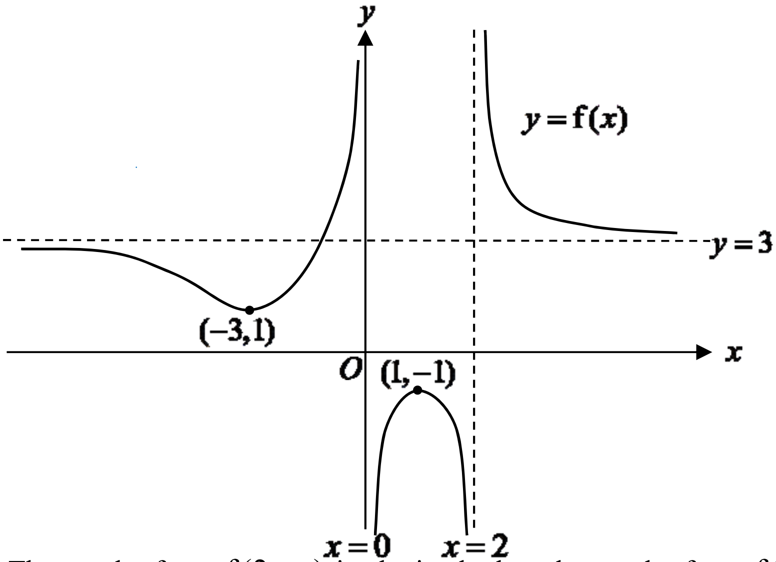
TJC Paper 1

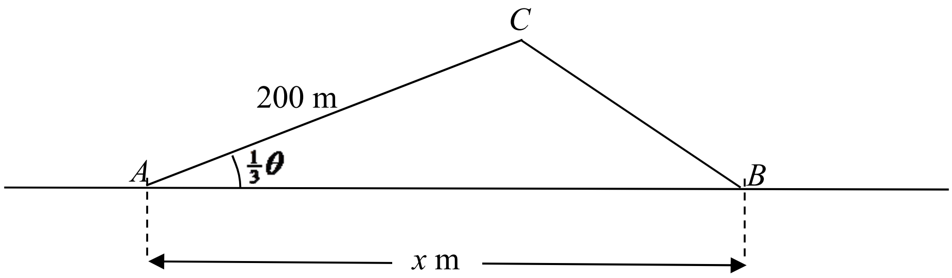
1 The graph of $y = \mathrm{f}(x)$ is shown below.



(a) The graph of $y = \mathrm{f}(2-x)$ is obtained when the graph of $y = \mathrm{f}(x)$ undergoes a sequence of transformations. Describe the sequence of transformations. [2]

(b) Sketch the graph of $y = \mathrm{f}'(x)$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

2



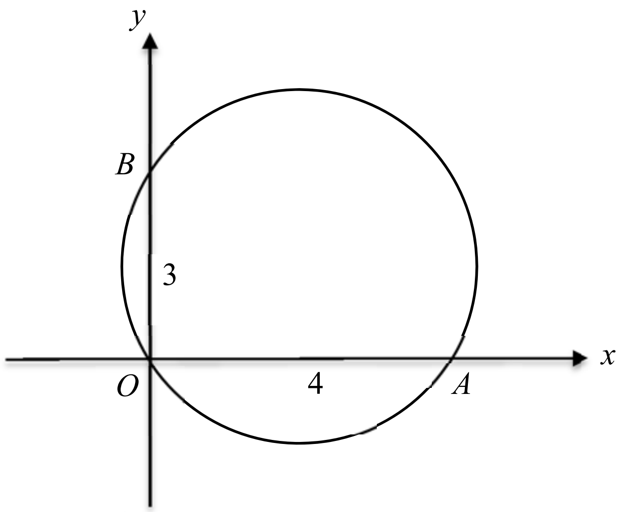
The diagram shows two points at ground level, A and B. The distance in metres between A and B is denoted by x. The angle of elevation of C from B is twice the angle of elevation of C from A. The distance AC is 200 m and $\angle BAC = \frac{1}{3}\theta$ radians. Show that

$$x = \frac{200\sin\theta}{\sin\frac{2}{3}\theta}.$$ [2]

It is given that θ is a small angle such that θ^4 and higher powers of θ are negligible. By using appropriate expansions from the List of Formulae (MF26), show that

$$x \approx \frac{2700 - 250\theta^2}{9}.$$ [4]

3



The diagram above shows a circle C which passes through the origin O and the points A and B. It is given that $OA = 4$ units and $OB = 3$ units.

(i) Show that the coordinates of the centre of C is $\left(2, \frac{3}{2}\right)$. Hence write down the equation of C in the form $(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = r^2$, where r is a constant to be determined. [2]

(ii) By adding a suitable line to the diagram above, find the range of values of m for which the equation $mx - \frac{3}{2} = \sqrt{\frac{25}{4} - (x-2)^2}$ has a solution. [4]

4 The curve C has equation $y = \sin 2x + 2\cos x$, $0 \le x \le 2\pi$.

(i) Using an algebraic method, find the exact x-coordinates of the stationary points. [You do not need to determine the nature of the stationary points.] [3]

(ii) Sketch the curve C, indicating clearly the coordinates of the turning points and the intersection with the axes. [1]

(iii) Find the area bounded by the curve C and the line $y = \frac{1}{\pi}x$. [3]

5 The curve C has equation $y = kx^3$. The tangent at the point P on C meets the curve again at point Q. The tangent at point Q meets the curve again at point R. It is given that the x-coordinates of P, Q and R are p, q, and r respectively, where $p \ne 0$.

(i) Show that p and q satisfy the equation $\left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right) - 2 = 0$. [4]

(ii) Show that p, q and r are three consecutive terms of a geometric progression. Hence determine if this geometric series is convergent. [4]

[You may use the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ for $a, b \in \mathbb{R}$]

6 **(a)** The vectors **a** and **b** are the position vector of points A and B respectively. It is given that $OA = 2\sqrt{7}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{a} \cdot \mathbf{b} = -14$.

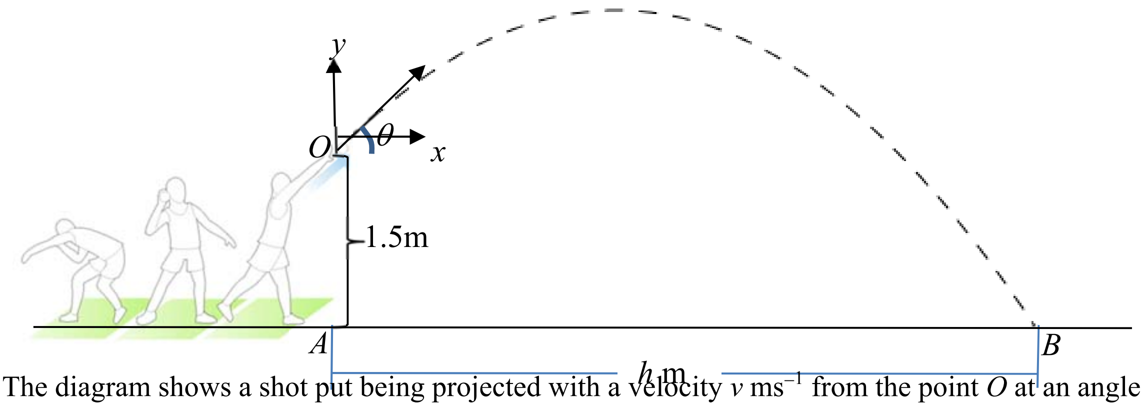
(i) Find angle AOB. [2]

(ii) State the geometrical meaning of $|\hat{\mathbf{a}} \cdot \mathbf{b}|$, where $\hat{\mathbf{a}}$ is the unit vector of **a**. [1]

(iii) Hence or otherwise, find the position vector of the foot of perpendicular from B to line OA in terms of **a**. [2]

(b) The non-zero vectors **p** and **q** are such that $|\mathbf{p} \times \mathbf{q}| = 2$. Given that **p** is a unit vector and $\mathbf{q} \cdot \mathbf{q} = 4$, show that **p** and **q** are perpendicular to each other. [3]

7



The diagram shows a shot put being projected with a velocity v ms^{-1} from the point O at an angle θ made with the horizontal. The point O is 1.5m above the point A on the ground. The x-y plane is taken to be the plane that contains the trajectory of this projectile motion with x-axis parallel to the horizontal and O being the origin. The equation of the trajectory of this projectile motion is known to be

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta},$$

where g ms^{-2} is the acceleration due to gravity.
The constant g is taken to be 10 and the distance between A and B is denoted by h m. Given that $v = 10$, show that h satisfies the equation

$$h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0$$ [3]

As θ varies, h varies. Show that stationary value of h occurs when θ satisfies the following equation

$$3\tan^2 2\theta - 20\sin 2\theta \tan 2\theta - 20\cos 2\theta - 20 = 0.$$ [5]

Hence find the stationary value of h. [2]

8 **(a)** In an Argand diagram, points P and Q represent the complex numbers $z_1 = 2 + 3\mathrm{i}$ and $z_2 = \mathrm{i}z_1$.

(i) Find the area of the triangle OPQ, where O is the origin. [2]

(ii) z_1 and z_2 are roots of the equation $(z^2 + az + b)(z^2 + cz + d) = 0$, where $a, b, c, d \in \mathbb{R}$. Find a, b, c and d. [4]

(b) Without using the graphing calculator, find in exact form, the modulus and argument of $v^* = \left(\frac{\sqrt{3} + \mathrm{i}}{-1 + \mathrm{i}}\right)^{14}$. Hence express v in exponential form. [5]

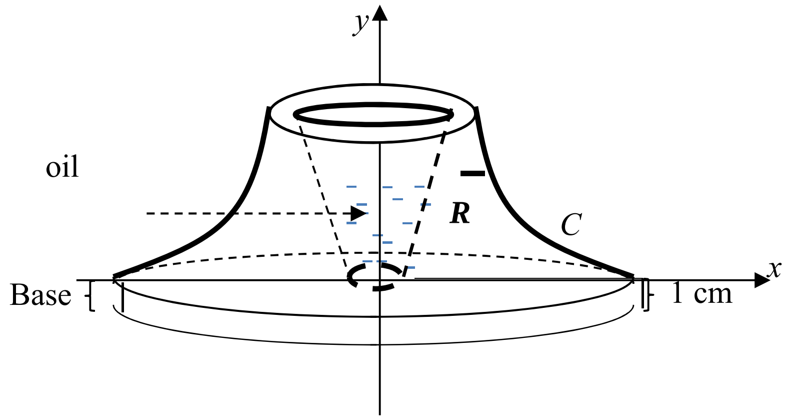
9 A curve C has parametric equation defined by

$$x = 4\sec t \text{ and } y = 8(1-\tan t), \text{ where } -\tfrac{1}{4}\pi \le t \le \tfrac{1}{4}\pi .$$

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t and hence show that the equation of tangent at the point $t = -\frac{1}{6}\pi$ is $y = 4x + 8(1-\sqrt{3})$. [3]

(ii) Find the Cartesian equation of C. [2]

R is the region bounded by C, the tangent in **(i)**, the normal to C at $t = 0$ and the x-axis. Part of an oil burner is formed by rotating R completely about the y-axis as shown in the diagram below (not drawn to scale). The base of the burner is a solid cylinder of thickness 1 cm. [You may assume each unit along the x and y axis to be 1 cm]



(iii) Find the volume of the material required to make the burner. [6]

10

The point A has coordinates (3, 1, 1). The line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, where λ is a parameter. P is a point on l when $\lambda = t$.

(i) Find cosine of the acute angle between AP and l in terms of t. Hence or otherwise, find the position vector of the point N on l such that N is the closest point to A. [6]

(ii) Find the coordinates of the point of reflection of A in l. [2]

The line L has equation $x = -1$, $2y = z + 2$.

(iii) Determine whether L and l are skew lines. [2]

(iv) Find the shortest distance from A to L. [3]

11 A hot air balloon rises vertically upwards from the ground as the balloon operator intermittently fires and turns off the burner. At time t minutes, the balloon ascends at a rate inversely proportional to $t + \lambda$, where λ is a positive constant. At the same time, due to atmospheric factors, the balloon descends at a rate of 2 km per minute. It is also known that initially the rate of change of the height of the balloon is 1 km per minute.

(i) Find a differential equation expressing the relation between H and t, where H km is the height of the hot air balloon above ground at time t minutes. Hence solve the differential equation and find H in terms of t and λ. [7]

Using $\lambda = 15$,

(ii) Find the maximum height of the balloon above ground in exact form. [3]

(iii) Find the total vertical distance travelled by the balloon when $t = 8$. [3]

(iv) Can we claim that the rate of change of the height of the balloon above the ground is decreasing? Explain your answer. [2]

H

TEMASEK JUNIOR COLLEGE, SINGAPORE
JC 2
Preliminary Examination 2017
Higher 2

TEMASEK JUNIOR COLLEGE

MATHEMATICS 9758/01

Paper 1

29 Aug 2017

Additional Materials: Answer paper
List of Formulae (MF26)

3 hours

READ THESE INSTRUCTIONS FIRST

Write your Civics group and name on all the work that you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use a soft pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

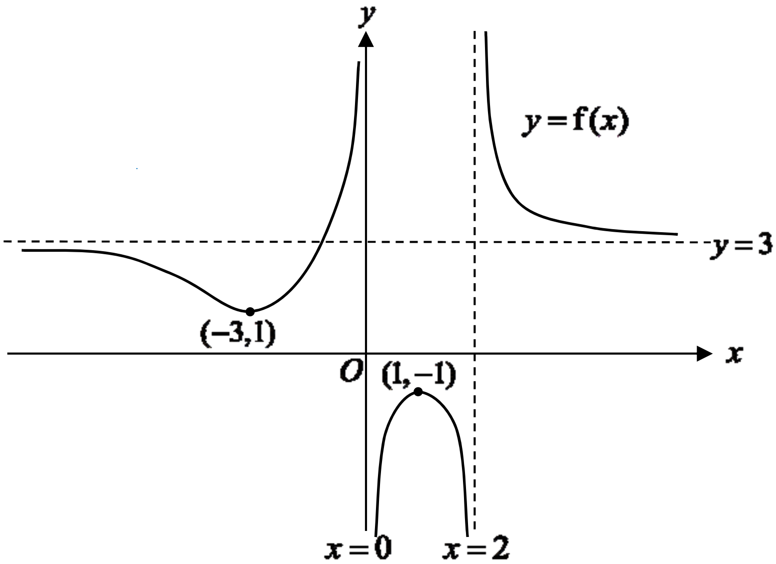
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, fasten all your work securely together.

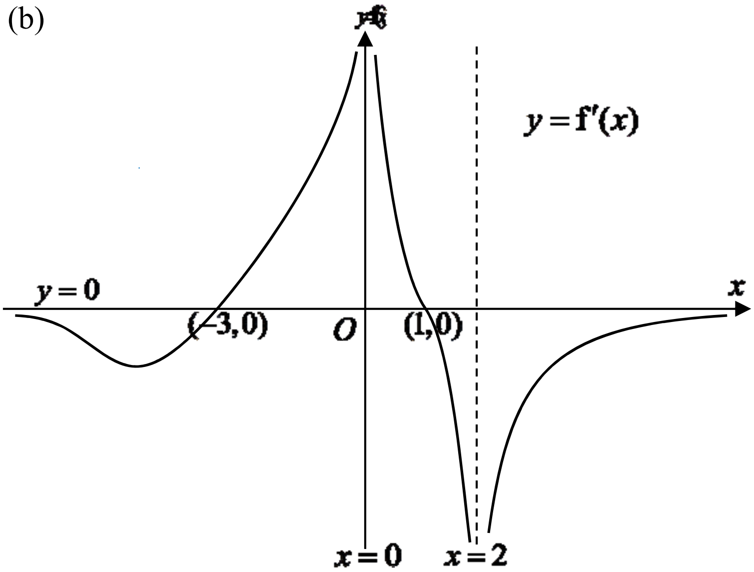
This document consists of 7 printed pages and 1 blank page.

1 The graph of $y = \mathrm{f}(x)$ is shown below.

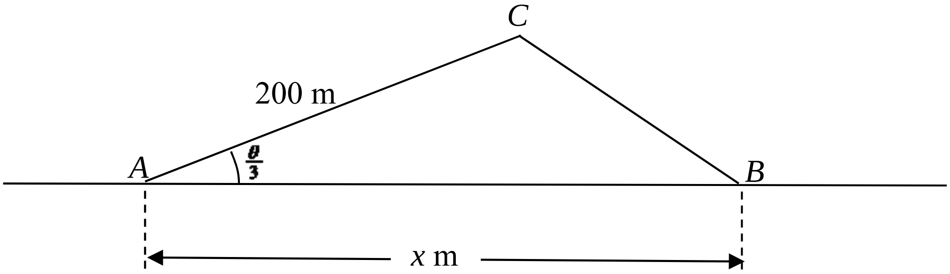


(a) The graph of $y = \mathrm{f}(2 - x)$ is obtained when the graph of $y = \mathrm{f}(x)$ undergoes a sequence of transformations. Describe the sequence of transformations. [2]

(b) Sketch the graph of $y = \mathrm{f}\,'(x)$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

(a) **Translation** of 2 **units** in the negative x-**direction**, followed by **reflection about** the y-axis. Alternative solution Reflection about the y-axis followed by translation of 2 units in the positive x-direction.	Need to use the correct words (**in bold**) when describing the transformations.
(b) 	All (3) asymptotes must be labelled, and intersections with axes written in coordinate form as instructed by the question.
Marker's comments This question is generally well-attempted.	

2



The diagram shows two points at ground level, A and B. The distance in metres between A and B is denoted by x. The angle of elevation of C from B is twice the angle of elevation of C from A. The distance AC is 200 m and $\angle BAC = \dfrac{\theta}{3}$ radians. Show that

$$x = \frac{200\sin\theta}{\sin\left(\frac{2}{3}\theta\right)}. \qquad [2]$$

It is given that θ is a small angle such that θ^4 and higher powers of θ are negligible. By using appropriate expansions from the List of Formulae (MF26), show that

$$x \approx \frac{2700 - 250\theta^2}{9}. \qquad [4]$$

$\angle ABC = 2 \times \angle BAC = \dfrac{2\theta}{3}$ $\Rightarrow \quad \angle ACB = \pi - \theta$ Using Sine rule, $\dfrac{x}{\sin(\pi-\theta)} = \dfrac{200}{\sin\left(\frac{2\theta}{3}\right)}$ Since $\sin(\pi-\theta) = \sin\pi\cos\theta - \cos\pi\sin\theta = \sin\theta$, $\dfrac{x}{\sin\theta} = \dfrac{200}{\sin\left(\frac{2\theta}{3}\right)}$ $\Rightarrow \quad x = \dfrac{200\sin\theta}{\sin\left(\frac{2}{3}\theta\right)}$ (Shown)	A common mistake is $\angle ACB = 2\theta - \dfrac{\theta}{3} - \dfrac{2\theta}{3} = \theta$. Students who made this mistake simply wanted θ to appear and do not ensure that the expression is true. ***Note:*** This is a "Show" question. Thus **<u>all</u>** working/explanation **<u>should</u>** be clearly stated, i.e., need to show $\angle ACB$ and $\angle ABC$, and state the method (sine rule) used.

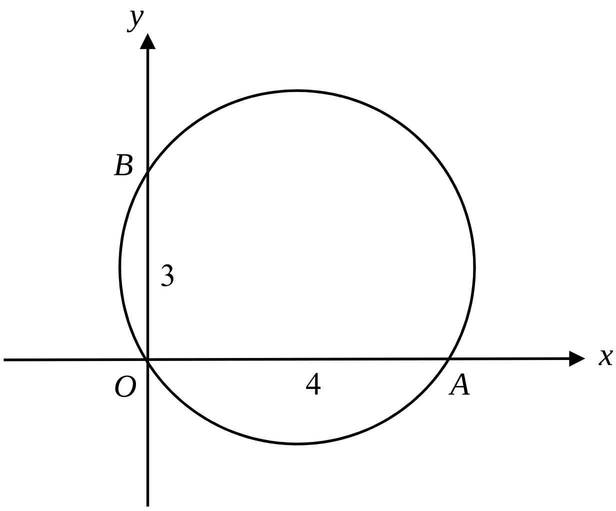
$x=\dfrac{200\sin\theta}{\sin\left(\frac{2}{3}\theta\right)}\approx\dfrac{200\left(\theta-\frac{\theta^3}{3!}\right)}{\left(\frac{2}{3}\theta-\frac{\left(\frac{2}{3}\theta\right)^3}{3!}\right)}$ since θ^4 and higher powers of θ are negligible $=\dfrac{200\,\theta\left(1-\frac{\theta^2}{6}\right)}{\frac{2}{3}\theta\left(1-\frac{2\theta^2}{27}\right)}$ $=300\left(1-\frac{\theta^2}{6}\right)\left(1-\frac{2\theta^2}{27}\right)^{-1}$ $=300\left(1-\frac{\theta^2}{6}\right)\left(1+\frac{2\theta^2}{27}+...\right)$ $=300\left(1-\frac{5\theta^2}{54}+...\right)$ $\approx\dfrac{2700-250\theta^2}{9}$	Note that since "+…" is dropped, the "$\approx$" sign should be used. Take out θ and cancel for easy computation. To ensure that the final expression is a polynomial, the denominator has to be "brought up" using power −1. Then use the expansion $(1+x)^{-1}$.

Marker's comments

For the 1st part, many students attempted to find *x* by considering the two triangles formed by drawing a line through *C* perpendicular to *AB*. This method is tedious.

The $\sin\theta$ and $\sin\left(\frac{2}{3}\theta\right)$ in the expression to be shown would suggest using sine rule.

3



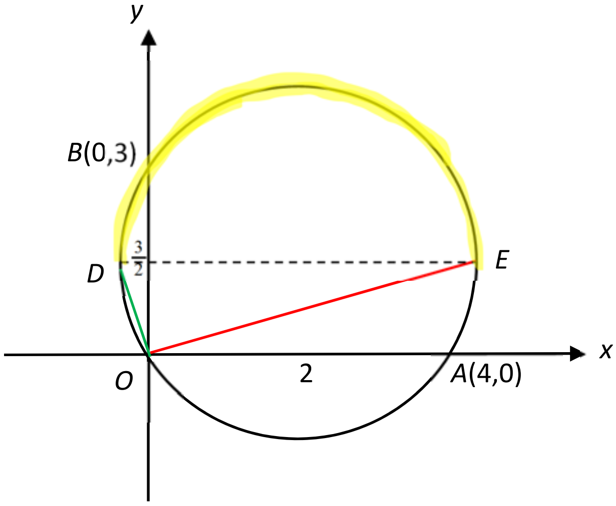
The diagram above shows a circle C which passes through the origin O and the points A and B.

It is given that $OA = 4$ units and $OB = 3$ units.

(i) Show that the coordinates of the centre of C is $\left(2, \frac{3}{2}\right)$. Hence write down the equation of C in the form $(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = r^2$, where r is a constant to be determined. [2]

(ii) By adding a suitable line to the diagram above, find the range of values of m for which the equation $mx - \frac{3}{2} = \sqrt{\frac{25}{4} - (x-2)^2}$ has a solution. [4]

(i)	Since ΔAOB is a right-angle in a semi-circle, AB forms the diameter of the circle. Hence, centre of circle is at the mid point of AB, i.e., $\left(2, \frac{3}{2}\right)$. $AB = \sqrt{3^2 + 4^2} = 5 \Rightarrow$ radius, $r = \frac{5}{2}$ units Therefore, equation of C is $(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{5}{2}\right)^2$.	Since this is a “Show” question, marks are awarded only if a clear explanation of how the centre coordinates are derived. Inefficient methods such as substituting coordinates into the circle equation and solving them simultaneously were used.

(ii) $$(x-2)^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{5}{2}\right)^2$$ $$\Rightarrow \quad y-\frac{3}{2}=\pm\sqrt{\frac{25}{4}-(x-2)^2}$$ Suitable line to add: $y=mx$  $$D=\left(-\frac{1}{2},\ \frac{3}{2}\right) \text{ and } E=\left(\frac{9}{2},\ \frac{3}{2}\right)$$ Gradient of line $OD=-\frac{3/2}{1/2}=-3$ Gradient of line $OE=\frac{3/2}{9/2}=\frac{1}{3}$ $\therefore\ m\le -3 \quad \text{or} \quad m\ge\frac{1}{3}$	Most students did not realise that the question only involves the top half of the circle (positive root). Realise that all possible $y=mx$ will lie between the green and red line. This motivates us to find the gradient of the green and red line. Since points D and E are the two (left/right) ends of the semicircle, their coordinates can be easily deduced using the centre coordinates and radius.

Marker's comments

Common mistakes:

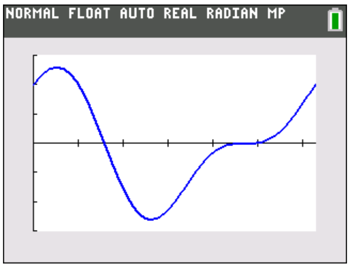
1. *Differentiating the equation of the circle to find the gradient of the tangent*:
   From the diagram, it is clear that $y=mx$ can intersect the semicircle even if it is not a tangent to the circle.
2. *Setting the discriminant to be 0*:
   The quadratic equation represents all the intersection points of $y=mx$ with the whole circle (instead of the semicircle).
3. *Stating that the range of m is* $-3\le m\le\frac{1}{3}$:
   Inaccurate deduction which can be avoided by using the diagram.

4 The curve C has equation $y = \sin 2x + 2\cos x$, $0 \le x \le 2\pi$.

(i) Using an algebraic method, find the exact x-coordinates of the stationary points. [You do not need to determine the nature of the stationary points.] [3]

(ii) Sketch the curve C, indicating clearly the coordinates of the turning points and the intersection with the axes. [1]

(iii) Find the area bounded by the curve C and the lines $y = \frac{1}{\pi}x$ and $x = \frac{5\pi}{6}$. [3]

(i) $y = \sin 2x + 2\cos x$ $\frac{dy}{dx} = 2\cos 2x - 2\sin x$ For stationary points, $\frac{dy}{dx} = 0$ $2\left[1 - 2\sin^2 x\right] - 2\sin x = 0$ $\Rightarrow 2\sin^2 x + \sin x - 1 = 0$ $\Rightarrow (2\sin x - 1)(\sin x + 1) = 0$ $\Rightarrow \sin x = 0.5$ or $\sin x = -1$ $\Rightarrow x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$ or $x = \frac{3\pi}{2}$	Differentiate and set $\frac{dy}{dx} = 0$ to find stationary points. As algebraic method is required, clear working of how the roots are arrived is expected, with usage of trigonometric identities along the way.
(ii) 	
(iii) From GC, the line $y = \frac{1}{\pi}x$ intersects the curve C at $x = 1.4544031$ Required area $= \int_{1..4544031}^{\frac{5\pi}{6}} \left[\frac{1}{\pi}x - (\sin 2x + 2\cos x)\right] dx$ $= 2.48$ (to 3 sig figs)	In order to find the area bounded by two curves, it is most important to find where the two curves intersect first, which can be done quickly using GC.

Marker's comments

Common mistakes:

1. In (i), it is unnecessary to convert $y=\sin 2x+2\cos x=2\sin x\cos x+2\cos x$ because it makes the differentiation more complicated. Students should have an awareness of the approach required by the question before manipulating the given information.

   An even more serious problem was that many students were unable to solve $2\cos 2x-2\sin x=0$ because identities were not used to convert it into a quadratic equation. Many were also unable to solve $\sin x=0.5$ (forgetting about the roots in other quadrants), or $\sin x=-1$ (rejecting it immediately without finding the basic angle).

2. Students were unable to identify the correct region, which resulted in them not finding the intersection between the two curves. Also, many students did not apply that the result $\int \mathrm{f}(x)-\mathrm{g}(x)\mathrm{d}x$ to find the area of the region bounded by two curves directly, and instead tried to find the area of the individual pieces which more often than not led to errors.

5 The curve C has equation $y = kx^3$. The tangent at the point P on C meets the curve again at point Q. The tangent at point Q meets the curve again at point R. If the x coordinates of P, Q and R are p, q, and r respectively where $p \neq 0$.

(i) Show that p and q satisfy the equation $\left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right) - 2 = 0$. [4]

(ii) Show that p, q and r are three consecutive terms of a geometric progression. Hence determine if this geometric series is convergent. [4]

[You may use the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ for $a, b \in \mathbb{R}$.]

(i) $y = kx^3$ $\frac{dy}{dx} = 3kx^2$ Point $P = (p,\ kp^3)$, Point $Q = (q,\ kq^3)$, Point $R = (r,\ kr^3)$	
Equation of tangent at point P: $y - kp^3 = 3kp^2(x-p)$ When tangent meets the curve again at Q: $kq^3 - kp^3 = 3kp^2(q-p)$ $q^3 - p^3 = 3p^2(q-p)$ $(q-p)(q^2+pq+p^2) = 3p^2(q-p)$ $(q-p)(q^2+pq-2p^2) = 0$ $q^2+pq-2p^2 = 0$ since $p \neq q$ because P and Q are different points Dividing by p^2: $\left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right) - 2 = 0$ (Shown)	Note that the gradient to tangent at point P is not $3kx^2$. You need to substitute x by p in $\frac{dy}{dx} = 3kx^2$ to get the gradient of tangent at point P.

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(ii) $\left(\frac{q}{p}\right)^2+\left(\frac{q}{p}\right)-2=0$

$\Rightarrow \quad \left(\frac{q}{p}+2\right)\left(\frac{q}{p}-1\right)=0$

$\Rightarrow \quad \frac{q}{p}=-2$ or $\frac{q}{p}=1$ (rejected since $q \neq p$)

Similarly for the other case,

$\frac{r}{q}=-2$

$\therefore \frac{q}{p}=\frac{r}{q}=-2$

Since the common ratio is the same, p, q and r are three consecutive terms of a geometric progression.

As $\left|\text{common ratio}\right|=2>1$, the geometric series is not convergent.

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Marker's comments

For part (i), while many students are able to find the equation of tangent, most students who had found the equation of tangent at P did not know how to continue from there. They need to observe more carefully what other information is given on the tangent to continue. In this case it is the fact that the tangent line cuts the curve again at point Q. This will lead to substituting x by q in the equation of tangent.

For part (ii), students must recall the condition for a sequence to be a GP, in this case $\frac{q}{p}=\frac{r}{q}$, and work towards it. As for the second part, students must be aware that the condition for geometric series to be convergent is $\left|\text{common ratio}\right|<1$.

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6 **(a)** The vectors **a** and **b** are the position vector of points A and B respectively. It is given that $OA = 2\sqrt{7}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{a} \cdot \mathbf{b} = -14$.

(i) Find angle AOB. [2]

(ii) State the geometrical meaning of $|\hat{\mathbf{a}} \cdot \mathbf{b}|$, where $\hat{\mathbf{a}}$ is the unit vector of **a**. [1]

(iii) Hence or otherwise, find the position vector of the foot of perpendicular from B to line OA in terms of **a**. [2]

(b) The non-zero vectors **p** and **q** are such that $|\mathbf{p} \times \mathbf{q}| = 2$. Given that **p** is a unit vector and $\mathbf{q} \cdot \mathbf{q} = 4$, show that **p** and **q** are perpendicular to each other. [3]

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(a)(i) <u>Given</u>: $|\underset{\sim}{a}| = 2\sqrt{7}$, $\underset{\sim}{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -14$
$\Rightarrow |\underset{\sim}{a}||\underset{\sim}{b}| \cos AOB = -14$
$\Rightarrow \left(2\sqrt{7}\right)\sqrt{1+4+9} \cos AOB = -14$
$\Rightarrow \cos AOB = -\dfrac{7}{\sqrt{7}\sqrt{14}} = -\dfrac{1}{\sqrt{2}}$
$\Rightarrow AOB = 135^\circ$
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Many students are confused between angles between two vectors and acute angles between 2 lines. Note that
(1) If θ is the angle between two vectors **a**, **b**, then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
(2) If θ is the acute angle between two lines with directional vectors **a**, **b**, then $\cos\theta = \dfrac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$
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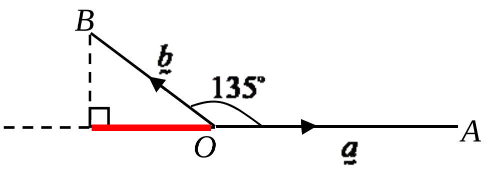
(a)(ii)
$|\underset{\sim}{\hat{a}} \cdot \underset{\sim}{b}|$
$= |\underset{\sim}{\hat{a}}||\underset{\sim}{b}||\cos AOB|$
$= |\underset{\sim}{b}||\cos AOB|$


$|\underset{\sim}{\hat{a}} \cdot \underset{\sim}{b}|$ is the length of projection of $\underset{\sim}{b}$ on $\underset{\sim}{a}$.
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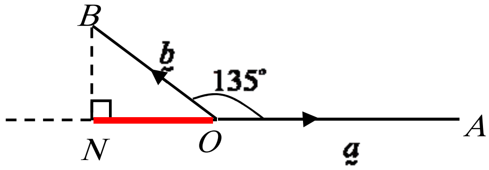
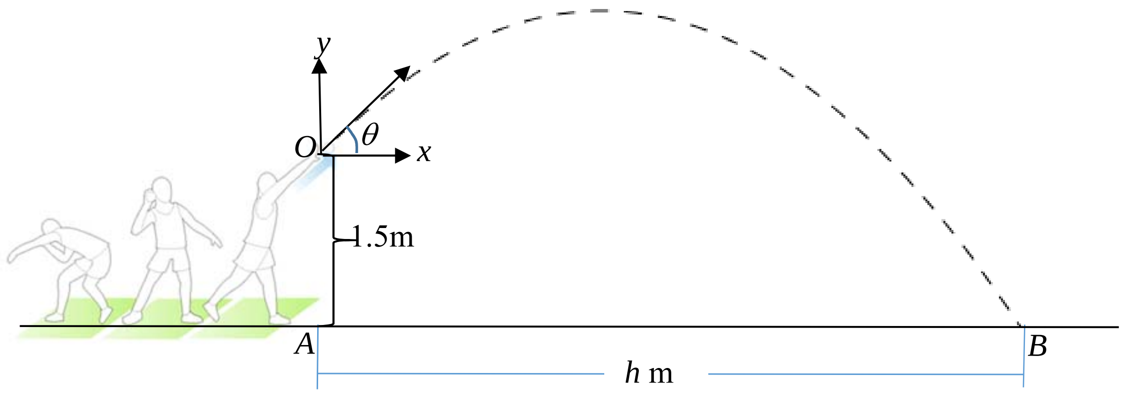
(a)(iii)  Let N be the foot of perpendicular from B to line OA. Length of projection, $ON = \lvert \hat{\underset{\sim}{a}} \cdot \underset{\sim}{b} \rvert = \dfrac{\lvert \underset{\sim}{a} \cdot \underset{\sim}{b} \rvert}{\lvert \underset{\sim}{a} \rvert} = \dfrac{\lvert -14 \rvert}{2\sqrt{7}} = \sqrt{7}$ Since $\angle AOB$ is an obtuse angle, $\overrightarrow{ON} = -\sqrt{7}\dfrac{\underset{\sim}{a}}{\lvert \underset{\sim}{a} \rvert} = -\frac{1}{2}\underset{\sim}{a}$	
Alternative method to (a)(iii) Let N be the foot of perpendicular from B to line OA. Since N lies on line OA, $\overrightarrow{ON} = \lambda \underset{\sim}{a}$ for some $\lambda \in \mathbb{R}$. Then, $\overrightarrow{BN} = \lambda \underset{\sim}{a} - \underset{\sim}{b}$ $\overrightarrow{BN} \perp \underset{\sim}{a}$ $\Rightarrow \quad \overrightarrow{BN} \cdot \underset{\sim}{a} = 0$ $\Rightarrow \quad (\lambda \underset{\sim}{a} - \underset{\sim}{b}) \cdot \underset{\sim}{a} = 0$ $\Rightarrow \quad \lambda \underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{a} \cdot \underset{\sim}{b}$ $\Rightarrow \quad \lambda \lvert \underset{\sim}{a} \rvert^2 = -14$ $\Rightarrow \quad \lambda (2\sqrt{7})^2 = -14$ $\Rightarrow \quad \lambda = -\frac{1}{2}$ Thus, $\overrightarrow{ON} = -\frac{1}{2}\underset{\sim}{a}$	
(b) <u>Given</u>: $\lvert \underset{\sim}{p} \rvert = 1$, $\underset{\sim}{q} \cdot \underset{\sim}{q} = 4$ $\Rightarrow \lvert \underset{\sim}{q} \rvert^2 = 4 \Rightarrow \lvert \underset{\sim}{q} \rvert = 2$ <u>Given</u>: $\lvert \underset{\sim}{p} \times \underset{\sim}{q} \rvert = 2$ $\Rightarrow \lvert \underset{\sim}{p} \rvert \lvert \underset{\sim}{q} \rvert \sin\theta = 2$, where θ is the angle between $\underset{\sim}{p}$ and $\underset{\sim}{q}$ $\Rightarrow (1)(2)\sin\theta = 2$ $\Rightarrow \sin\theta = 1$ $\Rightarrow \theta = 90^*$ Thus, $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are perpendicular to each other. (Shown)	Many students are not aware that $\lvert \underset{\sim}{p} \times \underset{\sim}{q} \rvert = \lvert \underset{\sim}{p} \rvert \lvert \underset{\sim}{q} \rvert \sin\theta$. Note that we do not need to mod sinθ as θ denotes the angle between vectors which means that θ is between 0 and π. Hence, sinθ will always be +ve.

Marker's comments

Students must know that the definition for both dot and cross product (i.e. $\left|\underset{\sim}{p}\cdot\underset{\sim}{q}\right| = \left|\underset{\sim}{p}\right|\left|\underset{\sim}{q}\right|\left|\cos\theta\right|$ and $\left|\underset{\sim}{p}\times\underset{\sim}{q}\right| = \left|\underset{\sim}{p}\right|\left|\underset{\sim}{q}\right|\sin\theta$) are very useful when solving problems that involve vectors which are not given in column vectors form.

Students who have applied using these definitions in this question had done well in this question.

7



The diagram shows a shot put being projected with a velocity v ms^{-1} from the point O at an angle θ made with the horizontal. The point O is 1.5m above the point A on the ground. The x-y plane is taken to be the plane that contains the trajectory of this projectile motion with x-axis parallel to the horizontal and O being the origin. The equation of the trajectory of this projectile motion is known to be

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta},$$

where g ms^{-2} is the acceleration due to gravity.

The constant g is taken to be 10 and the distance between A and B is denoted by h m. Given that $v = 10$, show that h satisfies the equation

$$h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0.$$ [3]

As θ varies, h varies. Show that stationary value of h occurs when θ satisfies the following equation

$$3\tan^2 2\theta - 20\sin 2\theta\tan 2\theta - 20\cos 2\theta - 20 = 0.$$ [5]

Hence find the stationary value of h. [2]

$y = x\tan\theta - \dfrac{10x^2}{2(10)^2\cos^2\theta}$ $\Rightarrow \quad y = x\tan\theta - \dfrac{x^2}{20\cos^2\theta}$ When $y = -1.5, \quad x = h$ $\therefore \quad -1.5 = h\tan\theta - \dfrac{h^2}{20\cos^2\theta}$ $\Rightarrow \quad -30\cos^2\theta = 20h\tan\theta\cos^2\theta - h^2$ $\Rightarrow \quad h^2 - 20h\sin\theta\cos\theta - 30\cos^2\theta = 0$ $\Rightarrow \quad h^2 - 10h\sin 2\theta - 15(1+\cos 2\theta) = 0$ $\Rightarrow \quad h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0$ --- (*) (Shown)	
Differentiate both sides w.r.t. θ, we have $2h\dfrac{\mathrm{d}h}{\mathrm{d}\theta} - 10\dfrac{\mathrm{d}h}{\mathrm{d}\theta}\sin 2\theta - 20h\cos 2\theta + 30\sin 2\theta = 0$ At stationary value, $\dfrac{\mathrm{d}h}{\mathrm{d}\theta} = 0$. $\therefore \quad -20h\cos 2\theta + 30\sin 2\theta = 0$ $\Rightarrow \quad h = \dfrac{30\sin 2\theta}{20\cos 2\theta} = \dfrac{3}{2}\tan 2\theta$ Sub into (*), we have $\left(\dfrac{3}{2}\tan 2\theta\right)^2 - 10\left(\dfrac{3}{2}\tan 2\theta\right)\sin 2\theta - 15\cos 2\theta - 15 = 0$ $\Rightarrow \quad \dfrac{9}{4}\tan^2 2\theta - 15\tan 2\theta\sin 2\theta - 15\cos 2\theta - 15 = 0$ $\Rightarrow \quad 3\tan^2 2\theta - 20\sin 2\theta\tan 2\theta - 20\cos 2\theta - 20 = 0$ (Shown) Using GC, $\theta = 0.71999$ (5 sig fig) Therefore, max $h = \dfrac{3}{2}\tan 2(0.71999) = 11.4$ (3 sig fig)	

Marker's comments

This question is poorly attempted in general.

(i) Students who fail to get credit for this part do not realise that $y = -1.5$ when $x = h$. There were also signs which indicate that students have difficulty applying the double-angle formula.

(ii) One common mistake made by students is to differentiate with respect to h. This is a conceptual error which indicates a poor understanding of derivatives. Many students on the other hand chose to make h the subject before differentiating, failing to realise that implicit differentiation would get the job done much easily. There was also a recurring problem of product rule when differentiating $10h\sin 2\theta$, with the erroneous result of $10\frac{\mathrm{d}h}{\mathrm{d}\theta}(-2\cos 2\theta)$.

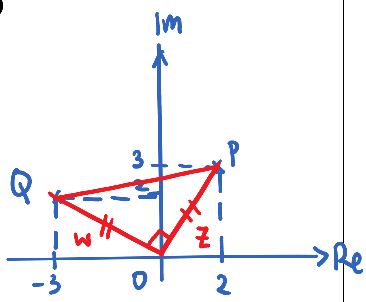
(iii) The equation can be easily solved using the GC, though there were many attempts to solve it algebraically. Students using the GC in degree mode would fail to obtain any credit for this part.

8 **(a)** In an Argand diagram, points P and Q represent the complex numbers $z_1 = 2+3\mathrm{i}$ and $z_2 = \mathrm{i}z_1$.

(i) Find the area of the triangle OPQ, where O is the origin. [2]

(ii) z_1 and z_2 are roots of the equation $\left(z^2+az+b\right)\left(z^2+cz+d\right)=0$, where $a, b, c, d \in \mathbb{R}$. Find a, b, c and d. [4]

(b) Without using the graphing calculator, find in exact form, the modulus and argument of $v^* = \left(\dfrac{\sqrt{3}+\mathrm{i}}{-1+\mathrm{i}}\right)^{14}$. Hence express v in exponential form. [5]

(a)(i) Since $w = \mathrm{i}z$, then $OP \perp OQ$ i.e. $\angle POQ = 90^o$. Area of triangle OPQ $=\dfrac{1}{2}\lvert z\rvert\lvert w\rvert$ $=\dfrac{1}{2}\lvert 2+3\mathrm{i}\rvert^2$ $=\dfrac{13}{2}\text{units}^2$ 	
(a)(ii) Since $\left(z^2+az+b\right)\left(z^2+cz+d\right)=0$ is a polynomial with constant coefficients, complex roots occur in conjugate pairs. Therefore, the four roots are $2+3\mathrm{i}$, $2-3\mathrm{i}$, $-3+2\mathrm{i}$ and $-3-2\mathrm{i}$. $[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})][z-(-3+2\mathrm{i})][z-(-3-2\mathrm{i})]$ $=\left(z^2-4z+13\right)\left(z^2+6z+13\right)$ Hence, $a=-4,\ b=13,\ c=6,\ d=13$.	Students should write out clearly the roots of the equation. There are 2 ways to expand $[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})]$ <u>Method 1</u> $[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})]$ $=z^2-(2+3\mathrm{i}+2-3\mathrm{i})z+(2+3\mathrm{i})(2-3\mathrm{i})$ <u>Method 2</u> $[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})]$ $=\left[(z-2)-(3\mathrm{i})\right]\left[(z-2)+(3\mathrm{i})\right]$ $=(z-2)^2-(3\mathrm{i})^2$

(b) $\lvert v^*\rvert = \dfrac{\lvert\sqrt{3}+\mathrm{i}\rvert^{14}}{\lvert -1+\mathrm{i}\rvert^{14}}$ $= \dfrac{2^{14}}{\left(\sqrt{2}\right)^{14}} = 2^7$ $\arg\left(v^*\right)$ $= \arg\left(\left(\dfrac{\sqrt{3}+\mathrm{i}}{-1+\mathrm{i}}\right)^{14}\right)$ $= 14\left[\arg\left(\sqrt{3}+\mathrm{i}\right) - \arg\left(-1+\mathrm{i}\right)\right]$ $= 14\left[\dfrac{\pi}{6} - \dfrac{3\pi}{4}\right]$ $= -\dfrac{49\pi}{6} \notin (-\pi, \pi]$ $\therefore\ \arg\left(v^*\right) = -\dfrac{\pi}{6}$ $\Rightarrow\ \arg\left(v\right) = \dfrac{\pi}{6}$ Since $\lvert v\rvert = \lvert v^*\rvert = 2^7$, then $v = 2^7 \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$.	Need to take note of how to present $\arg\left(v^*\right)$.

Marker's comments

(a)(i) This part was not well answered. Many students who were unclear/not aware that *OP* is perpendicular to *OQ* had problem arriving at the correct answer for the area of triangle *OPQ*. Many students used a variety of method (using vectors and cross product, shoelace method, area of trapezium/area of triangles) to find the area of triangle, some with more success than others.

(ii) Although many students were able to recognize that the complex roots occur in conjugate pairs since the coefficients of the equation are real, many students were unable to pair the factors $\left(z-(2+3\mathrm{i})\right)$ and $\left(z-(2-3\mathrm{i})\right)$, $\left(z-(-3+2\mathrm{i})\right)$ and $\left(z-(-3-2\mathrm{i})\right)$. Errors also occurred during the expansion of $\left(z-(2+3\mathrm{i})\right)\left(z-(2+3\mathrm{i})\right)$ and $\left(z-(-3+2\mathrm{i})\right)\left(z-(-3-2\mathrm{i})\right)$. A handful of students substituted $(2+3\mathrm{i})$ into the equation and managed to find a,b,c,d by comparing real and imaginary parts. Those who were unsuccessful in this method will not gain marks.

(b) Despite the statement at the start of the question, a large number of students used their GC to obtain modulus and argument to parts of the question. Many students (about 80%) of students rationalize the expression $\left(\dfrac{\sqrt{3}+\mathrm{i}}{-1+\mathrm{i}}\right)$ and soon realized that they did not have much success to solve the question except to use GC to obtain the modulus and argument. A number of students wrote down what they believed to be the argument of $(-1+\mathrm{i})$ without considering where the complex number was on an Argand diagram. This part clearly indicated that many students were weak in their understanding of the fundamental concepts of Complex Numbers.

9 A curve C has parametric equation defined by

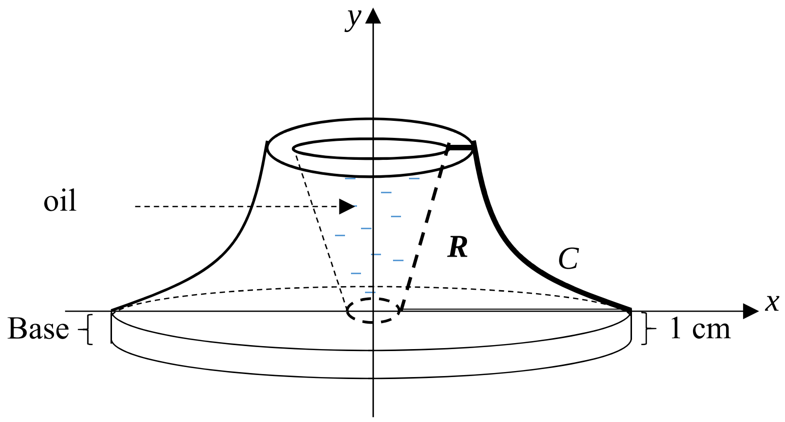
$$x = 4\sec t \text{ and } y = 8(1 - \tan t) \text{ where } -\frac{\pi}{4} \le t \le \frac{\pi}{4}.$$

(i) Find $\frac{dy}{dx}$ in terms of t and hence show that the equation of tangent at the point $t = -\frac{\pi}{6}$ is $y = 4x + 8\left(1 - \sqrt{3}\right)$. [3]

(ii) Find the Cartesian equation of C. [2]

R is the region bounded by C, the tangent in **(i)**, the normal to C at $t = 0$ and the x axis. Part of an oil burner is formed by rotating R 2π radians about the y-axis as shown in the diagram below (not drawn to scale). The base of the burner is a solid cylinder of thickness 1 cm.

[You may assume each unit along the x and y axis to be 1 cm]



Find the volume of the material required to make the burner. [6]

(i) $x = 4\sec t$ and $y = 8(1 - \tan t)$ $\frac{dx}{dt} = 4\sec t \tan t$, $\frac{dy}{dt} = -8\sec^2 t$ $\frac{dy}{dx} = -\frac{2}{\sin t}$ At $t = -\frac{\pi}{6}$, gradient of tangent $= 4$, $x = \frac{8}{3}\sqrt{3}$ and $y = 8\left(1 + \frac{\sqrt{3}}{3}\right)$ Equation of tangent is $y - 8\left(1 + \frac{\sqrt{3}}{3}\right) = 4\left(x - \frac{8\sqrt{3}}{3}\right)$ $y = 4x + 8\left(1 - \sqrt{3}\right)$ (Shown)	Students need to know that : $\sin(-x) = -x$ $\cos(-x) = x$ $\tan(-x) = -x$

(ii) $x = 4\sec t \Rightarrow \sec^2 t = \dfrac{x^2}{16}$

$y = 8(1-\tan t) \Rightarrow \tan^2 t = \left(1-\dfrac{y}{8}\right)^2$

Since $1+\tan^2 x = \sec^2 x$,

$1+\left(1-\dfrac{y}{8}\right)^2 = \dfrac{x^2}{16}$

$\dfrac{x^2}{16} - \dfrac{(y-8)^2}{64} = 1$

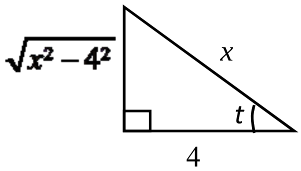
where $4 \le x \le 4\sqrt{2}$ and $0 \le y \le 16$ $\left(\because -\dfrac{\pi}{4} \le t \le \dfrac{\pi}{4}\right)$

<u>Alternative method:</u>

$\sec t = \dfrac{x}{4} \Rightarrow \cos t = \dfrac{4}{x}$

$y = 8(1-\tan t)$

$y = 8\left(1 \pm \dfrac{\sqrt{x^2-16}}{4}\right)$



(Note that $-\dfrac{\pi}{4} \le t \le \dfrac{\pi}{4} \Rightarrow \tan t = \dfrac{\sqrt{x^2-16}}{4}$ or $-\dfrac{\sqrt{x^2-16}}{4}$)

Note that $y = 8\left(1-\tan\left(\cos^{-1}\dfrac{4}{x}\right)\right)$ is not in its simplest form

When C intersects x-axis, $y = 0$,

$\dfrac{x^2}{16} - \dfrac{(0-8)^2}{64} = 1 \Rightarrow x^2 = 32$

$x = 4\sqrt{2}$ $(\because \text{radius} > 0)$

Volume of cylindrical base $= \pi\left(\sqrt{2}(4)\right)^2(1) = 32\pi$

<u>Method 1</u>

Volume of the solid that made the burner

$= \dfrac{\pi}{4}\displaystyle\int_0^8 64+(y-8)^2 \, dy - \dfrac{\pi}{16}\int_0^8 \left(y-8\left(1-\sqrt{3}\right)\right)^2 dy + 32\pi$

$\approx 475.718 = 476$ units^3 (using GC)

<u>Method 2</u>

Volume of solid that made the burner

$= \pi\displaystyle\int_{\frac{\pi}{4}}^{0} (4\sec t)^2(-8\sec^2 t)\, dt - \dfrac{\pi}{16}\int_0^8 \left(y-8\left(1-\sqrt{3}\right)\right)^2 dy + 32\pi \approx 476$

Method 3 Volume of solid that made the burner $= \frac{\pi}{4}\int_0^8 64+(y-8)^2 \; \mathrm{d}y + 32\pi$ $-\left[\underbrace{\frac{1}{3}\pi\left(2\sqrt{3}\right)^2\left(8+8\left(\sqrt{3}-1\right)\right)}_{\text{Volume of larger cone}}-\underbrace{\frac{1}{3}\pi\left(2\sqrt{3}-2\right)^2\left(8\left(\sqrt{3}-1\right)\right)}_{\text{Vol. of smaller cone}}\right]$ $\approx 476 \text{ units}^3$	Students who use Method 3 need to realise that when finding height of the two cones, for example, the height of the larger cone, they should not be using $8+8\left(1-\sqrt{3}\right)$ since $8\left(1-\sqrt{3}\right)$ is a negative y-intercept.

Marker's comments

(i) Generally well done.

Common errors is not knowing when to have negative sign when evaluating:

$$\tan\left(-\frac{\pi}{6}\right) \text{ and } \sec\left(-\frac{\pi}{6}\right)$$

(ii) Many students have forgotten the meaning of Cartesian equation, ended up with an equation that contains the parameter t which is wrong.

Quite a number of students leave answer as $y=8\left(1-\tan\left(\cos^{-1}\frac{4}{x}\right)\right)$ but this is not in the simplest form.

A serious mistake made by some students is to attempt to integrate $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2}{\sin t}$ but without realising that they cannot integrate $-\frac{2}{\sin t}$ with respect to x.

Last part

This part is very badly done.

Many either leave it blank or made a lot of careless/algebraic manipulation mistakes when trying to find x^2 in terms of y.

10 The point A has coordinates $(3, 1, 1)$. The line l has equation $\mathbf{r} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix}$, where λ is a parameter. P is a point on l when $\lambda = t$.

(i) Find cosine of the acute angle between AP and l in terms of t. Hence or otherwise, find the position vector of the point N on l such that N is the closest point to A. [6]

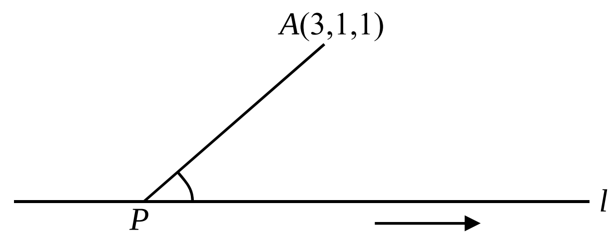
(ii) Find the coordinates of the point of reflection of A in l. [2]

The line L has equation $x = -1$, $2y = z + 2$.

(iii) Determine whether L and l are skew lines. [2]

(iv) Find the shortest distance from A to L. [3]

(i)



P is a point on l with parameter t. $\begin{pmatrix}2\\1\\1\end{pmatrix}$

$$\Rightarrow \quad \overrightarrow{OP} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}, \quad t \in \mathbb{R}$$

$$\overrightarrow{AP} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix} - \begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}-2\\0\\-2\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}$$

Let θ be the **acute** angle between BP and l.
Then,

$$\cos\theta = \frac{\left|\overrightarrow{AP}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}{\left|\overrightarrow{AP}\right|\left|\begin{pmatrix}2\\1\\1\end{pmatrix}\right|} = \frac{\left|\left[\begin{pmatrix}-2\\0\\-2\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}{\sqrt{(-2+2t)^2 + t^2 + (-2+t)^2}\sqrt{4+1+1}}$$

$$= \frac{|(-4-2) + t(4+1+1)|}{\sqrt{4t^2 - 8t + 4 + t^2 + t^2 - 4t + 4}\sqrt{4+1+1}}$$

$$= \frac{6|t-1|}{\sqrt{6}\sqrt{6t^2 - 12t + 8}} \quad \left(\text{or} \quad \frac{\sqrt{3}|t-1|}{\sqrt{3t^2 - 6t + 4}}\right)$$

Need to read Qn carefully and do not make careless mistakes.

θ is acute, $\cos\theta > 0$, so numerator needs to be positive.

$$\left[\begin{pmatrix}-2\\0\\-2\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}$$

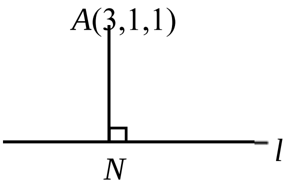
$$= \begin{pmatrix}-2\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$= (-4-2) + t(4+1+1) = -6 + 6t$$

Need to simplify the final answer, especially $\left|\overrightarrow{AP}\right|$

N is the closest point to A when $\theta = 90^\circ$.

$$\Rightarrow \cos 90^\circ = 0 = \frac{6|t-1|}{\sqrt{6}\sqrt{6t^2 - 12t + 8}}$$



$$\Rightarrow \quad t = 1$$

Thus, $\overrightarrow{ON} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}3\\2\\0\end{pmatrix}$

Alternative method

$\overrightarrow{ON} = \begin{pmatrix}1+2\lambda\\1+\lambda\\-1+\lambda\end{pmatrix}$, $\overrightarrow{AN} = \begin{pmatrix}2\lambda-2\\\lambda\\\lambda-2\end{pmatrix}$

$\overrightarrow{AN} \cdot \begin{pmatrix}2\\1\\1\end{pmatrix} = 0 \Rightarrow \lambda = 1$

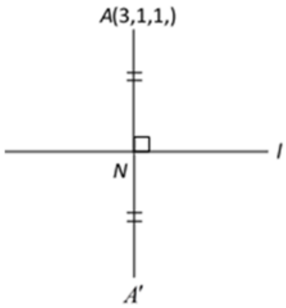
$\therefore \overrightarrow{ON} = \begin{pmatrix}3\\2\\0\end{pmatrix}$

(ii) Let A' be the point of reflection of A in l.
Using ratio theorem,

$$\overrightarrow{ON} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OA'}\right)$$

$$\Rightarrow \quad \overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA}$$

$$= 2\begin{pmatrix}3\\2\\0\end{pmatrix} - \begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}3\\3\\-1\end{pmatrix}$$



Thus, the coordinate of A' are $(3,3,-1)$.

Do not use long way to find point of reflection.
Eg. Begin with

$$\overrightarrow{BN} = \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{BA'}\right)$$

Must answer the Qn.

Must write r

(iii) l: $\quad \underset{\sim}{r} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}, \; t \in \mathbb{R}$

L: $\quad x = -1, \; 2y = z + 2 = \lambda$

i.e., L: $\quad \underset{\sim}{r} = \begin{pmatrix}-1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}0\\\frac{1}{2}\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\-2\end{pmatrix} + m\begin{pmatrix}0\\1\\2\end{pmatrix}$

At point of intersection of lines l and L:

$$\begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\-2\end{pmatrix} + m\begin{pmatrix}0\\1\\2\end{pmatrix} \Rightarrow t = -1, \; m = 0$$

Since the point $(-1,0,-2)$ lies on both l and L, the two lines **intersect** and thus cannot be skew lines. (Shown)

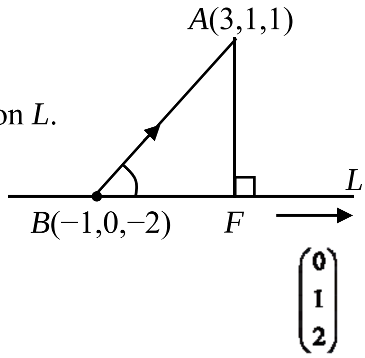
Cannot use the same parameter λ for both lines l and L.

You may also use this equation for L: $\underset{\sim}{r} = \begin{pmatrix}-1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\2\end{pmatrix}$

Need to know how to convert Cartesian equation to vector equation of a line.

Need to know how to use GC to solve the equation.

2 lines are not // and do not intersect $\Rightarrow$ they are skew lines

	Alternative method
(iv) $L: \underset{\sim}{r} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} + m \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$  Let B be the point $(-1,0,-2)$ on L. $\overrightarrow{BA} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$ Shortest distance from A to L $= \dfrac{\left\lvert \overrightarrow{BA} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rvert} = \dfrac{1}{\sqrt{1+4}} \left\lvert \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rvert$ $= \dfrac{1}{\sqrt{5}} \left\lvert \begin{pmatrix} -1 \\ -8 \\ 4 \end{pmatrix} \right\rvert = \dfrac{\sqrt{1+64+16}}{\sqrt{5}} = \dfrac{9\sqrt{5}}{5}$	Use $\times$ product not dot product Don't divide by $\lvert \overrightarrow{BA} \rvert$

Marker's comments

This is a straight forward question, but many students still did not score it well. They either made careless mistakes or cannot remember the correct formulae.

For (i), many students can get $\overrightarrow{AP} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ correctly but copied it down wrongly when they use it to find cos θ.

Many students make the following mistakes:

- $\left[\begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$
- Drop | | in the numerator part half way in the calculation or totally did not put.
- Some students used $\overrightarrow{OP}$ instead of $\overrightarrow{AP}$ to find cos θ.
- Not many students use $\theta = 90^0$ to find $\overrightarrow{ON}$.

(ii) Many students forgot to give coordinates of *A'*.

(iii) Badly done for this part.
- Quite a number of students cannot obtain the correct vector equation of line *L*.
- Of those who had the correct equation at the point of intersection, many of them gave no solution for the equation. (Do not know how to use GC to solve?)
- For those who can get the intersection point, many students conclude that: "Since there are intersection point, therefore they are skew lines."

(iv) Badly done for this part.
Careless mistake: Used line *l* instead of line *L*.
Use wrong formula: for e.g., used dot product instead of cross product or divide by $|\overrightarrow{BA}|$

11 A hot air balloon rises vertically upwards from the ground as the balloon operator intermittently fires and turns off the burner. At time t minutes, the balloon ascends at a rate inversely proportional to $t+\lambda$, where λ is a positive constant. At the same time, due to atmospheric factors, the balloon descends at a rate of 2 km per minute. It is also known that initially the rate of change of the height of the balloon is 1 km per minute.

(i) Find a differential equation expressing the relation between H and t, where H km is the height of the hot air balloon above ground at time t minutes. Hence solve the differential equation and find H in terms of t and λ. [7]

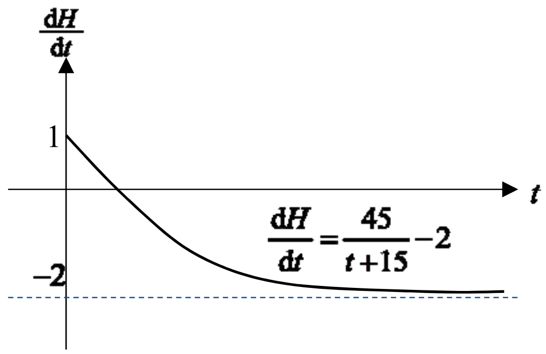
Using $\lambda = 15$,

(ii) Find the maximum height of the balloon above ground in exact form. [3]

(iii) Find the total vertical distance travelled by the balloon when $t = 8$. [3]

(iv) Can we claim that the rate of change of the height of the balloon above the ground is decreasing? Explain your answer. [2]

(i)	Rate of increase in height = $\dfrac{k}{t+\lambda}$ where k is a positive constant Rate of decrease in height = 2 Therefore, $\dfrac{\mathrm{d}H}{\mathrm{d}t}=\dfrac{k}{t+\lambda}-2$ Since $\dfrac{\mathrm{d}H}{\mathrm{d}t}=1$ when $t=0$, we have $1=\dfrac{k}{0+\lambda}-2$ $\Rightarrow 1=\dfrac{k-2\lambda}{\lambda}\;\therefore k=3\lambda$ Hence, $\dfrac{\mathrm{d}H}{\mathrm{d}t}=\dfrac{3\lambda}{t+\lambda}-2$ (Do not combine into one single fraction!) Integrating wrt t: $H=\int\left(\dfrac{3\lambda}{t+\lambda}-2\right)\mathrm{d}t=3\lambda\ln\|t+\lambda\|-2t+C$ Since $t+\lambda>0$, we have $H=3\lambda\ln(t+\lambda)-2t+C$ When $t=0$, $H=0$: $0=3\lambda\ln(\lambda)+C\;\therefore C=-3\lambda\ln\lambda$ $H=3\lambda\ln(t+\lambda)-2t-3\lambda\ln\lambda$ $\therefore H=3\lambda\ln\left(\dfrac{t}{\lambda}+1\right)-2t$	At least a few students in each class wrote rate of increase in height = $\dfrac{t+\lambda}{k}$, $k(t+\lambda)$ or $\dfrac{1}{k(t+\lambda)}$. Obviously they do not know the meaning of inversely proportional. Some leave their final answer as the general solution without finding the value of C.
(ii)	Using $\lambda=15$, at maximum height $\dfrac{\mathrm{d}H}{\mathrm{d}t}=\dfrac{45}{t+15}-2=0$ $\therefore t=7.5$ $\therefore H=45\ln\left(\dfrac{7.5}{15}+1\right)-2(7.5)=15\left(3\ln\dfrac{3}{2}-1\right)$	

(iii) When $t = 8$, $H = 45\ln\left(\frac{8}{15}+1\right)-2(8) = 45\ln\frac{23}{15}-16$ Total vertical distance travelled = Vertical distance travelled from $t=0$ to $t=7.5$ + Vertical distance travelled from $t=7.5$ to $t=8$ $= 15\left(3\ln\frac{3}{2}-1\right)+\left[15\left(3\ln\frac{3}{2}-1\right)-45\ln\frac{23}{15}+16\right]$ = 3.26 km (correct to 3 s.f.) (iv) $\frac{\mathrm{d}^2H}{\mathrm{d}t^2} = \frac{-45}{(t+15)^2} < 0$ for all real values of t, $t \geq 0$ i.e. the rate of change of the height of the balloon above ground is decreasing. Or from the graph of $\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{45}{t+15}-2$, we see that $\frac{\mathrm{d}H}{\mathrm{d}t}$ decreases as t increases. 	

Marker's comments

For part (i):

- For those who managed to get the correct DE, most are able to solve the DE using direct integration. Students lose marks if modulus is not included after integration or no reason is provided for dropping modulus.
- Students need to know that they are to find H in terms of t and λ, which means they need to find C by interpreting from the question that at $t = 0$, $H = 0$.

For part (ii):

Part (ii) was well done with only a few students not knowing how to approach the question. A few students did not read the question carefully and did not leave their answer for maximum height in exact form.

For part (iii):

Part (iii) was badly done. Many students did not realise that maximum height is reached at $t = 7.5$ (from (ii)), which means that H will decrease after 7.5 mins. Many students simply find H when $t = 8$. Some went to integrate $\frac{\mathrm{d}H}{\mathrm{d}t}$ from $t = 0$ to $t = 8$ which is incorrect.

For part (iv):

This part was also badly done. Many students conclude that as $t \to \infty$, $\frac{\mathrm{d}H}{\mathrm{d}t} \to -2$ and thus rate of change of height is decreasing, having the misconception that H decreases then rate of change of the height of the balloon is also decreasing. Some explain by drawing the graph of H instead of $\frac{\mathrm{d}H}{\mathrm{d}t}$.

Students need to know that if we want to show that H decreases with t, we need to show that $\frac{\mathrm{d}H}{\mathrm{d}t} < 0$. Similarly, if we want to show that rate of change of H, i.e. $\frac{\mathrm{d}H}{\mathrm{d}t}$ is decreasing, we need to show that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2H}{\mathrm{d}t^2} < 0$ or draw the graph of $\frac{\mathrm{d}H}{\mathrm{d}t}$ and show that it is decreasing with increasing t.

TJC Paper 2

1 Given that $\sin\left[(n+1)x\right]-\sin\left[(n-1)x\right]=2\cos nx\sin x$, show that

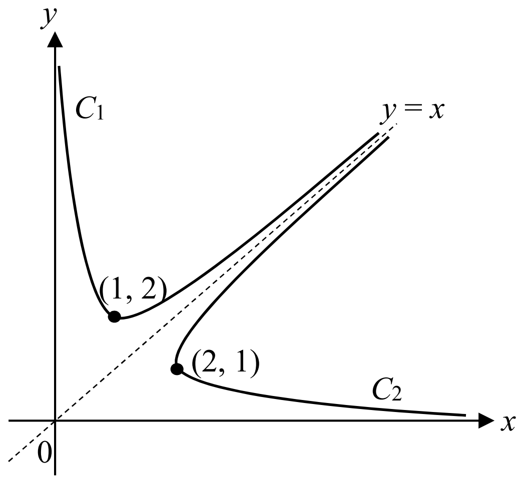
$$\sum_{r=1}^{n}\cos rx=\frac{\sin\left(n+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\ .\qquad [4]$$

Hence express

$$\cos^2\left(\frac{x}{2}\right)+\cos^2(x)+\cos^2\left(\frac{3x}{2}\right)+\ldots+\cos^2\left(\frac{11x}{2}\right)$$

in the form $a\left(\dfrac{\sin bx}{\sin cx}+d\right)$, where $a,\ b,\ c$ and d are real numbers. [3]

2



(a) The diagram above shows two curves C_1 and C_2 which are reflections of each other about the line $y = x$. State with justification, whether the following statement is true: "If C_1 is the graph of $y=\mathrm{f}(x)$, then C_2 is the graph of $y=\mathrm{f}^{-1}(x)$."

(b) The functions f and g are defined as follows

$$\mathrm{f}: x\mapsto\frac{1}{x^2-x-6},\quad x\in\mathbb{R}, x<0,\ x\neq-2$$

$$\mathrm{g}: x\mapsto\tan^{-1}\left(\frac{x}{2}\right),\quad x\in\mathbb{R}$$

(i) Sketch the graph of $y=\mathrm{f}(x)$. Determine whether f^2 exists. [3]

(ii) Find $\mathrm{f}^{-1}(x)$. [2]

(iii) Given that $\mathrm{gf}(a)=\dfrac{\pi}{4}$, find the exact value of a. [2]

3 Given that $\mathrm{e}^y=\sqrt{\mathrm{e}+x+\sin x}$. Show that

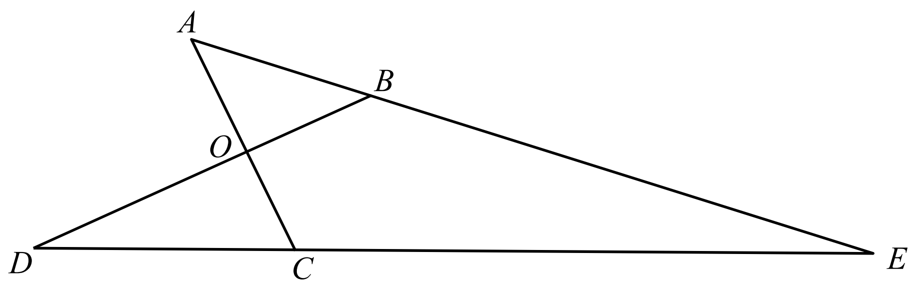
$$2\mathrm{e}^{2y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\mathrm{e}^{2y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\sin x=0.\qquad [2]$$

(i) Find the values of y, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x=0$. Hence, find in terms of e, the Maclaurin's series for $\ln(\mathrm{e}+x+\sin x)$, up to and including the term in x^2. [4]

(ii) By using appropriate standard series expansions from the List of Formulae (MF26), verify the correctness of the first three terms in the series expansion for $\ln(\mathrm{e}+x+\sin x)$ found in part **(i)**. [3]

(iii) Use your answer to part **(i)** to give an approximation for $\displaystyle\int_0^{\mathrm{e}^{-1}} \frac{2\mathrm{e}-4x}{\mathrm{e}^2\ln(\mathrm{e}+x+\sin x)}\,\mathrm{d}x$, giving your answer in terms of e. [3]

4



With reference to origin O, the points A, B, C and D are such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=-\mathbf{a}$ and $\overrightarrow{OD}=-2\mathbf{b}$. The lines AB and DC meet at E.

Find $\overrightarrow{OE}$ in terms of **a** and **b**. [4]

Hence show that $\dfrac{BE}{AB}=3$. [1]

It is given that A and E have coordinates (1, –4, 3) and (–3, 15, –5) respectively.

(i) Show that the lines AC and BD are perpendicular. [4]

(ii) Find the equation of the plane p that contains E and is perpendicular to the line BD. [2]

(iii) Find the distance between the line AC and p. [2]

5 Four classes CG40, CG41, CG42 and CG43 are tasked to organise a College event. Each class sends 3 representatives for a meeting.

(i) In how many different ways can the 12 representatives sit in a circle so that representatives from CG40 are not seated next to each other and representatives from other classes are seated with their respective classes? [3]

The 12 representatives are to be split up into 3 groups for bonding activities. Each group must consist of a representative from each class.

(ii) In how many ways can the groups be formed? [2]

6 In a game at the carnival, a player rolls discs onto a board containing squares, each of which bears one of the numbers 1, 2, 5 or 10. If a disc does not land within a square, the player receives nothing. The probability that the disc does not land within the square is $\dfrac{3}{4}$. If a disc lands within a square, the player receives the same amount (in dollars) as the number in the square. Given that a disc falls within a square, the probabilities of landing within a square with the numbers 1, 2, 5 and 10 are 0.5, 0.3, 0.12 and 0.08 respectively. It is assumed that the rolls of the discs are independent.

(i) A player pays \$5 to play the game and is given n discs. Find n if the game is fair. [4]

(ii) If a player is allowed to roll 3 discs for \$2, find the probability that the player will have a profit of \$10. [4]

7 A factory manufactures large number of pen refills. From past records, 3% of the refills are defective. A stationery store manager wishes to purchase pen refills from the factory. To decide whether to accept or reject a batch of refills, the manager designs a sampling process. He takes a random sample of 25 refills. The batch is accepted if there is no defective refill and rejected if there are more than 2 defective refills. Otherwise, a second random sample of 25 refills is taken. The batch is then accepted if the total number of defective refills in the two samples is fewer than 4 and rejected otherwise.

(i) Find the probability of accepting a batch. [4]

(ii) If a batch is accepted, find the probability that there are 2 defective refills found in the sampling process. [3]

The stationery store manager purchases 50 boxes of 25 refills each.

(iii) Find the probability that the mean number of defective refills in a box is less than 1. [2]

8 A study is done to find out the relationship between the age of women and the steroid levels in the blood plasma. Sample data collected from 10 females with ages ranging from 8 years old to 35 years old is as shown below.

Age (years) x	8	11	14	17	20	23	26	29	32	35
Steroid Level (mmol/litre) L	4.2	11.1	16.3	19.0	25.5	26.2	24.1	33.5	20.8	17.4

(i) Give a sketch of the scatter diagram for the data. Identify the outlier and suggest a reason, in the context of the question, why this data pair is an outlier. [3]

For the remaining part of the question, the outlier is to be removed from the calculation.

(ii) Comment on the suitability of each of the following models. Hence determine the best model for predicting the steroid level of a female based on her age.

Model A: $L = a + b\ln x$

Model B: $L = c + d\left(x-25\right)^2$

Model C: $L = e + f\left(x-25\right)^4$

where a, b, c, d, e and f are constants. [3]

(iii) Using the best model in **(ii)**, estimate the steroid level of a woman at age 40. Comment on the reliability of your estimate. [3]

(iv) It is known that body muscle mass and steroid level has a linear correlation. The muscle mass percentage m % of the 9 females were measured. An additional female, Jane, participated in the study. Jane has her muscle mass percentage and steroid level measured. The mean muscle mass percentage of the 10 females is now found to be 26.28 %. The equation of the least squares regression line of m on L for the 10 pairs of data is

$$m = 2.22 + 1.25L.$$

Calculate Jane's steroid level. [3]

9 A flange beam is a steel beam with a "H"-shaped cross section, and is used as a supporting structure in construction and civil engineering. A factory manufactures both Grade X and Grade Y flange beams. The load that can be supported by a Grade X flange beam follows a normal distribution with mean 2.43×10^5 kN and standard deviation 4.5×10^4 kN. The load that can be supported by a Grade Y flange beam is 1.5 times of the load that can be supported by a Grade X flange beam.

(i) Find the probability that the combined load that can be supported by two randomly chosen Grade Y flange beams is within 1×10^4 kN of the combined load that can be supported by three randomly chosen Grade X flange beams. [4]

(ii) A construction company wants to buy 100 sets of three Grade X flange beams. Find the probability that fewer than 95 of these sets can support more than 6×10^5 kN. [3]

The company decides to place an order with the factory for a custom-made flange beam such that the probability of being able to support a load of at least 6×10^5 kN must be at least 0.999. It is also assumed that the load that can be supported by the custom-made flange beam also follows a normal distribution.

(iii) By taking the standard deviation of a custom-made flange beam to be 3×10^4 kN, find the smallest possible mean load in kN, giving your answer correct to the nearest thousand, for the factory to meet the company's requirements for the custom-made flange beam. [5]

10 **(a)** College students intending to further their studies overseas have to sit for a mandatory Overseas Universities Test (OUT). Researcher Mr Anand wishes to find out if male college students tend to score higher for OUT compared to female college students. Mr Anand's colleague randomly selects 150 male and 150 female students from the combined student population of three particular colleges near his home to form a sample of 300 college students for the research. Explain whether this sample is a random sample. [2]

(b) The mean OUT score for all college students in 2016 is 66.
Mr Anand randomly selects 240 college students taking OUT in 2017 and their scores, x, are summarised in the following table:

Score, x	60	65	68	70	75	80
Frequency, f	40	90	63	27	18	2

(i) Write down the unbiased estimates of the population mean and variance of the OUT scores for the college students in 2017. [1]

(ii) Test, at the 10% level of significance, whether the mean OUT score for all college students in 2017 is higher than the mean score attained in 2016. [4]

(iii) Explain what is meant by the phrase "10% level of significance" in this context. [1]

(iv) Mr Anand draws a new sample of 240 male college students. Using the unbiased estimate for the population variance computed in **(i)**, find the range of values for the sample mean $\overline{x}$ that is required for this new sample to achieve a different conclusion from that in **(ii)**. [4]

(c) The 2017 OUT scores of the male and female college students are independent and assumed to be normally distributed with means and standard deviations as shown in the following table:

	Mean	Standard deviation
Male College Students	64	5.5
Female College Students	66	3.5

Mr Beng and Miss Charlene both scored 70. Explain who performed better relative to their respective gender cohort. [2]

Section A: Pure Mathematics [40 marks]

1 Given that $\sin\left[(n+1)x\right]-\sin\left[(n-1)x\right]=2\cos nx\sin x$, show that

$$\sum_{r=1}^{n}\cos rx=\frac{\sin\left(n+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\quad.$$ [4]

Hence express

$$\cos^2\left(\frac{x}{2}\right)+\cos^2(x)+\cos^2\left(\frac{3x}{2}\right)+\ldots+\cos^2\left(\frac{11x}{2}\right)$$

in the form $a\left(\dfrac{\sin bx}{\sin cx}+d\right)$, where a,b,c and d are real numbers. [3]

<u>Given</u>: $2\cos nx\sin x=\sin(n+1)x-\sin(n-1)x$

Thus,
$$2\cos x\ \sin x=\sin 2x-\sin 0x$$
$$2\cos 2x\sin x=\sin 3x-\sin x$$
$$2\cos 3x\sin x=\sin 4x-\sin 2x$$
$$\cdots$$
$$2\cos(n-2)x\sin x=\sin(n-1)x-\sin(n-3)x$$
$$2\cos(n-1)x\sin x=\sin(n)x-\sin(n-2)x$$
$$2\cos nx\sin x=\sin(n+1)x-\sin(n-1)x$$

Adding the n equations above, we have

$$2\sin x\sum_{r=1}^{n}\cos rx=\sin(n+1)x+\sin nx-\sin x$$

$$2\sin x\sum_{r=1}^{n}\cos rx=2\sin\left(\frac{2n+1}{2}\right)x\cos\tfrac{1}{2}x-\sin x$$

$$\cancel{2}\left(2\sin\tfrac{1}{2}x\cancel{\cos\tfrac{1}{2}x}\right)\sum_{r=1}^{n}\cos rx=\cancel{2}\sin\left(n+\tfrac{1}{2}\right)x\cancel{\cos\tfrac{1}{2}x}-\cancel{2}\sin\tfrac{1}{2}x\cancel{\cos\tfrac{1}{2}x}$$

$$2\sin\tfrac{1}{2}x\sum_{r=1}^{n}\cos rx=\sin\left(n+\tfrac{1}{2}\right)x-\sin\tfrac{1}{2}x$$

□ $$\sum_{r=1}^{n}\cos rx=\frac{\sin\left(n+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\quad\text{(Shown)}$$

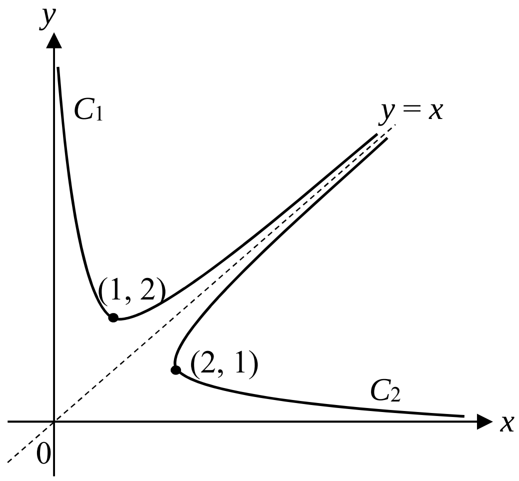
$\cos^2\left(\frac{x}{2}\right)+\cos^2(x)+\cos^2\left(\frac{3x}{2}\right)+...+\cos^2\left(\frac{11x}{2}\right)$ $=\frac{1+\cos x}{2}+\frac{1+\cos 2x}{2}+\frac{1+\cos 3x}{2}+...+\frac{1+\cos 11x}{2}$ $=\frac{1}{2}\left(11+\sum_{r=1}^{11}\cos rx\right)$ $=\frac{1}{2}\left(11+\frac{\sin\left(11+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\right)$ $=\frac{1}{2}\left(11+\frac{\sin\frac{23}{2}x}{2\sin\frac{1}{2}x}-\frac{1}{2}\right)$ $=\frac{1}{2}\left(\frac{\sin\frac{23}{2}x}{2\sin\frac{1}{2}x}+\frac{21}{2}\right)=\frac{1}{4}\left(\frac{\sin\frac{23}{2}x}{\sin\frac{1}{2}x}+21\right)$	

Marker's comments

Most students were able to make use of the given result and apply the method of differences to solve for $\sum_{r=1}^{n}\cos rx=\sum_{r=1}^{n}\frac{\sin(r+1)x-\sin(r-1)x}{2\sin x}=\frac{\sin(n+1)x+\sin nx-\sin x}{2\sin x}$. Thereafter, many students fail to apply the appropriate factor formula and double-angle formula to obtain the desired answer.

The second part of the question involves the use of double-angle formula to convert $\cos^2\left(\frac{r}{2}\right)$ into $\frac{\cos(rx)+1}{2}$, but many students chose to replace the index r by $\frac{r}{2}$, which would not allow them to achieve anything. Some students lost credit by failing to express their answer in the form as stated in the question.

2(a)



The diagram above shows two curves C_1 and C_2 which are reflections of each other about the line $y = x$. State with justification, whether the following statement is true:

"If C_1 is the graph of $y = \mathrm{f}(x)$, then C_2 is the graph of $y = \mathrm{f}^{-1}(x)$." [1]

(b) The functions f and g are defined as follows

$$\mathrm{f} : x \mapsto \frac{1}{x^2 - x - 6}, \quad x \in \mathbb{R}, x < 0,$$

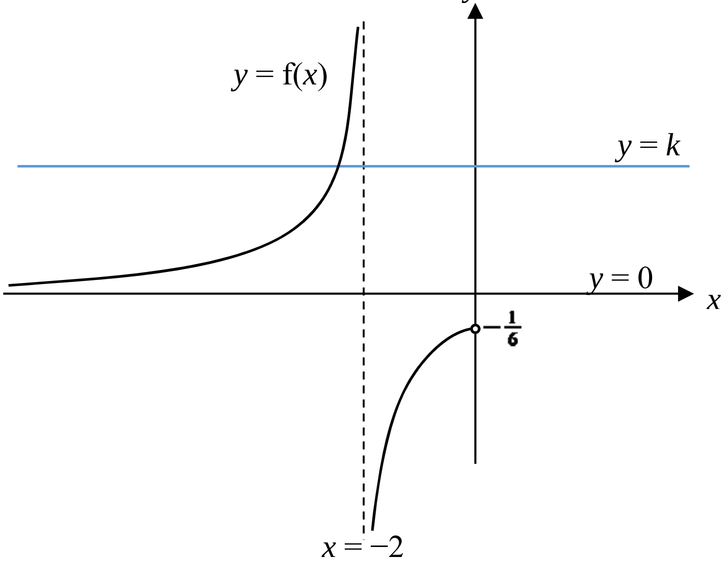
$$\mathrm{g} : x \mapsto \tan^{-1}\left(\frac{x}{2}\right), \quad x \in \mathbb{R}.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$. Determine whether f^2 exists. [3]

(ii) Find $\mathrm{f}^{-1}(x)$. [2]

(iii) Given that $\mathrm{gf}(a) = \frac{\pi}{4}$, find the exact value of a. [2]

(a) From the graph of $y = \mathrm{f}(x)$ which is C_1, there exists a horizontal line $y = 3$ which cuts the graph of $y = \mathrm{f}(x)$ at 2 points. f is not one to one and thus f^{-1} does not exist. Since f^{-1} does not exist, C_2 is not the graph of $\mathrm{f}^{-1}(x)$.	

(b)(i)  Since $R_{\mathrm{f}} = \left(-\infty, -\frac{1}{6}\right) \cup (0, \infty)$ and $D_{\mathrm{f}} = (-\infty, 0)$ i.e. $R_{\mathrm{f}} \not\subset D_{\mathrm{f}}$ Therefore, f^2 does not exist.	
(b)(ii) Let $y = \dfrac{1}{x^2 - x - 6}$ $\Rightarrow \quad yx^2 - xy - 6y - 1 = 0$ $\Rightarrow \quad x = \dfrac{y \pm \sqrt{y^2 + 4y(6y+1)}}{2y}$ $\Rightarrow \quad x = \dfrac{y \pm \sqrt{25y^2 + 4y}}{2y}$ Since $x < 0$, $x = \dfrac{y - \sqrt{25y^2 + 4y}}{2y} = \dfrac{1}{2} - \dfrac{\sqrt{25y^2 + 4y}}{2y}$ Thus, $\mathrm{f}^{-1}(x) = \dfrac{1}{2} - \dfrac{\sqrt{25x^2 + 4x}}{2x}$.	

(b)(iii) $\mathrm{gf}(a) = \frac{\pi}{4}$ $\tan^{-1}\left(\frac{\mathrm{f}(a)}{2}\right) = \frac{\pi}{4}$ $\frac{\mathrm{f}(a)}{2} = 1$ $\mathrm{f}(a) = 2 \quad \Rightarrow \quad a = \mathrm{f}^{-1}(2)$ $\therefore a = \frac{1}{2} - \frac{\sqrt{25(2)^2 + 4(2)}}{2(2)} = \frac{1}{2} - \frac{3\sqrt{3}}{2}$ **Alternative solution** $\mathrm{gf}(a) = \frac{\pi}{4}$ $\tan^{-1}\left(\frac{\mathrm{f}(a)}{2}\right) = \frac{\pi}{4}$ $\Rightarrow \quad \frac{\mathrm{f}(a)}{2} = 1$ $\Rightarrow \quad \mathrm{f}(a) = 2$ $\Rightarrow \quad \frac{1}{a^2 - a - 6} = 2$ $\Rightarrow \quad 2a^2 - 2a - 13 = 0$ $\Rightarrow \quad a = \frac{2 \pm \sqrt{4 + 4(2)(13)}}{4} = \frac{2 \pm 6\sqrt{3}}{4} = \frac{1}{2} \pm \frac{3}{2}\sqrt{3}$ Since $a < 0$, $a = \frac{1}{2} - \frac{3}{2}\sqrt{3}$	

Marker's comments

(a) This part is generally well-answered by students who recognised that f is not 1-1 and so the inverse cannot exist. Students who gave no or incorrect justification to why the statement is false fail to gain any credit.

(b) The graph of $y = \mathrm{f}(x)$ is generally well-drawn and most students were able to present their sketches within the correct domain. The main issue for this part is that many students tried to justify whether f^2 exists or not in relation to whether the inverse function exists or not, showing a misconception between composite and inverse functions. Many students proceeded to obtain full credits for the desired results of parts (ii) and (iii), but students need to first be aware that no credit was deducted when the domain or range was presented wrongly.

3 Given that $e^y = \sqrt{e + x + \sin x}$. Show that

$$2e^{2y}\frac{d^2y}{dx^2} + 4e^{2y}\left(\frac{dy}{dx}\right)^2 + \sin x = 0 .$$ [2]

(i) Find the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 0$. Hence, find in terms of e, the Maclaurin's series for $\ln(e + x + \sin x)$, up to and including the term in x^2. [4]

(ii) By using appropriate standard series expansions from the List of Formulae (MF26), verify the correctness of the first three terms in the series expansion for $\ln(e + x + \sin x)$ found in part **(i)**. [3]

(iii) Use your answer to part **(i)** to give an approximation for $\int_0^{e^{-1}} \frac{2e - 4x}{e^2 \ln(e + x + \sin x)} \, dx$, giving your answer in terms of e. [3]

$e^y = \sqrt{e + x + \sin x}$ $\Rightarrow \quad e^{2y} = e + x + \sin x$ Differentiate wrt x: $e^{2y}\left(2\frac{dy}{dx}\right) = 1 + \cos x$ i.e., $\quad 2e^{2y}\frac{dy}{dx} = 1 + \cos x$ Differentiate wrt x: $2\left[e^{2y}\frac{d^2y}{dx^2} + \frac{dy}{dx}e^{2y}\left(2\frac{dy}{dx}\right)\right] = -\sin x$ i.e., $\quad 2e^{2y}\frac{d^2y}{dx^2} + 4e^{2y}\left(\frac{dy}{dx}\right)^2 + \sin x = 0$ (Shown)	Square both sides first. Do not use the tedious method of working out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ directly from given equation.

(i) When $x=0$, $e^{2y}=e+0+0 \Rightarrow y=\frac{1}{2}$ $2e\frac{dy}{dx}=1+1 \Rightarrow \frac{dy}{dx}=\frac{1}{e}$ $2e\frac{d^2y}{dx^2}+4e\left(\frac{1}{e}\right)^2+0=0 \Rightarrow \frac{d^2y}{dx^2}=\frac{-2}{e^2}$ $\therefore \quad y=\frac{1}{2}+\frac{1}{e}x-\frac{2}{e^2}\left(\frac{x^2}{2!}\right)+\cdots$ $e^{2y}=e+x+\sin x$ $\Rightarrow \ln(e+x+\sin x)=2y$ $=2\left(\frac{1}{2}+\frac{1}{e}x-\frac{2}{e^2}\frac{x^2}{2!}+\cdots\right)$ i.e., $\ln(e+x+\sin x)=1+\frac{2}{e}x-\frac{2}{e^2}x^2+\cdots$	This series expansions is for y, and **not** for $\ln(e+x+\sin x)$ or e^{2y}.
(ii) $\ln(e+x+\sin x)$ $=\ln(e+x+x+\ldots)$ $=\ln(e+2x+\ldots)$ $=\ln\left(e\left(1+\frac{2}{e}x+\ldots\right)\right)$ $=\ln e+\ln\left(1+\frac{2}{e}x+\ldots\right)$ $=1+\ln\left(1+\frac{2}{e}x+\ldots\right)$ $=1+\left[\left(\frac{2}{e}x\right)-\frac{1}{2}\left(\frac{2}{e}x\right)^2+\cdots\right]$ $=1+\frac{2}{e}x-\frac{2}{e^2}x^2+\cdots$ (Verified)	Apply the following standard series expansions. $\sin x=x+\ldots$ and $\ln(1+x)=x-\frac{x^2}{2}+\ldots$. ($x^3$ term can be ignored as the result in part (i) is only up to x^2 term). *Note:* $\ln(e+x+\sin x)\neq \ln e+\ln x+\ln\sin x$ $\ln(e+x+\sin x)\neq(\ln e)\ln(x+\sin x)$, $\ln(e+x+\sin x)\neq\ln\left(1+\frac{x}{e}+\frac{\sin x}{e}\right)$ You can use this (ii) result to check whether you make mistakes in part (i) or (ii) if the results are different.

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(iii) $\displaystyle\int_0^{e^{-1}} \frac{2e-4x}{e^2\ln\left(e+x+\sin x\right)}\,dx$

$\displaystyle\approx \int_0^{e^{-1}} \frac{2e-4x}{e^2\left(1+\frac{2}{e}x-\frac{2}{e^2}x^2\right)}\,dx$

$\displaystyle= \int_0^{e^{-1}} \frac{2e-4x}{e^2+2ex-2x^2}\,dx$

$\displaystyle= \Big[\ln\left(e^2+2ex-2x^2\right)\Big]_0^{\frac{1}{e}}$

$\displaystyle= \ln\left(e^2+2-2\left(\tfrac{1}{e}\right)^2\right)-\ln\left(e^2\right)$

$\displaystyle= \ln\left(e^2+2-\frac{2}{e^2}\right)-2$

$\displaystyle= \ln\left(e^4+2e^2-2\right)-4$

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Use $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right|+c$

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Marker's comments

About 20% of the students use tedious method to show first part.

Badly done for part (ii). Many students leave blank for this part. For those who tried, many of them get different answers for (i) and (ii), and still wrote (verified). They should use it to check their own mistakes.

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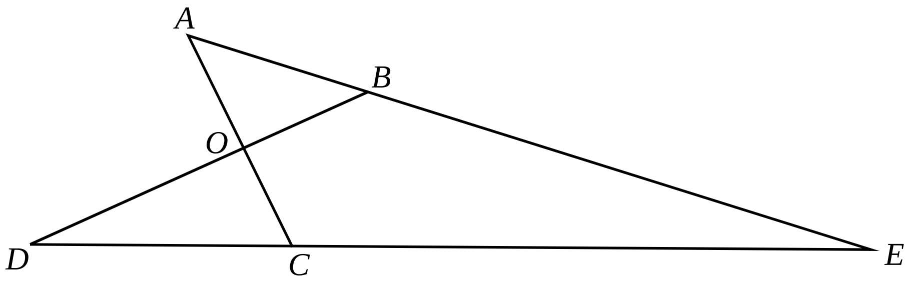

With reference to origin O, the points A, B, C and D are such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=-\mathbf{a}$ and $\overrightarrow{OD}=-2\mathbf{b}$. The lines AB and DC meet at E.

Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

Hence show that $\dfrac{BE}{AB}=3$. [1]

It is given that A and E have coordinates $(1, -4, 3)$ and $(-3, 15, -5)$ respectively.

(i) Show that the lines AC and BD are perpendicular. [4]

(ii) Find the equation of the plane p that contains E and is perpendicular to the line BD. [2]

(iii) Find the distance between the line AC and p. [2]

Equation of line AB is $\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})$. Equation of line DC is $\underset{\sim}{r} = -\underset{\sim}{a} + \mu(-2\underset{\sim}{b} - (-\underset{\sim}{a}))$, i.e., $\underset{\sim}{r} = -\underset{\sim}{a} + \mu(-2\underset{\sim}{b} + \underset{\sim}{a})$. To find E, the point of intersection of lines AB and CD, consider $\underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a}) = -\underset{\sim}{a} + \mu(-2\underset{\sim}{b} + \underset{\sim}{a})$ $\Rightarrow \quad (1-\lambda)\underset{\sim}{a} + \lambda\underset{\sim}{b} = (-1+\mu)\underset{\sim}{a} - 2\mu\underset{\sim}{b}$ $\Rightarrow \quad (2-\lambda-\mu)\underset{\sim}{a} = (-2\mu-\lambda)\underset{\sim}{b}$ Since $\underset{\sim}{a}$ is not parallel to $\underset{\sim}{b}$, $\begin{cases} 2-\mu-\lambda=0 & \cdots(1) \\ -2\mu-\lambda=0 & \cdots(2) \end{cases}$ Solving (1) and (2), we have $\mu=-2$ and $\lambda=4$ $\therefore \overrightarrow{OE} = \underset{\sim}{a} + 4(\underset{\sim}{b} - \underset{\sim}{a}) = -3\underset{\sim}{a} + 4\underset{\sim}{b}$ $\therefore \overrightarrow{BE} = \overrightarrow{OE} - \overrightarrow{OB} = -3\underset{\sim}{a} + 4\underset{\sim}{b} - \underset{\sim}{b} = 3(\underset{\sim}{b} - \underset{\sim}{a}) = 3\overrightarrow{AB}$ $\therefore \dfrac{BE}{AB} = 3$	Students must know that there is no such things as $\dfrac{vector}{vector}$. In this case, students who have written $\dfrac{3(\mathbf{b}-\mathbf{a})}{(\mathbf{b}-\mathbf{a})} = 3$ will not be given any credit.
(i) $\overrightarrow{OE} = -3\underset{\sim}{a} + 4\underset{\sim}{b} = \begin{pmatrix} -3 \\ 15 \\ -5 \end{pmatrix}$ $\Rightarrow \quad -3\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + 4\underset{\sim}{b} = \begin{pmatrix} -3 \\ 15 \\ -5 \end{pmatrix}$ $\Rightarrow \quad 4\underset{\sim}{b} = \begin{pmatrix} -3 \\ 15 \\ -5 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \quad \Rightarrow \quad \underset{\sim}{b} = \begin{pmatrix} 0 \\ \frac{3}{4} \\ 1 \end{pmatrix}$ $\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \frac{3}{4} \\ 1 \end{pmatrix} = -4\left(\frac{3}{4}\right) + 3 = 0$ $\Rightarrow$ OA and OB are perpendicular $\Rightarrow$ AC and BD are perpendicular (as AC is parallel to OA and BD is parallel to OB)	Students must give clear explanation for every step. In this case, students must explain clearly why $OA \perp OB$ implies $AC \perp BD$.

(ii) Equation of the plane p is $\underset{\sim}{r}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}=\begin{pmatrix}-3\\15\\-5\end{pmatrix}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}$, i.e. $\underset{\sim}{r}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}=25$	
(iii) Distance between the line AC and the plane p = distance of O from p $=\dfrac{\begin{pmatrix}-3\\15\\-5\end{pmatrix}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}}{\sqrt{3^2+4^2}}=5$	

Marker's comments

The first part of this question is badly done. Students must know that problems involving vectors that are not given in the column vector way are very common in this syllabus. This question is just one example which requires you to find the intersection between two lines, in which position vectors of points on the lines are as generic vectors **a** and **b**. Students are advised to do more such practices from MSM and all other vectors revision resources that are given out.

Section B: Statistics [60 marks]

5 Four classes CG40, CG41, CG42 and CG43 are tasked to organise a College event. Each class sends 3 representatives for a meeting.

(i) In how many different ways can the 12 representatives sit in a circle so that representatives from CG40 are not seated next to each other and representatives from other classes are seated with their respective classes? [3]

The 12 representatives are to be split up into 3 groups for bonding activities. Each group must consist of a representative from each class.

(ii) In how many ways can the groups be formed? [2]

(i) Number of ways to arrange the 3 classes except CG40 $=(3!)^3(3-1)!$ Number of ways to arrange reps from CG40 for a particular arrangement of the other 3 classes = 3! Required number $=(3!)^3(3-1)!3!=2592$	
(ii) Required number $=\dfrac{(3!)^4}{3!}=216$	

Marker's comments

Students are advised to present their working for P&C questions clearly, step by step. Many students are not able to get any credit at all for this question because their answer is a one-liner answer and they got the answer wrong. Such students may be able to at least obtain one or two marks if they have presented and explained their working more clearly.

6 In a game at the carnival, a player rolls discs onto a board containing squares, each of which bears one of the numbers 1, 2, 5 or 10. If a disc does not land within a square, the player receives nothing. The probability that the disc does not land within the square is $\frac{3}{4}$. If a disc lands within a square, the player receives the same amount (in dollars) as the number in the square. Given that a disc falls within a square, the probabilities of landing within a square with the numbers 1, 2, 5 and 10 are 0.5, 0.3, 0.12 and 0.08 respectively. It is assumed that the rolls of the discs are independent.

(i) A player pays \$5 to play the game and is given n discs. Find n if the game is fair.[4]

(ii) If a player is allowed to roll 3 discs for \$2, find the probability that the player will have a profit of \$10. [4]

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(i) Let Y (in dollars) be the amount received by a player for each roll.

y	0	1	2	5	10
$\mathrm{P}(Y=y)$	$\frac{3}{4}$	$\frac{1}{4}\times\frac{1}{2}$ $=\frac{1}{8}$	$\frac{1}{4}\times\frac{3}{10}$ $=\frac{3}{40}$	$\frac{1}{4}\times\frac{3}{25}$ $=\frac{3}{100}$	$\frac{1}{4}\times\frac{8}{100}$ $=\frac{1}{50}$

$\mathrm{E}(Y) = \left(0\times\frac{3}{4}\right)+\left(1\times\frac{1}{8}\right)+\left(2\times\frac{3}{40}\right)+\left(5\times\frac{3}{100}\right)+\left(10\times\frac{1}{50}\right)$
$= 0.625$

For the game to be fair,

$\mathrm{E}(Y_1+Y_2+\cdots+Y_n-5)=0$

$\Rightarrow \quad n\mathrm{E}(Y)-5=0$

$\Rightarrow \quad 0.625n=5$

$\Rightarrow \quad n=8$

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For DRV questions, it is often useful to write out the probability distribution table. You should check that the probabilities add up to 1.

Since there are n discs, for the game to be fair, the total expected earnings from the n discs minus the cost of 1 game must be 0.

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(ii) Pay \$2 for 3 rolls with a gain of \$10 implies that the player needs to receive \$12 from 3 rolls.

Required probability

$= 3!\times P\left(Y_1=0,Y_2=2,Y_3=10\right)+3\times P\left(Y_1=1,Y_2=1,Y_3=10\right)$
$\quad +3\times P\left(Y_1=2,Y_2=5,Y_3=5\right)$

$= 3!\times\left(\frac{3}{4}\right)\left(\frac{3}{40}\right)\left(\frac{1}{50}\right)+3\times\left(\frac{1}{8}\right)^2\left(\frac{1}{50}\right)+3\times\left(\frac{3}{40}\right)\left(\frac{3}{100}\right)^2$

$= 0.0789$ (exact)

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There are 3 cases to gain a total \$12 from 3 discs, and students should consider the order of appearance of the different scores.

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Marker's comments

Common Mistakes:

Part (i):

1. Many students were unable to understand the statement "**Given that** a disc falls within a square…". They should realise that it is a conditional probability, which can be easily visualised using a tree diagram.
2. It is incorrect to subtract \$5 off from the score of each disc directly. This is because it implies that every disc thrown costs \$5, which is incorrect since *n* discs costs \$5 (fixed).
3. In general, we say that $\mathrm{E}(X)=0$ when a game is fair, instead of $\mathrm{E}(X)\geq 0$ which many students wrote. $\mathrm{E}(X)\geq 0$ in this question implies that the player is expected to win more than \$5, which is unfair for the game stall owner.

Part (ii):

4. Many students failed to consider the order of appearance of the scores. Many were also unable to consider all cases which could lead to \$12. Students who did not realise that the player must receive \$12 should read the question carefully.

7 A factory manufactures large number of pen refills. From past records, 3% of the refills are defective.

A stationery store manager wishes to purchase pen refills from the factory. To decide whether to accept or reject a batch of refills, the manager designs a sampling process. He takes a random sample of 25 refills. The batch is accepted if there is no defective refill and rejected if there are more than 2 defective refills. Otherwise, a second random sample of 25 refills is taken. The batch is then accepted if the total number of defective refills in the two samples is fewer than 4 and rejected otherwise.

(i) Find the probability of accepting a batch. [4]

(ii) If a batch is accepted, find the probability that there are 2 defective refills found in the sampling process. [3]

The stationery store manager purchases 50 boxes of 25 refills each.

(iii) Find the probability that the mean number of defective refills in a box is less than 1. [2]

Let X be the number of defective refills in the sample of 25 refills drawn from a batch which contains 3% defective refills. Then, $X \sim \mathrm{B}(25, 0.03)$ (i) P(accepting a batch) $= \mathrm{P}(X=0)+\mathrm{P}(X=1)\mathrm{P}(X \le 2)+\mathrm{P}(X=2)\mathrm{P}(X \le 1)$ = 0.4669747053 + 0.3473570958 + 0.1109593034 ≈ 0.9252911 = 0.925 (correct to 3 s.f.)	The cases in which the batch can be accepted should be thought through carefully.
(ii) Required probability = P(2 defective refills | batch is accepted) $= \dfrac{\mathrm{P}(X_1=1)\mathrm{P}(X_2=1)+\mathrm{P}(X_1=2)\mathrm{P}(X_2=0)}{0.9252911}$ =0.209 (correct to 3 s.f.)	The question is asking for the conditional probability of having 2 defective refills given that the batch is accepted.

(iii) $X \sim \mathrm{B}\ (25,\ 0.03)$ Since *sample size* = 50 is large, by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(25(0.03), \frac{25(0.03)(0.97)}{50}\right)$ approximately $\bar{X} \sim \mathrm{N}(0.75, 0.1455)$ Required probability $= P\left(\bar{X} < 1\right) = 0.981$ (correct to 3 s.f.) **Alternative solution** $X_1 + \ldots + X_{50} \sim \mathrm{B}(50 \times 25, 0.03)$ $X_1 + \ldots + X_{50} \sim \mathrm{B}(1250, 0.03)$ $\mathrm{P}(X_1 + \ldots + X_{50} < 50)$ $= \mathrm{P}(X_1 + \ldots + X_{50} \le 49)$ $= 0.973$	In general, when the question asks for a mean number of *X* when *X* is a discrete random variable, students should consider applying Central Limit Theorem. This is especially so if the question has keywords such as "approximate/estimate the probability".

Marker's comments

Part (i):

1. Quite a large number of students did not understand the first line and hence did not realise that the number of defective refills follow a Binomial Distribution. This leads to an attempt to list out all the cases manually. While computing the individual cases, most students using this approach did not consider the order of appearance of the "defective" refills (as per Binomial formula).
2. For students who considered the Binomial Distribution, many did not understand the selection process if a second batch is required. Many took the question at face value, i.e. $\mathrm{P}(1 \le X_1 \le 2) \cdot \mathrm{P}(X_1 + X_2 < 4)$. Students need to realise that the number of defects in the first sample affects the allowable number of defects in the second sample.

Part (ii):

3. Apart from not realising that the question is asking for the conditional probability, many students were unable to identify the cases of having P(2 defective refills $\cap$ batch is accepted). They either forgot that we can have $\mathrm{P}\left(X_1 = 2\right)\mathrm{P}\left(X_2 = 0\right)$, or thought that $\mathrm{P}\left(X_1 = 0\right)\mathrm{P}\left(X_2 = 2\right)$ was possible. The latter is not possible because if $\mathrm{P}(X_1 = 0)$, there the sample would have been accepted immediately and a second sample would not be taken.

Part (iii):

4. The computation of the parameters for $\overline{X}$ was poorly done. There was a lot of confusion about what *n* is. In this case, $X \sim \mathrm{B}\ (25,\ 0.03)$ and we have 50 samples. Hence $X_1 + \ldots + X_{50} \sim N(50 \times 25 \times 0.03, 50 \times 25 \times 0.03 \times 0.97)$ approx. by CLT, and hence $\overline{X} = \dfrac{X_1 + \ldots + X_{50}}{50} \sim N(25 \times 0.03, \dfrac{25 \times 0.03 \times 0.97}{50})$.

8 A study is done to find out the relationship between the age of women and the steroid levels in the blood plasma. Sample data collected from 10 females with ages ranging from 8 years old to 35 years old is as shown below.

Age (years) x	8	11	14	17	20	23	26	29	32	35
Steroid Level (mmol/litre) L	4.2	11.1	16.3	19.0	25.5	26.2	24.1	33.5	20.8	17.4

(i) Give a sketch of the scatter diagram for the data. Identify the outlier and suggest a reason, in the context of the question, why this data pair is an outlier. [3]

For the remaining part of the question, the outlier is to be removed from the calculation.

(ii) Comment on the suitability of each of the following models. Hence determine the best model for predicting the steroid level of a female based on her age.

Model A: $L = a + b\ln x$

Model B: $L = c + d\left(x - 25\right)^2$

Model C: $L = e + f\left(x - 25\right)^4$

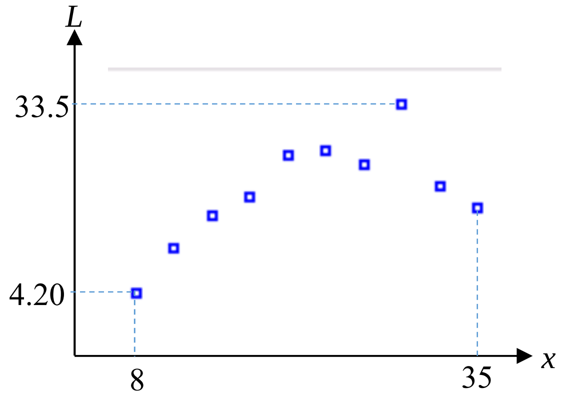
where a, b, c, d, e and f are constants. [3]

(iii) Using the best model in **(ii)**, estimate the steroid level of a woman at age 40. Comment on the reliability of your estimate. [3]

(iv) It is known that body muscle mass and steroid level has a linear correlation. The muscle mass percentage m % of the 9 females were measured. An additional female, Jane, participated in the study. Jane has her muscle mass percentage and steroid level measured. The mean muscle mass percentage of the 10 females is now found to be 26.28 %. The equation of the least squares regression line of m on L for the 10 pairs of data is

$$m = 2.22 + 1.25L.$$

Calculate Jane's steroid level. [3]

(i)  Outlier is x = 29, L = 33.5 because from age 23 onwards, there is a decreasing steroid level as age increases. However, at x = 29, the steroid level suddenly increases and this could be due to reasons such as illness/medication/pregnancy/intake of additional steroids by athlete/..etc (give any one of these reasons)	Students need to take note of what to indicate on scatter diagram: - Label of axes - Spacing and different in “height” between data points - Label min and max values The identification of outlier cannot be just circling of the point. Student must clearly states the x and L value of the outlier. Explanation of why (29, 33.5) is an outlier must be provided with a suggested possible contextual reason as well as an explanation of the kind of data trend that is resulted from this reason.
(ii) Model A is not suitable because as x increases, L is either only increasing (if $b > 0$) or only decreasing (if $b < 0$) which does not resemble the data trend of x and L whereby L increases but when it reaches about 23 years old, the steroid level decreases. Model B and C have similar trend as the given data set when $d < 0$ and $f < 0$ respectively and both are suitable models. But for Model C, the r-value of L and $(x-25)^4$ is -0.929 And for model B, the r-value of L and $(x-25)^2$ is -0.987 Since for model B, the r value is closer to -1, therefore model B is a better model.	Suitability of model must take into consideration the difference between the model and the data trend, using appropriate (sign) of b, d and f. As it is not possible to gauge the steepness of gradient base on the data points in the scatter diagram, thus the use of steepness to decide on whether model B or C is better is not accepted. Calculation of r must omit the outlier.

(iii) Least squares regression line equation is $$L = 25.238 - 0.073667(x-25)^2$$ $$L = 25.2 - 0.0737(x-25)^2$$ When $x = 40$, $L = 25.2 - 0.0737(40-25)^2 \approx 8.7$ The prediction is unreliable because $x = 40$ is outside the data range of 8 to 35 years old.	Calculation of equation of regression line must omit the outlier. Student must note that they cannot use the command $Y_1(40)$ directly to find L as the GC treat $(x-25)^2$ as X. Answer should be left to 1 decimal place, following that in the given table of L values.
(iv) Since $\overline{m} = 2.22 + 1.25\overline{L}$, $$26.28 = 2.22 + 1.25\overline{L}$$ $$\Rightarrow \overline{L} = 19.248$$ $\sum_{i=1}^{10} L_i = 192.48$ and since $\sum_{i=1}^{9} L_i = 164.6$, therefore Jane's steroid level is $27.88 \approx 27.9$	Students must realise that even if they are able to find the value of Jane's muscle mass t be 37.07, they cannot substitute this value into the equation to find Jane's steroid level. So even if the answer obtained is also 27.9, they are wrongly assuming that the data point lies on the regression line. Only $(\overline{L}, \overline{m})$ lies on the regression line.

Marker's comments

(i) The scatter diagram is quite well drawn but the labelling of axes, minimum and maximum values of x and L are often left out or wrongly labelled. Many students are mainly describing the data trend and the high L level of the point (29. 33.5) and did not give a contextual reason on why the data trend is as such. On the other hand, another group of students gave a very brief contextual reason but did not provide any elaborate on what this reason led to.

(ii) Many students did not explore the different possible sign of b, d and f. Most students wrongly assume that b, d and f are positive and rejected model B and C. One serious mistake that some students made is that they associate the power n in the expression $\left(x-25\right)^n$ to the number of turning points that the graph has, not realising that for all even integer n, there is only one turning point. Many students also did not read the question instruction to comment on the suitability of **each** model, they mainly compute values of r for all 3 models and conclude the one best model.

(iii) Many students left the estimated value of L to 3 s.f. instead of 1 decimal place. Some forgot to write down the equation of the regression line. Some wrongly write the equation as $L = 25.2 - 0.0737x^2$. Many forgot to omit the outlier in both part (iii) and in (ii) when finding r value.

Although most students are able to answer this part correctly, their answers are rather vague. Phrasing such as "it is an extrapolation" or "It is within data range" is not acceptable as it is unclear whether the student is referring to x or L within data range. Students must also remember to answer the question using the given term "not reliable" instead of "not accurate".

(iv) This part is generally well done but the notation of $\overline{L}$ is often not used, many just write it as L even if their subsequent workings show that they know that the value 19.248 is the mean steroid level.

9 A flange beam is a steel beam with a "H"-shaped cross section, and is used as a supporting structure in construction and civil engineering. A factory manufactures both Grade X and Grade Y flange beams. The load that can be supported by a Grade X flange beam follows a normal distribution with mean 2.43×10^5 kN and standard deviation 4.5×10^4 kN. The load that can be supported by a Grade Y flange beam is 1.5 times of the load that can be supported by a Grade X flange beam.

(i) Find the probability that the combined load that can be supported by two randomly chosen Grade Y flange beams is within 1×10^4 kN of the combined load that can be supported by three randomly chosen Grade X flange beams. [4]

(ii) A construction company wants to buy 100 sets of three Grade X flange beams. Find the probability that fewer than 95 of these sets can support more than 6×10^5 kN. [3]

The company decides to place an order with the factory for a custom-made flange beam such that the probability of being able to support a load of at least 6×10^5 kN must be at least 0.999.

It is assumed that the load that can be supported by the custom-made flange beam also follows a normal distribution.

(iii) By taking the standard deviation of a custom-made flange beam to be 3×10^4 kN, find the smallest possible mean load in kN, giving your answer correct to the nearest thousand, for the factory to meet the company's requirements for the custom-made flange beam. [5]

Let A and B be the load that can be supported (in kN) by a Grade X and Grade Y flange beam respectively. Then, $A \sim N\left(2.43\times10^5, \left(4.5\times10^4\right)^2\right)$.	Many students did not define the random variables. Some defined it wrongly and just wrote it as "Let A be the Grade X and B be the Grade Y." There are some who took $\left(4.5\times10^4\right)$ as the variance of A.
Since $B=1.5A$, then $B \sim N\left(1.5\left(2.43\times10^5\right), 1.5^2\left(4.5\times10^4\right)^2\right)$ i.e., $B \sim N\left(3.645\times10^5, \left(6.75\times10^4\right)^2\right)$ leave in exact decimal!	Common mistakes for Var(B): 1. Var(B) = $\left(4.5\times10^4\right)^2$ 2. Var(B) = $1.5\left(4.5\times10^4\right)^2$
(i) Want to find $P\left(\left\|\left(B_1+B_2\right)-\left(A_1+A_2+A_3\right)\right\|<1\times10^4\right)$ $E\left(\left(B_1+B_2\right)-\left(A_1+A_2+A_3\right)\right)$ $=2\times3.645\times10^5-3\times2.43\times10^5=0$ $\mathrm{Var}\left(\left(B_1+B_2\right)-\left(A_1+A_2+A_3\right)\right)$ $=2\times\left(6.75\times10^4\right)^2+3\left(4.5\times10^4\right)^2=1.51875\times10^{10}$	There are still students who wrote $2B-3A$ instead of $\left(B_1+B_2\right)-\left(A_1+A_2+A_3\right)$. Students are advised not to correct their working answer to 3 s.f especially if the exact decimal answer is obtained. For e.g. in this case, if students round of their answer for E(B) to 3.65×10^5, their answer for $E\left(\left(B_1+B_2\right)-\left(A_1+A_2+A_3\right)\right)$ is 1000 instead of 0.
i.e. $\left(B_1+B_2\right)-\left(A_1+A_2+A_3\right)\sim N\left(0, 1.51875\times10^{10}\right)$	Must always write down the distribution after finding expectation and variance!
Required probability $=P\left(\left\|\left(B_1+B_2\right)-\left(A_1+A_2+A_3\right)\right\|<1\times10^4\right)$ $=P\left(-1\times10^4<\left(B_1+B_2\right)-\left(A_1+A_2+A_3\right)<1\times10^4\right)$ = 0.0647 (to 3 s.f.)	Some students do not understand what is meant by A within 1×10^4 kN of B. *Note:* In general, $P\left(\left\|T\right\|<1\times10^4\right)\neq P\left(T<1\times10^4\right)+P\left(T>-1\times10^4\right)$

(ii) $A_1 + A_2 + A_3 \sim N\left(3\times 2.43\times 10^5, 3\left(4.5\times 10^4\right)^2\right)$ $\mathrm{P}\left(A_1 + A_2 + A_3 > 6\times 10^5\right) \approx 0.951045$ Let T be the number of sets (out of 100 sets) of three Grade X flange beams that can support more than $6\times 10^5\,\text{kN}$. Then, $T \sim \mathrm{B}(100,\ 0.951045)$ Required probability $= \mathrm{P}\left(T < 95\right)$ $= \mathrm{P}\left(T \le 94\right)$ $= 0.365$ (to 3 s.f.)	Some students missed out on the three in the "100 sets of three Grade X flange beams" and went to find $\mathrm{P}\left(A > 6\times 10^5\right)$ Did not change $\mathrm{P}\left(T < 95\right)$ to $\mathrm{P}\left(T \le 94\right)$ when using binomcdf
(iii) Let W be the load that can be supported (in kN) by a custom-made flange beam. Given: $W \sim N\left(\mu,\ \left(3\times 10^4\right)^2\right)$ $\mathrm{P}(W \ge 6\times 10^5) \ge 0.999$ $\Rightarrow$ $1 - \mathrm{P}(W < 6\times 10^5) \ge 0.999$ $\Rightarrow$ $\mathrm{P}(W < 6\times 10^5) \le 0.001$ $\Rightarrow$ $\mathrm{P}\left(Z \le \dfrac{6\times 10^5 - \mu}{3\times 10^4}\right) \le 0.001$ By GC, $\dfrac{6\times 10^5 - \mu}{3\times 10^4} \le -3.0902$ $6\times 10^5 - \mu \le -92760$ $-\mu \le -92760 - 6\times 10^5$ $\mu \ge 692760$ Thus, smallest mean = 693 kN (to nearest thousand)	Some denote the smallest possible mean as $\bar{x}$ which is incorrect as $\bar{x}$ always denote sample mean. A few standardised wrongly. Wrote $\dfrac{\mu - 6\times 10^5}{3\times 10^4}$. *Common mistake:* Took invNorm of the right area which is 0.999 instead of left area, 0.001 Some started off with inequality but ended up with an equation after taking invNorm.

Marker's comments

- Students are advised not to use Z to denote the random variable as Z denotes the Standard Normal Variable, i.e $Z \sim \mathrm{N}(0,\ 1)$
- Students are advised to use exact decimal workings answer or working answers with more decimal places to avoid loss of accuracy in their final answer.
- There are at least a few in each class who do not know how to correct their answer to the nearest thousand.
- For part(iii), students are advised not to use GC Table although the unknown is to be corrected to the nearest thousand. Students who use GC table but did not show their workings clearly do not get the full marks.

10(a) College students intending to further their studies overseas have to sit for a mandatory Overseas Universities Test (OUT). Researcher Mr Anand wishes to find out if male college students tend to score higher for OUT compared to female college students. Mr Anand's colleague randomly selects 150 male and 150 female students from the combined student population of three particular colleges near his home to form a sample of 300 college students for the research. Explain whether this sample is a random sample.[2]

(b) The mean OUT score for all college students in 2016 is 66.

Mr Anand randomly selects 240 college students taking OUT in 2017 and their scores, x, are summarised in the following table:

Score, x	60	65	68	70	75	80
Frequency, f	40	90	63	27	18	2

(i) Write down the unbiased estimates of the population mean and variance of the OUT scores for the college students in 2017. [1]

(ii) Test, at the 10% level of significance, whether the mean OUT score for all college students in 2017 is higher than the mean score attained in 2016. [4]

(iii) Explain what is meant by the phrase "10% level of significance" in this context.[1]

(iv) Mr Anand draws a new sample of 240 male college students. Using the unbiased estimate for the population variance computed in **(i)**, find the range of values for the sample mean $\overline{x}$ that is required for this new sample to achieve a different conclusion from that in **(ii)**. [4]

(c) The 2017 OUT scores of the male and female college students are independent and assumed to be normally distributed with means and standard deviations as shown in the following table:

	Mean	Standard deviation
Male College Students	64	5.5
Female College Students	66	3.5

Mr Beng and Miss Charlene both scored 70. Explain who performed better relative to their respective gender cohort. [2]

(a) Sample is non-random/biased since students from other colleges do not have any chance of being selected.	Need to mention that the probability of a student being selected into the sample is not the same for every student taking OUT in 2017 since students from other colleges do not have any chance of being selected.
(b)(i) Using GC, unbiased estimate of population mean, $\overline{x} = 66.391$ $= 66.4$ (to 3 s.f.) and unbiased estimate of population variance, $s^2 = 4.1048^2 = 16.8$ (to 3 s.f.)	A number of students forgot to square the value 4.1048.
(b)(ii) Let μ be the population mean OUT score of students in 2017 . $\mathrm{H}_0 : \mu = 66$ $\mathrm{H}_1 : \mu > 66$ Level of significance: 10% Test Statistic: $\dfrac{\overline{X} - \mu}{s/\sqrt{n}} \sim \mathrm{N}(0, 1)$ by Central Limit Theorem since $n = 240$ is large. Under H_0, with $\overline{x} = 66.391$, $s = 4.1048$, $n = 240$, we have $p = 0.0697$ Since p-value < 0.1, we reject H_0 There is sufficient evidence at the 10% level of significance to conclude that the mean OUT score of male college students is higher than 66.	
(iii) There is a probability of 0.1 of wrongly concluding that the mean OUT score of male college students is higher than 66.	10% chance or probability of 0.1

(iv) $H_0 : \mu = 66$ $H_1 : \mu > 66$ Level of significance: 10% Do not reject H_0, $p > 0.10$ $$\frac{\overline{x} - 66}{4.1048 / \sqrt{240}} < 1.28155$$ $\Rightarrow \quad \overline{x} > 66.3396$ $\therefore \quad \overline{x} > 66.3$ (to 3 s.f.)	A number of students wrote the critical value as -1.28155 (invnorm(0.10)), without paying attention to H_1.
(c) Let M and F be the OUT scores of male and female college students respectively Given: $M \sim \mathrm{N}(64, 5.5^2)$ and $F \sim \mathrm{N}(66, 3.5^2)$ $\mathrm{P}(M \le 70) = 0.86234$ $\Rightarrow$ Mr Beng is in the 86$^{\text{th}}$ percentile of male students (or Mr Beng scored higher than 86% of the male cohort) $\mathrm{P}(F \le 70) = 0.87345$ $\Rightarrow$ Miss Charlene is in the 87$^{\text{th}}$ percentile of female students $\therefore$ Miss Charlene performed better relative to her gender cohort.	

Marker's comments

(a) Many students were able to recognise that the sample is not random. However, many of them were not able to give precise explanation. Wrong responses included mentioning the proportion of males and females, abilities of students in colleges which were not apparent in the question.

(b) (i) This part requires students to write out the unbiased estimates of the population mean and variance upon entering the data into the GC. Many students were not able to retrieve the correct unbiased estimate of population variance, they wrote down the sample variance instead. A number of students applied the formulas to find the unbiased estimates using the statistics, some with more success obtaining the values, some used the wrong statistics or wrong formula and did not obtain the correct values.

(b) (ii) Most students gained full marks here. Those who did not get (b)(i) correct would have lost some marks but not all if they have written the correct hypotheses, and conclusion given in context.

(b) (iii) This part was badly done. Many students were not able to explain precisely the phrase "10% level of significance", some students seemed to have problem remembering the definition.

(b) (iv) Students have some grasp of what was required, there were many varied errors in setting up the inequality to achieve a different conclusion from (b)(ii).

(c) Many students attempted to answer this part with lengthy paragraphs about standard deviations of the distribution of OUT scores of the male and female students. Many failed to explain using percentiles or probabilities of Beng and Charlene scoring 70 marks and above/or below. A large number of students thought they were computing the probability of Beng/Charlene scoring 70 marks using the normalpdf function. They did not understand that the probability is defined as area under the normal curve and hence the value they obtained were not able to explain who performed better in their cohort.