frac{3}{2} + 2(3) - 3\left(\frac{5}{2}\right) = -6 + 6 = 0$, M lies in p_1. (shown)
10 (iii)	$\overrightarrow{OB} = \begin{pmatrix} 1+\lambda \\ 2-\lambda \\ 4+2\lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$. Since B lies on p_1, $(1+\lambda)+2(2-\lambda)-3(4+2\lambda)=0$ $-7-7\lambda=0$ $\lambda=-1$ $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$ Coordinates of B are $(0, 3, 2)$.

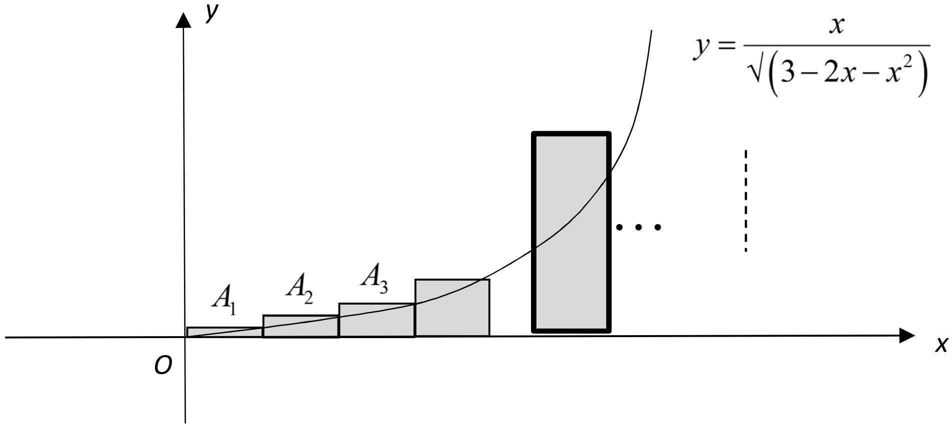
Likewise for part (**vi**).

10 (iv)	$\theta = \cos^{-1}\left\|\frac{\overrightarrow{BA}\cdot\overrightarrow{A'B}}{\left\|\overrightarrow{BA}\right\|\left\|\overrightarrow{A'B}\right\|}\right\|$ $= \cos^{-1}\left\|\frac{\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\\1\end{pmatrix}}{\sqrt{6}\sqrt{6}}\right\|$ $= \cos^{-1}\left\|\frac{1}{6}\right\|$ $= 80.4°$ (1 d.p.) **Note:** You are expected to recognize that $\overrightarrow{A'B} = \overrightarrow{BC}$. Hence, acute angle between the line AB and p_1 $= \frac{180° - 80.4°}{2}$ $= 49.8°$ (1 d.p.)
10 (v)	Possible cartesian equations of p_2: $x + 2y - 3z = -\frac{\sqrt{14}}{2}$ or $x + 2y - 3z = \frac{\sqrt{14}}{2}$
10 (vi)	As incident ray AD varies, D is nearest to origin when OD is the shortest. Note that p_1 contains the origin. $AB = \left\|\begin{pmatrix}-1\\1\\-2\end{pmatrix}\right\| = \sqrt{6}$ $\cos 49.8° = \frac{\sqrt{6}}{BD}$ $\Rightarrow BD = \frac{\sqrt{6}}{\cos 49.8°} = 3.79$ units (3 s.f.)
10 (vii)	Let γ be the required angle of inclination: $\cos^2 60° + \cos^2 45° + \cos^2 \gamma = 1$ $\frac{1}{4} + \frac{1}{2} + \cos^2 \gamma = 1$ $\cos \gamma = \pm\frac{1}{2}$ $\therefore \gamma = 60°$ (since γ is acute)

End of Paper

H2 2017 Preliminary Exam Paper 2 Question
Section A: Pure Matheatics [40 marks].

1	The cubic equation $az^3 - 31z^2 + 212z + b = 0$, where a and b are real numbers, has a complex root $z = 1 - 3\text{i}$. (i) Explain why the equation must have a real root. [2] (ii) Find the values of a and b and the real root, showing your working clearly. [5]
2	Relative to the origin O, the points A, B and C have position vectors **a**, $\mathbf{a}+\mathbf{c}$ and **c** respectively. The point X is on AC produced such that $AC:CX$ is $2:3$ and the point Y is such that $AXYB$ is a parallelogram. **(i)**The lines AY and BX intersect at the point N. Show that $\overrightarrow{ON} = \frac{1}{4}(\mathbf{7c}-\mathbf{a})$. [3] **(ii)**Given that the area of triangle OAB is 4 square units, find the area of triangle OAN. [4] **(iii)**Give a geometrical interpretation of $\left\lvert \overrightarrow{OA} \times \frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert} \right\rvert$. Use the results from part **(ii)** to show that $\left\lvert \overrightarrow{OA} \times \frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert} \right\rvert = \frac{56}{\lvert 7\mathbf{c} - 5\mathbf{a}\rvert}$. [3]
3	**(a)** Find the series expansion of $\text{e}^{2x}\ln(1+3x)$, where $-\frac{1}{3} < x \le \frac{1}{3}$, in ascending powers of x, up to and including the term in x^3. [3] **(b)** In the triangle PQR as shown in the diagram below, $PR = 1$, angle $QPR = \frac{3\pi}{4}$ radians and angle $PRQ = 2\theta$ radians.  **(i)** Show that $QR = \frac{1}{\cos 2\theta - \sin 2\theta}$. [4] **(ii)** Given that θ is sufficiently small angle, show that $QR \approx 1 + a\theta + b\theta^2$, for constants a and b to be determined. [4]
4	**(a)** Find $\int \text{e}^x \sin x \, \text{d}x$. [3] **(b)**



The diagram shows the curve with eq$k$uation $y=\dfrac{x}{\sqrt{\left(3-2x-x^2\right)}}$ for $0\le x<1$.

The region bounded by the curve, the x-axis and the line $x=k,\ 0<k<1$ is denoted by S.
It is given that n rectangles of equal width are drawn between $x=0$ and $x=k$.

(i) Show that the area of the first rectangle, $A_1=\dfrac{k^2}{n\sqrt{3n^2-2nk-k^2}}$. [1]

(ii) Show that the total area of all the n rectangles is

$$\sum_{r=1}^{n}\frac{rk^2}{n\sqrt{\left(3n^2-anrk-br^2k^2\right)}},$$

where a and b are constants to be determined. [2]

It is now given that $k=\left(\sqrt{3}\right)-1$.

(iii) Use integration to find the actual area of region S. Hence state the exact value of

$$\sum_{r=1}^{\infty}\frac{rk^2}{n\sqrt{\left(3n^2-anrk-br^2k^2\right)}}.$$ [6]

Section B: Probability and Statistics [60 marks]

5 An unbiased six-sided die is rolled twice. The random variable X represents the higher of the two values if they are different, and their common value if they are the same. The probability distribution of X is given by the formula

$$\mathrm{P}(X=r)=k(2r-1)\quad \text{for } r=1,\ 2,\ 3,\ 4,\ 5,\ 6.$$

(i) Find the exact value of k, giving your answer as a fraction in its simplest form. [2]

(ii) Find the expectation of X. [2]

A round of the game consists of rolling the unbiased six-sided die twice, and X is taken as the score for the round. A player plays three rounds of the game.

(iii) Find the probability that the total score for the three rounds is 16. [2]

6 A geologist splits rocks to look for fossils. On average 7% of the rocks selected from a particular area contain fossils.

The geologist selects a random sample of 20 rocks from this area.

(i) Find the probability that at least three of the rocks contain fossils. [2]

A random sample of n rocks is selected from this area.

(ii) The geologist wants to have a probability of 0.8 or greater of finding fossils in at least three of these rocks. Find the least possible value of n. [3]

In early 2017, geologists found the fossils of *zilantophis schuberti*, a new discovered species of winged serpent. On average, the proportion of rocks that contain fossils of *zilantophis schuberti* in this area is p. It is known that the modal number of fossils of *zilantophis schuberti* in a random sample of 10 rocks is 3.

(iii) Use this information to find exactly the range of values that p can take. [4]

7 A pilot records the take-off distance, S metres, for his private aircraft on runways at various altitudes of h metres. The data are shown in the table below.

h	0	300	600	900	1200	1500	1800
S	635	690	750	840	950	1080	1250

(i) Plot a scatter diagram on graph paper for these values, labelling the axes, using a scale of 2 cm to represent a take-off distance of 100 metres on the y-axis and an appropriate scale for the x-axis. [2]

It is thought that the take-off distance S can be modelled by one of the formulae

$$S = ah + b \qquad \text{or} \qquad S = ch^2 + d\,,$$

where a, b, c and d are constants.

(ii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) h and S,

(b) h^2 and S. [2]

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $S = ah + b$ or $S = ch^2 + d$ is the better model. [2]

(iv) Find the equation of the least-square regression line for the model you have chosen in part **(iii)**. [1]

(v) Use the equation of your regression line to estimate the take-off distance for altitude of 2200 metres. Comment on the reliability of your estimate when $h = 2200$. [2]

8 A manufacturing plant processes raw material for a supplier. An order placed with the plant is considered to be a bulk order when a worker is expected to process more than 300 kg (kilograms) of raw material.

Albert uses a machine to process X kg of raw material and Bob uses a separate machine to process Y kg of raw material on a working day. X and Y are independent random variables with the distributions $\mathrm{N}(296,\ 8^2)$ and $\mathrm{N}(290,\ 12^2)$ respectively.

(i) Find the probability that Albert processes more than 300 kg of raw material on a randomly selected working day. [2]

(ii) Find the probability that, over a period of 15 independent working days, there are exactly four working days on which Albert processes more than 300 kg of raw material. [2]

(iii) Find the probability that the total amount of raw material Bob processes over two working days exceeds twice the amount of raw material Albert processes on one working day. [4]

The plant receives a bulk order and Albert wants to have a probability of at least 0.95 of meeting the order.

(iv) This can be done by changing the value of μ, the mean amount of raw material Albert processes using the machine, but the standard deviation remains unchanged. Find the least value of μ. [3]

9 The town council is investigating the mass of rubbish in domestic dustbins. In 2016, the mean mass of rubbish in domestic dustbins was 20.0 kg per household per week. The town council starts a recycling initiative and wishes to determine whether there has been a reduction in the mass of rubbish in domestic dustbins.

The mass of rubbish in a domestic dustbin is denoted by X kg. A random sample of 50 domestic dustbins is selected and the results are summarised as follows.

$$n = 50 \qquad \sum x = 924.5 \qquad \sum x^2 = 18249.2$$

(i) Explain what is meant in this context by the term 'a random sample'. [2]

(ii) Explain why the town council is able to carry out a hypothesis test without knowing anything about the distribution of the mass of rubbish in domestic dustbins. [2]

(iii) Find the unbiased estimates of the population mean and variance and carry out the test at the 1% level of significance for the town council. [6]

(iv) Use your results in part **(iii)** to find the range of values of n for which the result of the test would be that the null hypothesis is rejected at the 1% level of significance. [3]

10 The number of employees of a statutory board, classified by department and years of working experience, is shown below.

	5 years or less	5 to 10 years	10 years or more	Total
Human Resource Department	20	50	30	100
Legal Department	15	60	45	120
Finance Department	25	30	45	100
Total	60	140	120	320

The Managing Director of the statutory board wishes to select three employees to participate in an overseas conference. The Managing Director selects one employee from each department to participate in the conference.

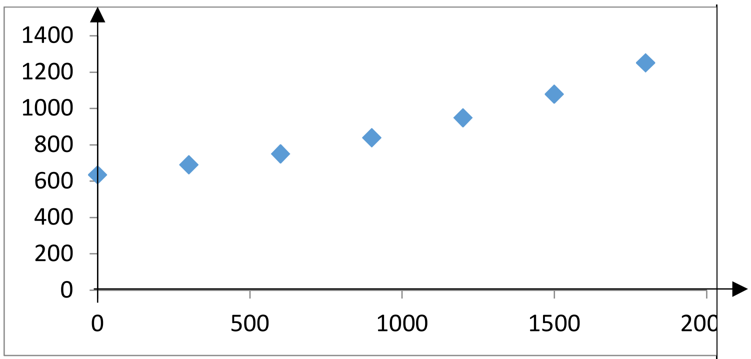
	(i) Find the probability that two of the selected employees have years of working experience ‘10 years or more’ and the remaining one has years of working experience ‘5 years or less’. [3] **(ii)** Given that exactly one of the selected employees has years of working experience ‘5 years or less’, find the probability that one of the selected employees is from the Legal Department and has years of working experience ‘5 to 10 years’. [3]

− End Of Paper −

ANNEX B

TPJC H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Complex numbers	(i)Since the **coefficients** of $az^3 - 31z^2 + 212z + b = 0$ are **all real**, **complex roots occur in conjugate pair**. Since a **cubic equation has three roots**, the third root must be a real root. (ii) $a = 25$, $b = 190, -\frac{19}{25}$
2	Vectors	(ii) 7 (iii) length of perpendicular from O to AN.
3	Maclaurin series	(a) $3x + \frac{3}{2}x^2 + 6x^3 + \ldots$ (b)(ii) $a = 2$, $b = 6$
4	Application of Integration	(a) $\frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + D$ (b)(iii) $\sqrt{3} - 1 - \frac{\pi}{6}$
5	DRV	(i) $\frac{1}{36}$ (ii) $\frac{161}{36}$ (iii) 0.112
6	Binomial Expansion	(i)0.161 (ii)60 (iii) $\therefore \frac{3}{11} < p < \frac{4}{11}$
7	Correlation & Linear Regression	S

		 (ii)(a)0.9809 (b)0.9960 (iii) The scatter diagram shows that S increases at an increasing rate as h increases, and for $S = ch^2 + d$, $r \approx 0.9960$ which is closer to 1, so the model $S = ch^2 + d$ is a better model. (iv) $S = 0.000182h^2 + 672$ (v)1550 Estimate for when $h = 2200$ metres is not reliable since $h = 2200$ metres is outside the range of the given data and extrapolation is not a good practice.
8	Normal Distribution	(i)0.309 (ii)0.214 (iii)0.303 (iv) 314
9	Hypothesis Testing	(i) Every dustbin has **an equal probability of being selected** and the selections of each dustbin are **made independently**. (ii) Since $n = 50$ is large, by Central Limit Theorem, the **mean** mass of rubbish in dustbins will be approximately normally distributed. (iii) 18.49, 23.6 Since p-value = 0.013937 > 0.01, we do **not** reject H_0 and conclude that there is **insufficient** evidence at 1% level of significance to claim that there has been a reduction in the mass of rubbish in dustbins. (iv) $n \geq 56,\ n \in \mathbb{Z}^{+}$

10	P&C, Probability	(i) $\frac{63}{800}$ (ii) $\frac{28}{61}$ (iii) 504 (iv) 3360

H2 Mathematics 2017 Preliminary Exam Paper 2 Solutions

1(i)	Since the **coefficients** of $az^3-31z^2+212z+b=0$ are **all real**, **complex roots occur in conjugate pair**. Since a **cubic equation has three roots**, the third root must be a real root.
1(ii)	Since $1-3\mathrm{i}$ is a root of $az^3-31z^2+212z+b=0$, $a(1-3\mathrm{i})^3-31(1-3\mathrm{i})^2+212(1-3\mathrm{i})+b=0$ $a(-26+18\mathrm{i})-31(-8-6\mathrm{i})+212(1-3\mathrm{i})+b=0$ $(-26a+460+b)+(18a-450)\mathrm{i}=0$ Comparing real and imaginary parts: $-26a+460+b=0$ ----------- (1) $18a-450=0$ -----------------(2) From (2), $a=25$, $b=190$ $(z-(1-3\mathrm{i}))(z-(1+3\mathrm{i}))$ $=z^2-2z+10$ $25z^3-31z^2+212z+190=(z^2-2z+10)(cz+d)$ Comparing coefficient of z^3: $c=25$ Comparing constant: $190=10d$ $d=19$ The real root is $-\dfrac{19}{25}$.
2(i)	$\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{a}+\mathbf{c}$, $\overrightarrow{OC}=\mathbf{c}$ $\overrightarrow{OX}=\overrightarrow{OA}+\overrightarrow{AX}$ $=\overrightarrow{OA}+\dfrac{5}{2}\overrightarrow{AC}$ $=\mathbf{a}+\dfrac{5}{2}(\mathbf{c}-\mathbf{a})$ $=\dfrac{1}{2}(5\mathbf{c}-3\mathbf{a})$ **Alternatively:** **By Ratio Theorem:** $\overrightarrow{OC}=\dfrac{2\overrightarrow{OX}+3\overrightarrow{OA}}{5}$ $\overrightarrow{OX}=\dfrac{5\overrightarrow{OC}-3\overrightarrow{OA}}{2}$ $\overrightarrow{OX}=\dfrac{1}{2}(5\mathbf{c}-3\mathbf{a})$ By midpoint theorem: $\overrightarrow{ON}=\dfrac{\overrightarrow{OB}+\overrightarrow{OX}}{2}$ $\overrightarrow{ON}=\dfrac{1}{2}\left[\mathbf{a}+\mathbf{c}+\dfrac{1}{2}(5\mathbf{c}-3\mathbf{a})\right]$ $=\dfrac{1}{4}(7\mathbf{c}-\mathbf{a})$
2(ii)	Area of triangle OAB = $\dfrac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$

	$4=\frac{1}{2}\lvert\mathbf{a}\times(\mathbf{a}+\mathbf{c})\rvert$ $=\frac{1}{2}\lvert\mathbf{a}\times\mathbf{c}\rvert \quad (\because \mathbf{a}\times\mathbf{a}=\underset{\sim}{0})$ $\Rightarrow \lvert\mathbf{a}\times\mathbf{c}\rvert=8$ Area of triangle OAN = $\frac{1}{2}\left\lvert\overrightarrow{OA}\times\overrightarrow{ON}\right\rvert$ $=\frac{1}{2}\left\lvert\mathbf{a}\times\frac{1}{4}(7\mathbf{c}-\mathbf{a})\right\rvert$ $=\frac{7}{8}\lvert\mathbf{a}\times\mathbf{c}\rvert \quad (\because \mathbf{a}\times\mathbf{a}=\underset{\sim}{0})$ $=\frac{7}{8}(8)$ $=7$ square units
2(iii)	$\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert$ is the length of perpendicular from O to AN. **<u>Alternative answer:</u>** $\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert$ is the shortest distance from O to AN. $\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert$ is the area of a parallelogram formed with vector $\overrightarrow{OA}$ and unit vector $\overrightarrow{AN}$ as its adjacent sides. (**Not recommended here**) Area of triangle $OAN=7$ $\frac{1}{2}\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert\left\lvert\overrightarrow{AN}\right\rvert=7$ $\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert=\frac{14}{\lvert\overrightarrow{AN}\rvert}$ $=\frac{14}{\lvert\overrightarrow{ON}-\overrightarrow{OA}\rvert}$ $=\frac{14}{\left\lvert\frac{1}{4}(7\mathbf{c}-\mathbf{a})-\mathbf{a}\right\rvert}$ $=\frac{56}{\lvert 7\mathbf{c}-5\mathbf{a}\rvert}$ **(shown)**
3(a)	$\mathrm{e}^{2x}\ln(1+3x)$ $=\left(1+2x+\frac{(2x)^2}{2!}+...\right)\left(3x-\frac{(3x)^2}{2}+\frac{(3x)^3}{3}-\ldots\right)$ where $-1<3x\le 1$ $=(1+2x+2x^2+...)\left(3x-\frac{9}{2}x^2+9x^3-...\right)$

	$=3x-\frac{9}{2}x^2+9x^3+6x^2-9x^3+6x^3+\ldots$ $=3x+\frac{3}{2}x^2+6x^3+\ldots$ where $-\frac{1}{3}<x\le\frac{1}{3}$
3(b)(i)	$\frac{QR}{\sin\frac{3\pi}{4}}=\frac{PR}{\sin\left(\pi-\frac{3\pi}{4}-2\theta\right)}$ $\frac{QR}{\sin\frac{3\pi}{4}}=\frac{PR}{\sin\left(\frac{\pi}{4}-2\theta\right)}$ $QR=\frac{\sin\frac{3\pi}{4}}{\sin\frac{\pi}{4}\cos2\theta-\cos\frac{\pi}{4}\sin2\theta}$ $QR=\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\cos2\theta-\frac{1}{\sqrt{2}}\sin2\theta}$ $QR=\frac{1}{\cos2\theta-\sin2\theta}$ (shown)
3(b)(ii)	When θ is small, $QR\approx\frac{1}{\left(1-\frac{(2\theta)^2}{2!}\right)-2\theta}$ $=\frac{1}{1-2\theta-2\theta^2}$ $=\left(1-\left(2\theta+2\theta^2\right)\right)^{-1}$ $=1+\left(2\theta+2\theta^2\right)+\left(2\theta+2\theta^2\right)^2+\ldots$ $=1+2\theta+2\theta^2+4\theta^2+\ldots$ $=1+2\theta+6\theta^2+\ldots$ $a=2,\ b=6$
4(a)	$\int e^x\sin x\ dx$ $=e^x\sin x-\int e^x\cos x\ dx$ $=e^x\sin x-\left[e^x\cos x+\int e^x\sin x\ dx\right]$ $=e^x\sin x-e^x\cos x-\int e^x\sin x\ dx$ Hence, $\int e^x\sin x\ dx=e^x\sin x-e^x\cos x-\int e^x\sin x\ dx+C$ $2\int e^x\sin x\ dx=e^x\sin x-e^x\cos x+C$ $\int e^x\sin x\ dx=\frac{1}{2}\left(e^x\sin x-e^x\cos x\right)+D$

4(b)(i)	Area of first rectangle, $x=\frac{k}{n}$: $A_1=\dfrac{k/n}{\sqrt{3-2\left(k/n\right)-\left(k/n\right)^2}}\cdot\dfrac{k}{n}=\dfrac{k^2/n^2}{\sqrt{\dfrac{3n^2-2nk-k^2}{n^2}}}=\dfrac{k^2}{n\sqrt{3n^2-2nk-k^2}}$
4(b)(ii)	Area of second rectangle, $x=\dfrac{2k}{n}$: $A_2=\dfrac{2k/n}{\sqrt{3-2\left(2k/n\right)-\left(2k/n\right)^2}}\cdot\dfrac{k}{n}=\dfrac{2k^2}{n\sqrt{3n^2-2n(2k)-(2k)^2}}$ Area of third rectangle, $x=\dfrac{3k}{n}$: $A_3=\dfrac{3k/n}{\sqrt{3-2\left(3k/n\right)-\left(3k/n\right)^2}}\cdot\dfrac{k}{n}=\dfrac{3k^2}{n\sqrt{3n^2-2n(3k)-(3k)^2}}$ By observation, combined area of n rectangles: $A=\displaystyle\sum_{r=1}^{n}\frac{rk^2}{n\sqrt{3n^2-2nrk-r^2k^2}}$, where $a=2$ *and* $b=1$
4(b)(iii)	$\displaystyle\sum_{r=1}^{\infty}\frac{rk^2}{n\sqrt{\left(3n^2-anrk-br^2k^2\right)}}$ $=$ Area under curve from $x=0$ to $x=\sqrt{3}-1$ $=\displaystyle\int_0^{\sqrt{3}-1}\frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$ $=\displaystyle\int_0^{\sqrt{3}-1}\frac{-\frac{1}{2}(-2-2x)-1}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$ $=-\dfrac{1}{2}\displaystyle\int_0^{\sqrt{3}-1}\frac{-2-2x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x-\int_0^{\sqrt{3}-1}\frac{1}{\sqrt{4-(x+1)^2}}\,\mathrm{d}x$ $=-\dfrac{1}{2}\left[\dfrac{\sqrt{3-2x-x^2}}{1/2}\right]_0^{\sqrt{3}-1}-\left[\sin^{-1}\left(\dfrac{x+1}{2}\right)\right]_0^{\sqrt{3}-1}$ $=-\left[\sqrt{3-2x-x^2}\right]_0^{\sqrt{3}-1}-\left[\sin^{-1}\left(\dfrac{x+1}{2}\right)\right]_0^{\sqrt{3}-1}$ $=-\left[1-\sqrt{3}\right]-\left[\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)-\sin^{-1}\dfrac{1}{2}\right]$ $=\sqrt{3}-1-\dfrac{\pi}{3}+\dfrac{\pi}{6}$ $=\sqrt{3}-1-\dfrac{\pi}{6}$ (exact)

5(i)

$$\sum_{r=1}^{6} \mathrm{P}(X = r) = 1$$

$$k + 3k + 5k + 7k + 9k + 11k = 1$$

$$k = \frac{1}{36}$$

5(ii)

$$\mathrm{E}(X) = 1(k) + 2(3k) + 3(5k) + 4(7k) + 5(9k) + 6(11k)$$

$$= 161k$$

$$= \frac{161}{36}$$

5(iii)

Required Probability

$= \mathrm{P}(\{6,6,4\}) + \mathrm{P}(\{6,5,5\})$

$$= \left(\frac{11}{36}\right)^2 \left(\frac{7}{36}\right)\frac{3!}{2!} + \left(\frac{11}{36}\right)\left(\frac{9}{36}\right)^2 \frac{3!}{2!}$$

$= 0.112$ (3 s.f.) Accept: $\dfrac{1738}{15552} = \dfrac{869}{7776}$

6(i)

Let X be the number of rocks containing fossils out of 20 rocks.

$X \sim \mathrm{B}(20,\ 0.07)$

$$\mathrm{P}(X \geq 3) = 1 - \mathrm{P}(X \leq 2)$$

$$= 0.161 \quad \text{(3 s.f.)}$$

6(ii)

Let Y be the number of rocks containing fossils out of 20 rocks.

$Y \sim \mathrm{B}(n,\ 0.07)$

$\mathrm{P}(Y \geq 3) \geq 0.8$

<u>Method 1a: Using GC Table</u>

n	$\mathrm{P}(Y \geq 3)$	
59	0.79085	< 0.8
60	0.80023	> 0.8
61	0.80925	> 0.8

Hence, least $n = 60$.

<u>Method 1b: Using GC Table</u>

$\mathrm{P}(Y \leq 2) \leq 0.2$

n	$\mathrm{P}(Y \leq 2)$	
59	0.20915	> 0.2
60	0.19977	< 0.2
61	0.19075	< 0.2

Hence, least $n = 60$.

<u>Method 2: Using the binomial distribution function</u>

$\mathrm{P}(Y \leq 2) \leq 0.2$

$\mathrm{P}(Y = 0) + \mathrm{P}(Y = 1) + \mathrm{P}(Y = 2) \leq 0.2$

$$0.93^n + n(0.07)(0.93)^{n-1} + \frac{n(n-1)}{2}(0.07^2)(0.93)^{n-2} \leq 0.2$$

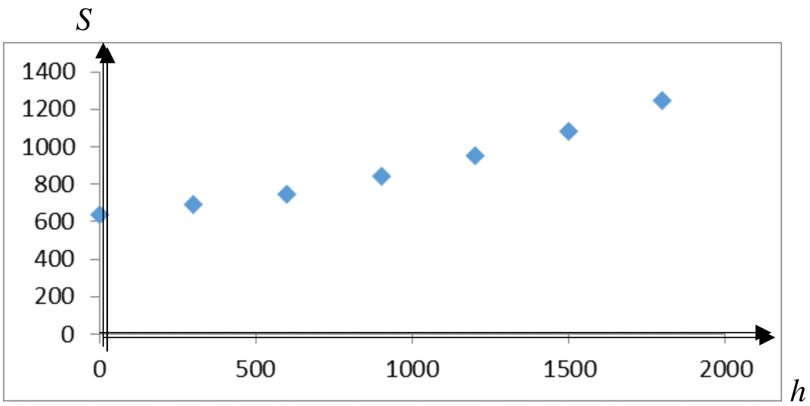
Using GC to sketch the graph:

Hence, least $n = 60$.

6(iii)

Let W be the number of fossils of *zilantophis schuberti* in a random sample of 10 rocks.

$W \sim \mathrm{B}(10,\ p)$

	$\mathrm{P}(W=3) > \mathrm{P}(W=2)$ $\frac{10!}{3!7!}p^3(1-p)^7 > \frac{10!}{2!8!}p^2(1-p)^8$ $120p^3(1-p)^7 > 45p^2(1-p)^8$ $8p > 3(1-p)$ (Since $0<p<1$) $\frac{8}{3}p > 1-p$ $p > \frac{3}{11}$ $\mathrm{P}(W=3) > \mathrm{P}(W=4)$ $\frac{10!}{3!7!}p^3(1-p)^7 > \frac{10!}{4!6!}p^4(1-p)^6$ $120p^3(1-p)^7 > 210p^4(1-p)^6$ $4(1-p) > 7p$ (Since $0<p<1$) $1-p > \frac{7}{4}p$ $p < \frac{4}{11}$ $\therefore \frac{3}{11} < p < \frac{4}{11}$
7(i)	
7(ii)	**(a)** $r = 0.980867 \approx 0.9809$ (4 d.p.) **(b)** $r = 0.996039 \approx 0.9960$ (4 d.p.)
7(iii)	The scatter diagram shows that <u>S increases at an increasing rate as h increases</u>, and for $S = ch^2 + d$, <u>$r \approx 0.9960$ which is closer to 1</u>, so the model $S = ch^2 + d$ is a better model.
7 (iv)	The equation of regression line is $S = 0.0001822853073h^2 + 671.7261905$ i.e. $S = 0.000182h^2 + 672$ (3 s.f.)
7 (v)	$S = 0.00018229(2200)^2 + 671.73$ $= 1554.0136$ $= 1550$ metres (3 s.f.)

	Estimate for when h = 2200 metres is <u>not</u> reliable since h = 2200 metres is outside the range of the given data and <u>extrapolation</u> is not a good practice.
8(i)	$X \sim \mathrm{N}(296,\ 8^2)$ $\quad Y \sim \mathrm{N}(290,\ 12^2)$ Required probability = [P(X > 300)] $= 0.30854$ (5 s.f.) (0.3085375322) $= 0.309$ (3 s.f.)
8(ii)	Let W be the number of days in which Albert processes more than 300 kg of raw material on that day out of 15 days. $W \sim \mathrm{B}(15, 0.30854)$ $\mathrm{P}(W = 4) = 0.214$ (3 s.f.)
8(iii)	Let $S = Y_1 + Y_2 - 2X$ $\mathrm{E}(S) = \mathrm{E}(Y_1) + \mathrm{E}(Y_2) - 2\mathrm{E}(X) = 2\times290-2\times296=-12$ $\mathrm{Var}(S) = 2\mathrm{Var}(Y) + 2^2\mathrm{Var}(X) = 2\times12^2+2^2\times8^2=544$ Hence, $S \sim \mathrm{N}(-12,\ 544)$ $\mathrm{P}(S > 0) = 0.303$ (3 s.f.) (0.3034526994)
8(iv)	$X \sim \mathrm{N}(\mu,\ 8^2)$ $\mathrm{P}(X>300)=\mathrm{P}\left(Z>\frac{300-\mu}{8}\right)\geq 0.95$ $\mathrm{P}\left(Z\leq\frac{300-\mu}{8}\right)\leq 0.05$ $\frac{300-\mu}{8}\leq -1.6449$ $\mu \geq 313.1592$ Least value of $\mu = 314$ kg (3 s.f.)
9(i)	Every dustbin has <u>**an equal probability of being selected**</u> and the selections of each dustbin are <u>**made independently**</u>.
9(ii)	Since $n=50$ is <u>large</u>, by <u>Central Limit Theorem</u>, the <u>**mean**</u> mass of rubbish in dustbins will be approximately normally distributed.
9 (iii)	Unbiased estimate of population mean, $\bar{x}=\frac{924.5}{50}=18.49$ Unbiased estimate of population variance, $s^2=\frac{1}{49}\left[18249.2-\frac{924.5^2}{50}\right]=23.575$ (5 s.f.) $=23.6$ (3 s.f.) Let μ be the population mean mass of rubbish, in kg, in a domestic dustbin. To test: $\mathrm{H_0}$: $\mu=20$ against $\mathrm{H_1}$: $\mu<20$ at 1% level of significance Since $n=50$ is large, by Central Limit Theorem, $\overline{X}\sim\mathrm{N}\left(20,\ \frac{23.575}{50}\right)$ approximately under $\mathrm{H_0}$. Test Statistic: $Z=\frac{\overline{X}-20}{\sqrt{23.575/50}}\sim\mathrm{N}(0,1)$ approximately under $\mathrm{H_0}$.

	Using GC, $[\bar{x}=18.49, s^2=23.575, n=50]$ $z_{test}=-2.199$, p-value = 0.013937 (5 s.f.) Since p-value = 0.013937 > 0.01, we do **not** reject H_0 and conclude that there is **insufficient** evidence at 1% level of significance to claim that there has been a reduction in the mass of rubbish in dustbins.
9 (iv)	For H_0 to be rejected, $z_{test}=\dfrac{18.49-20}{\sqrt{23.575}}\times\sqrt{n}<-2.3263$ $n > 55.954$ Range of values of n is $n\geq 56,\ n\in\mathbb{Z}^+$ [Also Accept: $n>55,\ n\in\mathbb{Z}$ (or equivalent form)]
10(i)	Required probability $=\dfrac{30}{100}\times\dfrac{45}{120}\times\dfrac{25}{100}+\dfrac{30}{100}\times\dfrac{15}{120}\times\dfrac{45}{100}+\dfrac{20}{100}\times\dfrac{45}{120}\times\dfrac{45}{100}$ $=\dfrac{63}{800}$
10(ii)	Required probability $=\dfrac{(0.2)(0.5)(0.75)+(0.8)(0.5)(0.25)}{(0.2)(0.875)(0.75)+(0.8)(0.125)(0.75)+(0.8)(0.875)(0.25)}$ $=\dfrac{28}{61}$
10 (iii)	Number of different possible codes $={}^{9}C_2\times 2!\ \times{}^{7}C_1$ $=504$
10 (iv)	**<u>Method 1: Complementary Method</u>** Number of possible arrangements $=\left[{}^{4}C_3\times{}^{5}C_2\times 5!\right]-\left[\left({}^{4}C_3\times 3!\right)\times{}^{5}C_2\times 3!\right]$ $=3360$ **<u>Method 2: List by Cases</u>** <u>Case 1: All the even digits are separated</u> ${}^{4}C_3\times{}^{5}C_2\times 2!\ \times 3!=480$ <u>Case 2: Exactly two even digits are next to each other (and the third even digit is separated)</u> ${}^{4}C_3\times\left({}^{3}C_2\times 2!\right)\times{}^{5}C_2\times 3!\ \times{}^{2}C_1=2880$ Number of possible arrangements $=480+2880$ $=3360$

End of Paper