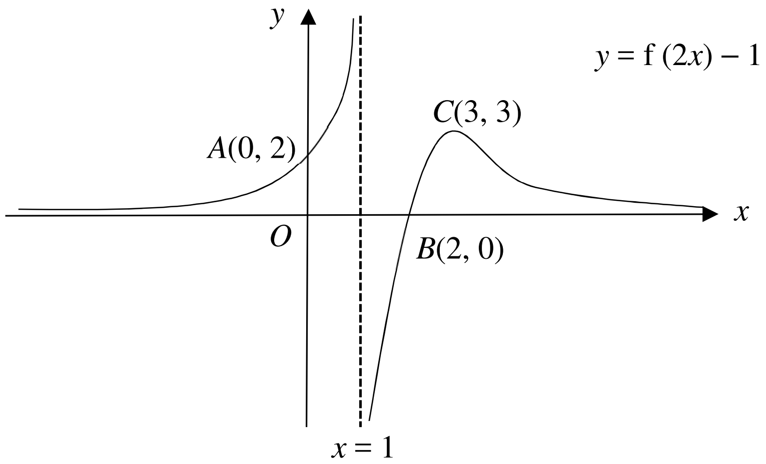
H2 Mathematics 2017 Preliminary Exam Paper 1 Question (9758)
Answer all questions [100 marks]

1



The diagram shows the curve $y = \mathrm{f}(2x) - 1$ with a maximum point at $C(3, 3)$. The curve crosses the axes at the points $A(0, 2)$ and $B(2, 0)$. The line $x = 1$ and the x-axis are the asymptotes of the curve.

On separate diagrams, sketch the graphs of

(i) $y = \mathrm{f}(x)$, [2]

(ii) $y = \mathrm{f}'(x)$, [2]

stating clearly the equations of the asymptotes and the coordinates of the points corresponding to A, B and C where appropriate.

2 **(i)** Without using a calculator, solve the inequality $\dfrac{x}{x^2 - 5} \le 0$, giving your answer in exact form. [2]

(ii) Hence, find the set of values of x for which $\dfrac{\sqrt{x}}{x - 5} \le 0$. [2]

3 Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P on OA is such that $OP : PA = 2 : 3$, and the point Q on OB is such that $OQ : QB = 1 : 2$. Given that M is the mid-point of PQ, state the position vector of M in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

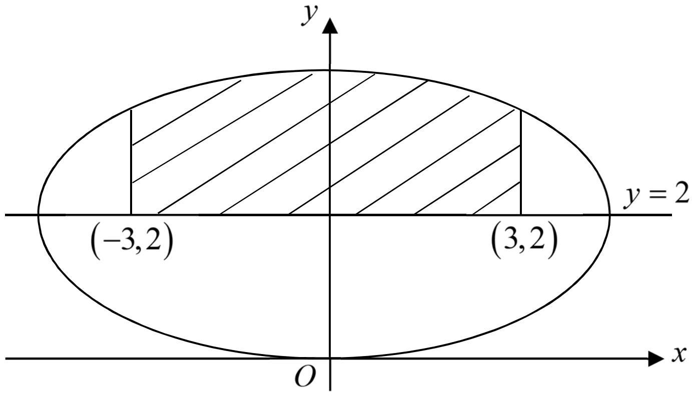
Show that the area of triangle OMP can be written as $k\left|\mathbf{a} \times \mathbf{b}\right|$, where k is a constant to be determined. [4]

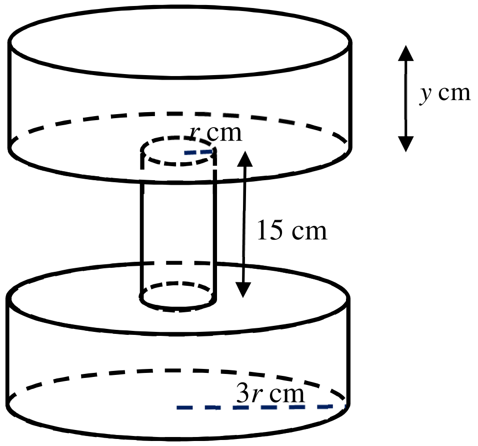
4 Find

(a) $\displaystyle\int \cos(\ln x)\,\mathrm{d}x$, [3]

(b) $\displaystyle\int \frac{1 - 2x}{2x^2 + 1}\,\mathrm{d}x$. [3]

5	It is given that $z=\sqrt{3}+\mathrm{i}$ and $w=-1+\mathrm{i}$. **(i)** Without using a calculator, find an exact expression for $\dfrac{z^2}{w^*}$. Give your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [4] **(ii)** Find the exact value of the real number q such that $\arg\left(1-\dfrac{q}{z}\right)=\dfrac{\pi}{12}$. [3]
6	It is given that $y=\ln\left(3+\mathrm{e}^{x}\right)$. **(i)** Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [3] **(ii)** By differentiating the above result, find the first four non-zero terms of the Maclaurin series for y. Give the coefficients in exact form. [3] **(iii)** Hence find the Maclaurin series for $\dfrac{\mathrm{e}^{-2x}}{3+\mathrm{e}^{-2x}}$, up to and including the term in x^2. [2]
7	The curve C has equation $y=\dfrac{ax^2+bx-8}{x-2}$, where a and b are constants. It is given that C has asymptote $y=3-2x$. **(i)** Find the value of a and show that $b=7$. [3] **(ii)** Sketch C, stating clearly the equations of any asymptotes and the coordinates of any stationary points and any points of intersection with the axes. [3] **(iii)** By drawing another suitable curve on the same diagram, deduce the number of real roots of the equation $\left(-2x^2+7x-8\right)^2-25\left(x-2\right)^3=0$. [3]
8	Emily has 1016 toy bricks. **(i)** Emily wishes to build a brick structure with one brick in the first row, two bricks in the second row, three bricks in the third row and so on. What is the maximum number of rows that she can build and how many bricks will be left unused? [4] **(ii)** Emily keeps all her 1016 bricks in $(2k-1)$ bags of different sizes. She packs m bricks into the smallest bag. For each subsequent bag, she packs double the number of bricks she packs in the previous bag. Given that she has 64 bricks in the kth bag, find the value of m and the number of bags. [5]

9	**(a)**	By using the substitution $x = 3\sec\theta$, evaluate $\displaystyle\int_{3\sqrt{2}}^{6} \frac{3x+1}{\sqrt{x^2-9}}\,dx$ exactly. [5]
	(b)	 The diagram shows an ellipse with equation $\dfrac{x^2}{16}+\dfrac{(y-2)^2}{4}=1$. **(i)** Find the area of the shaded region, giving your answer correct to 3 decimal places. [2] **(ii)** Find the exact volume of the solid generated when the shaded region is rotated 180° about the y-axis. [4]
10	**(a)**	By using the substitution $z = x - y$, solve the differential equation $\dfrac{dy}{dx}=\dfrac{x-y-1}{x-y+1}$. Find the particular solution for which $y = 1$ when $x = 1$. [4]
	(b)	A sky diver jumped out of an aeroplane over a certain mountainous valley with zero speed and t seconds later, the speed of his descent was v metres per second. He experienced gravitational force and air resistance which affect v. Gravity would increase his speed by a constant 10 metres per second^2 and the air resistance would decrease his speed at a rate proportional to the square of his speed. It is given that when his speed reaches 50 metres per second, the rate of change of his speed is 7.5 metres per second^2. By setting up and solving a differential equation, show that $v=\dfrac{100\left(1-e^{-mt}\right)}{1+e^{-mt}}$, where m is a constant to be found. [7] Describe briefly what his speed would be after he had descended for a long time and just before he deployed his parachute. [1]
11		



A plastic water dumbbell consists of a cylinder as a handle and two cylinders as the weights. The handle has a radius r cm and height 15 cm. Each weight has radius $3r$ cm and height y cm. The dumbbell is made of plastic of negligible thickness and the volume of the dumbbell is a fixed value k cm^3.

(i) Given that $r = r_1$ is the value of r which gives the minimum external surface area, show that r_1 satisfies the equation $102\pi r^3 + 30\pi r^2 - k = 0$. [6]

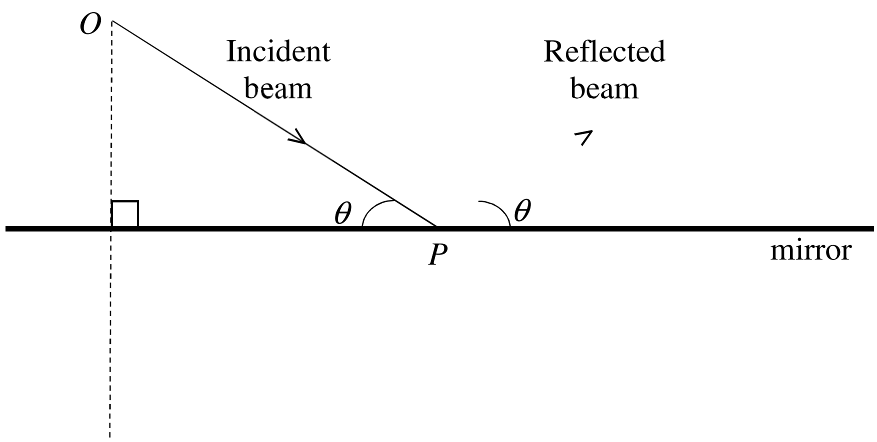
(ii) Find the value of r_1 if $k = 450$. [1]

(iii) It is given instead that $r = 2$ and $y = 7$. Water is pumped into an empty dumbbell through an opening from the top at a rate of 15 cm^{3}s^{-1}. Find the exact rate at which the depth of the water is increasing after 1 minute. [5]

[Question 12 is printed on the next page.]

12	Laser (Light Amplification by Stimulated Emission of Radiation) has many applications including medicine, data storage, military and industrial uses. It has the property of spatial coherence, which allows the laser beam to stay narrow over long distances. When a laser beam is projected onto a mirror at an angle, it reflects off the mirror at the same angle.

An engineer is designing a device that does industrial cutting using a laser beam. To make the device compact, the device has a mirror to reflect the beam before it leaves the device. The laser beam source is located at the origin O. It projects an incident beam with direction vector $\mathbf{i}+2\mathbf{j}+\mathbf{k}$. The beam hits the mirror at the point P with angle θ. The mirror has an equation $-x+2y+3z=12$.



(i) Find the acute angle θ that the beam makes with the mirror. [2]

(ii) By finding O', the image of O in the mirror, find a vector equation of the line that the reflected beam is on. [7]

(iii) The engineer plans to install a sensor at (3, 1, 0) to monitor the heat produced by the laser. For the sensor to work properly, the sensor must be less than 2 units away from either the incident or the reflected beam. Determine if the sensor will work properly. [4]

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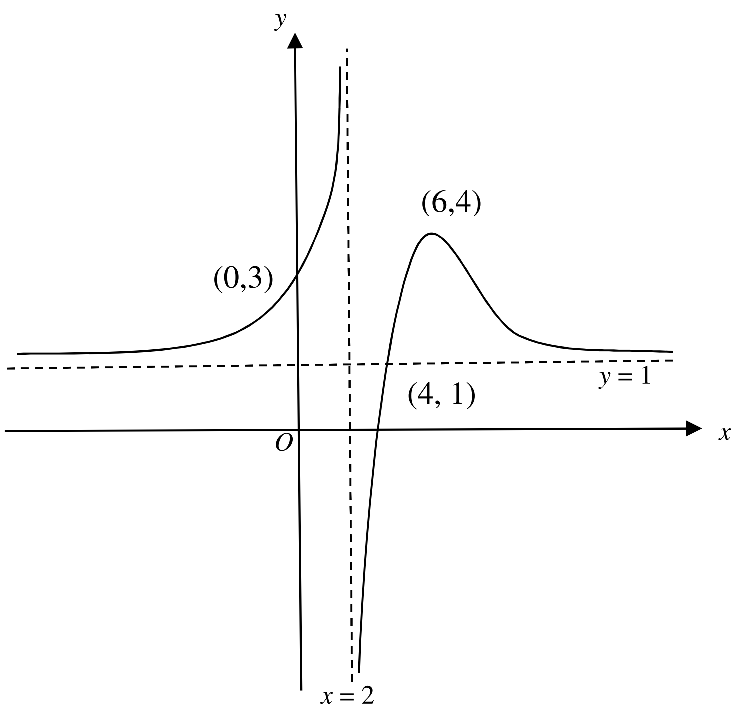
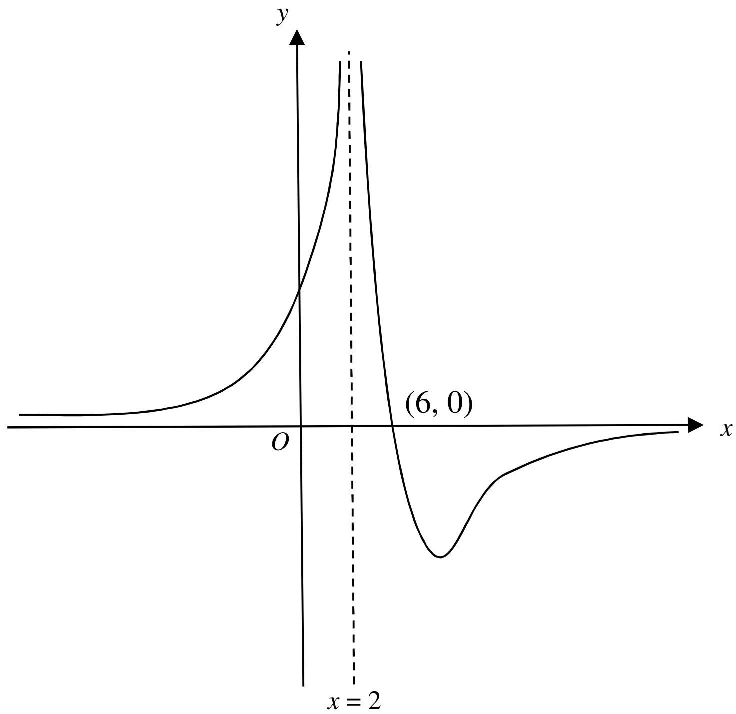
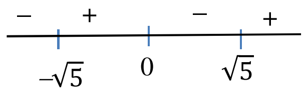
YISHUN JUNIOR COLLEG5E

Mathematics Department

PRELIM Solutions

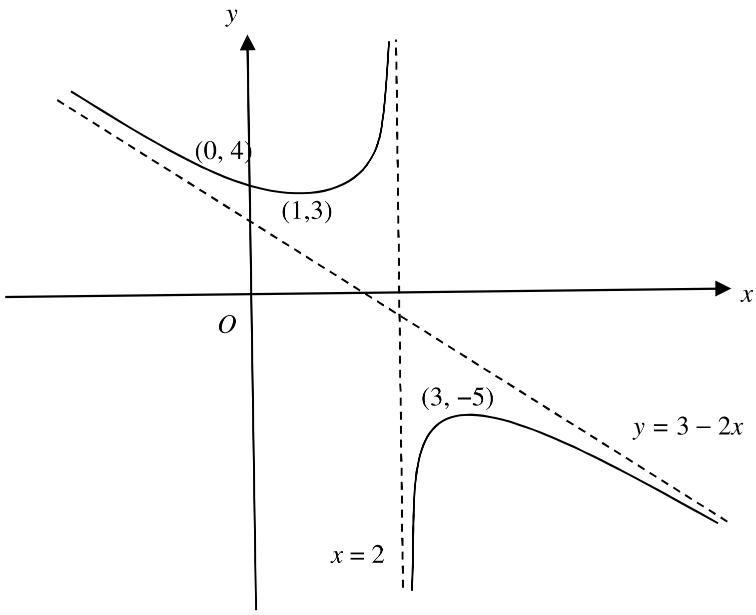
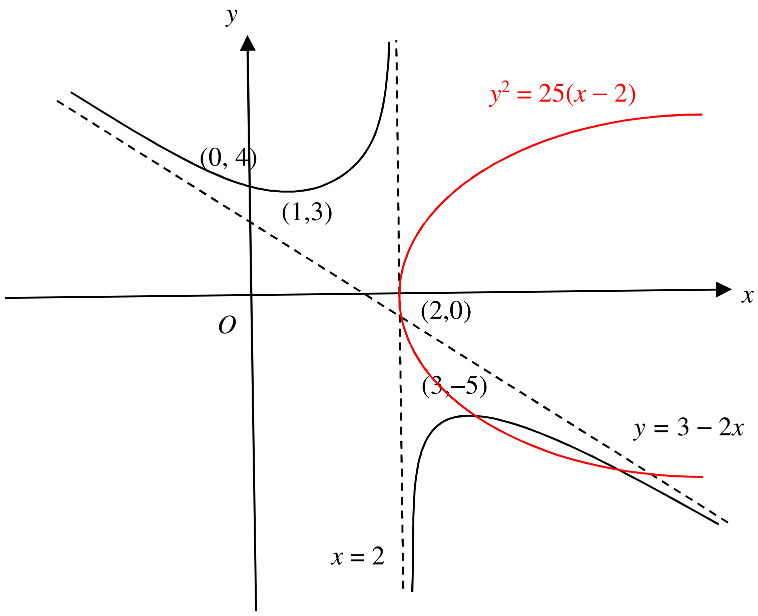
Subject : JC2 H2 MATHEMATICS 9758/9740 P1 **Date** :

Qn	Solution
1(i)	
(ii)	

Qn	Solution
2(i)	$\frac{x}{x^2-5} \le 0$ $\frac{x}{\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)} \le 0$  $\therefore x < -\sqrt{5}$ or $0 \le x < \sqrt{5}$
(ii)	$\frac{\sqrt{x}}{x-5} \le 0$ $\frac{\sqrt{x}}{\left(\sqrt{x}\right)^2-5} \le 0$ Replace x by $\sqrt{x}$ in the result from (i), $\sqrt{x} < -\sqrt{5}$ or $0 \le \sqrt{x} < \sqrt{5}$ (Reject $\because \sqrt{x} \ge 0$) or $0 \le x < 5$ Required set = $\left\{x \in \mathbb{R}: \ 0 \le x < 5\right\}$
3	$\overrightarrow{OP} = \frac{2}{5}\mathbf{a}$ $\overrightarrow{OQ} = \frac{1}{3}\mathbf{b}$ $\overrightarrow{OM} = \frac{1}{2}\left(\frac{2}{5}\mathbf{a} + \frac{1}{3}\mathbf{b}\right)$ Area of triangle *OMP* $= \frac{1}{2}\left\|\left(\frac{1}{2}\left(\frac{2}{5}\mathbf{a} + \frac{1}{3}\mathbf{b}\right)\right) \times \frac{2}{5}\mathbf{a}\right\|$ $= \frac{1}{2}\left\|\left(\left(\frac{1}{5}\mathbf{a} + \frac{1}{6}\mathbf{b}\right)\right) \times \frac{2}{5}\mathbf{a}\right\|$ $= \frac{1}{2}\left\|\frac{2}{25}\mathbf{a} \times \mathbf{a} + \frac{1}{15}\mathbf{b} \times \mathbf{a}\right\|$ $= \frac{1}{2}\left\|\frac{1}{15}\mathbf{b} \times \mathbf{a}\right\|$ $= \frac{1}{30}\left\|-\mathbf{a} \times \mathbf{b}\right\|$ $= \frac{1}{30}\left\|\mathbf{a} \times \mathbf{b}\right\|$

Qn	Solution
4(a)	$\int \cos(\ln x)\ \mathrm{d}x = x\cos(\ln x) - \int -x\sin(\ln x).\frac{1}{x}\ \mathrm{d}x$ $= x\cos(\ln x) + \int \sin(\ln x)\ \mathrm{d}x$ $= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\ \mathrm{d}x$ $2\int \cos(\ln x)\ \mathrm{d}x = x\cos(\ln x) + x\sin(\ln x) + \text{constant}$ $\int \cos(\ln x)\ \mathrm{d}x = \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right] + C$
(b)	$\int \frac{1-2x}{2x^2+1}\ \mathrm{d}x$ $= \frac{1}{2}\int \frac{1}{x^2+\frac{1}{2}}\ \mathrm{d}x - \frac{1}{2}\int \frac{4x}{2x^2+1}\ \mathrm{d}x$ $= \frac{\sqrt{2}}{2}\tan^{-1}\sqrt{2}x - \frac{1}{2}\ln\left(2x^2+1\right) + c$
5(i)	$\lvert z\rvert = \lvert\sqrt{3}+\mathrm{i}\rvert = \sqrt{3+1} = 2$, $\lvert w\rvert = \lvert -1+\mathrm{i}\rvert = \sqrt{1+1} = \sqrt{2}$ $\arg(z) = \arg(\sqrt{3}+\mathrm{i})$ $= \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ $\arg(w) = \arg(-1+\mathrm{i})$ $= \pi - \tan^{-1}1 = \frac{3\pi}{4}$ $\frac{z^2}{w^*} = \frac{\left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^2}{\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}}$ $=2^{\frac{3}{2}}\mathrm{e}^{\mathrm{i}\frac{13\pi}{12}}$ $=2^{\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\frac{11\pi}{12}}$
(ii)	$\arg\left(1-\frac{q}{z}\right) = \arg\left(\frac{z-q}{z}\right)$ $= \arg(z-q) - \arg(z) = \frac{\pi}{12}$ $\arg(z-q) = \frac{\pi}{12} + \frac{\pi}{6} = \frac{\pi}{4}$ $\arg\left(\left(\sqrt{3}-q\right)+\mathrm{i}\right) = \frac{\pi}{4}$ $\sqrt{3}-q = 1 \Rightarrow q = \sqrt{3}-1$

Qn	Solution
6(i)	$y=\ln(3+\mathrm{e}^x)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{e}^x}{3+\mathrm{e}^x}$ $\left(3+\mathrm{e}^x\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(3+\mathrm{e}^x\right)+\mathrm{e}^x\dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{e}^x}{\left(3+\mathrm{e}^x\right)}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{e}^x}{\left(3+\mathrm{e}^x\right)}$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\dfrac{\mathrm{d}y}{\mathrm{d}x}$ (proved)
(ii)	$\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ When $x=0,\ y=\ln 4,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{4},\ \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{3}{16},\ \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=\dfrac{3}{32}$ $y=\ln 4+x\left(\dfrac{1}{4}\right)+\dfrac{x^2}{2!}\left(\dfrac{3}{16}\right)+\dfrac{x^3}{3!}\left(\dfrac{3}{32}\right)+\dots$ $=\ln 4+\dfrac{1}{4}x+\dfrac{3}{32}x^2+\dfrac{1}{64}x^3+\dots$
(iii)	$\dfrac{\mathrm{e}^x}{3+\mathrm{e}^x}=\dfrac{1}{4}+\dfrac{3}{16}x+\dfrac{3}{64}x^2+\dots$ $\dfrac{\mathrm{e}^{-2x}}{3+\mathrm{e}^{-2x}}=\dfrac{1}{4}+\dfrac{3}{16}(-2x)+\dfrac{3}{64}(-2x)^2+\dots$ $=\dfrac{1}{4}-\dfrac{3}{8}x+\dfrac{3}{16}x^2+\dots$
7(i)	$a=-2$ By long division, $y=(b-4)-2x+\dfrac{2b-16}{x-2}$. $b-4=3\Rightarrow b=7$ (shown)

Qn	Solution
(ii)	The equation is $y = \dfrac{-2x^2 + 7x - 8}{x - 2}$ 
(iii)	$(-2x^2 + 7x - 8)^2 - 25(x-2)^3 = 0$ $(-2x^2 + 7x - 8)^2 = 25(x-2)^3$ $\left(\dfrac{-2x^2 + 7x - 8}{x - 2}\right)^2 = 25(x-2)$ Add graph of $y^2 = 25(x-2)$  From the graphs, the number of real roots is 2.

Qn	Solution
8(i)	This is an AP with $a = 1$, $d = 1$. For $S_n \le 1016$ $\frac{n}{2}(1+n) \le 1016$ $n^2 + n - 2032 \le 0$ From GC, $-45.58 \le n \le 44.58$ She can complete a maximum of 44 rows. $S_{44} = \frac{44}{2}(1+44) = 990$ Number of bricks left = 1016 − 990 = 26
(ii)	The sequence is a GP with common ratio 2 $S_{2k-1} = 1016$ $\frac{m\left[2^{2k-1}-1\right]}{2-1} = 1016$ $m\left[2^{2k-1}-1\right] = 1016 \quad ---(1)$ $T_k = 64 \Rightarrow m2^{k-1} = 64$ ---- (2) (1) ÷ (2): $\frac{2^{2k-1}-1}{2^{k-1}} = \frac{1016}{64}$ From GC, $k = 4$ Sub. into (2): $m2^{4-1} = 64$ $\Rightarrow m = 8$ No. of bags = 2(4) − 1 = 7
9(a)	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sec\theta\tan\theta$ $\int_{3\sqrt{2}}^{6} \frac{3x+1}{\sqrt{x^2-9}}\,\mathrm{d}x$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{9\sec\theta+1}{\sqrt{9\sec^2\theta-9}}(3\sec\theta\tan\theta)\,\mathrm{d}\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{9\sec\theta+1}{3\tan\theta}(3\sec\theta\tan\theta)\,\mathrm{d}\theta$

Qn	Solution
	$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 9\sec^2\theta+\sec\theta \ \mathrm{d}\theta$ $=\left[9\tan\theta+\ln\left\|\sec\theta+\tan\theta\right\|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $=9\tan\frac{\pi}{3}+\ln\left\|\sec\frac{\pi}{3}+\tan\frac{\pi}{3}\right\|-\left(9\tan\frac{\pi}{4}+\ln\left\|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right\|\right)$ $=9\sqrt{3}+\ln\left\|2+\sqrt{3}\right\|-\left(9+\ln\left\|\sqrt{2}+1\right\|\right)$ $=9\sqrt{3}-9+\ln\frac{2+\sqrt{3}}{\sqrt{2}+1}$
(b)(i)	Consider $y=2\pm2\sqrt{1-\frac{x^2}{16}}$ Required area $=\int_{-3}^{3} 2+2\sqrt{1-\frac{x^2}{16}}\ \mathrm{d}x-2(6)$ $=10.753$ (3 dp) Alternative Consider $\frac{x^2}{16}+\frac{y^2}{4}=1\Rightarrow y=\pm2\sqrt{1-\frac{x^2}{16}}$ Required area $=\int_{-3}^{3} 2\sqrt{1-\frac{x^2}{16}}\ \mathrm{d}x$ or $4\int_{0}^{3}\sqrt{1-\frac{x^2}{16}}\ \mathrm{d}x$ $=10.753$ (3 dp)
(ii)	When $x=3$, $y=2+2\sqrt{1-\frac{9}{16}}=2+\frac{1}{2}\sqrt{7}$ When $x=0$, $y=4$ Required Volume $=\frac{\sqrt{7}}{2}\pi(3^2)+\pi\int_{2+\frac{1}{2}\sqrt{7}}^{4} 16\left(1-\frac{(y-2)^2}{4}\right)\mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\int_{2+\frac{1}{2}\sqrt{7}}^{4} 1-\frac{(y-2)^2}{4}\ \mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\left[y-\frac{(y-2)^3}{12}\right]_{2+\frac{1}{2}\sqrt{7}}^{4}$ $=18\pi+16\pi\left[4-\frac{2}{3}-2-\frac{\sqrt{7}}{2}+\frac{7\sqrt{7}}{96}\right]$ $=\frac{1}{3}\left(64-7\sqrt{7}\right)\pi$

Qn	Solution
	<u>Alternative</u> When $x=3$, $y=2\sqrt{1-\frac{9}{16}}=\frac{1}{2}\sqrt{7}$ When $x=0$, $y=2$ Required Volume $=\frac{\sqrt{7}}{2}\pi(3^2)+\pi\int_{\frac{1}{2}\sqrt{7}}^{2}16\left(1-\frac{y^2}{4}\right)\,\mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\int_{\frac{1}{2}\sqrt{7}}^{2}1-\frac{y^2}{4}\,\mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\left[y-\frac{y^3}{12}\right]_{\frac{1}{2}\sqrt{7}}^{2}$ $=\frac{9\sqrt{7}}{2}\pi+\frac{1}{6}\left[128-41\sqrt{7}\right]\pi$ $=\frac{1}{3}\left(64-7\sqrt{7}\right)\pi$
10(a)	$z=x-y\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=1-\frac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{\mathrm{d}z}{\mathrm{d}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x-y-1}{x-y+1}$. $\Rightarrow 1-\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{z-1}{z+1}$ $\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=1-\frac{z-1}{z+1}$ $\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{2}{z+1}$ $\int(z+1)\,\mathrm{d}z=\int 2\,\mathrm{d}x$ $\frac{z^2}{2}+z=2x+C$ $\frac{(x-y)^2}{2}+x-y=2x+C$ where C is a constant $\frac{(x-y)^2}{2}-x-y=C$ When $x=1$, $y=1$, $\Rightarrow C=-2$ Therefore $\frac{(x-y)^2}{2}-x-y=-2$

Qn	Solution
(b)	$\frac{\mathrm{d}v}{\mathrm{d}t}=10-kv^2$, where $k>0$ When $v=50, \frac{\mathrm{d}v}{\mathrm{d}t}=7.5$ $7.5=10-k(50)^2$ $\Rightarrow k=0.001$ $\therefore \frac{\mathrm{d}v}{\mathrm{d}t}=10-0.001v^2$ $\int \frac{1}{10-0.001v^2}\ \mathrm{d}v=\int 1\ \mathrm{d}t$ $\frac{1}{0.001}\int \frac{1}{10000-v^2}\ \mathrm{d}v=\int 1\ \mathrm{d}t$ $1000\int \frac{1}{100^2-v^2}\ \mathrm{d}v=\int 1\ \mathrm{d}t$ $\frac{1000}{200}\ln\left\|\frac{100+v}{100-v}\right\|=t+C$ $\ln\left\|\frac{100+v}{100-v}\right\|=\frac{1}{5}t+\frac{1}{5}C$ $\left\|\frac{100+v}{100-v}\right\|=\mathrm{e}^{\frac{1}{5}t+\frac{1}{5}C}$ $\frac{100+v}{100-v}=\pm\mathrm{e}^{\frac{1}{5}t+\frac{1}{5}C}$ $\frac{100+v}{100-v}=A\mathrm{e}^{0.2t}$ where $A=\pm\mathrm{e}^{0.2C}$ When $t=0$, $v=0$ then $A=1$ $\frac{100+v}{100-v}=\mathrm{e}^{0.2t} \Rightarrow \frac{100-v}{100+v}=\mathrm{e}^{-0.2t}$ $\mathrm{e}^{-0.2t}(100+v)=100-v$ $v\left(1+\mathrm{e}^{-0.2t}\right)=100\left(1-\mathrm{e}^{-0.2t}\right)$ $v=\frac{100\left(1-\mathrm{e}^{-0.2t}\right)}{1+\mathrm{e}^{-0.2t}}$ As $t\to\infty$, $\mathrm{e}^{-0.2t}\to 0$ and $v\to 100$ The sky diver’s speed would increase to a limit of 100 m/s long after he has descended and before he deployed his parachute.

Qn	Solution
11(i)	$V=2\left[\pi(3r)^2 y\right]+\left(\pi r^2\times 15\right)$ $k=18\pi yr^2+15\pi r^2$ $y=\frac{1}{18\pi r^2}\left(k-15\pi r^2\right)$ $A=4\left[\pi(3r)^2\right]-2\pi r^2+15(2\pi r)+2y\left[2\pi(3r)\right]$ $=34\pi r^2+30\pi r+2\left[\frac{1}{18\pi r^2}\left(k-15\pi r^2\right)\right]\left[2\pi(3r)\right]$ $=34\pi r^2+30\pi r+\frac{2}{3r}\left(k-15\pi r^2\right)$ $=34\pi r^2+20\pi r+\frac{2k}{3r}$ $\frac{\mathrm{d}A}{\mathrm{d}r}=68\pi r+20\pi-\frac{2k}{3r^2}$ At minimum area, $\frac{\mathrm{d}A}{\mathrm{d}r}=0$ $68\pi r+20\pi-\frac{2k}{3r^2}=0$ $204\pi r^3+60\pi r^2-2k=0$ $102\pi r^3+30\pi r^2-k=0$ (shown)
(ii)	$102\pi r^3+30\pi r^2-450=0$ From GC, $r=1.03$ (3 s.f.)
(iii)	Volume of water pumped after 1 min = 15 (60) = 900 cm^3 Volume of a weight = $\pi(3\times 2)^2\times 7=791.68$ cm^3 Volume of the handle = $\pi(2)^2\times 15=188.50$ cm^3 Since $900<791.68+188.50=980.18$, the water level is at the handle at 1 min. Let W = volume of water in the handle and h = depth of water from the base of the handle $W=\pi(2)^2 h=4\pi h$ $\frac{\mathrm{d}W}{\mathrm{d}h}=4\pi$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}W}{\mathrm{d}t}\times\frac{\mathrm{d}h}{\mathrm{d}W}$ $=15\times\frac{1}{4\pi}$ Thus the depth of the water is increasing at a rate of $\frac{15}{4\pi}$ cm s^{-1}.

Qn	Solution
12(i)	Let θ be the acute angle between the plane and the incident beam. $\sin\theta = \dfrac{\left\| \begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\2\\3\end{pmatrix} \right\|}{\sqrt{1+4+1}\sqrt{1+4+9}}$ $= \dfrac{6}{\sqrt{84}}$ Therefore $\theta = 40.9°$
(ii)	Let F be the foot of the perpendicular from O to the plane. $\overrightarrow{OF} = \lambda \begin{pmatrix}-1\\2\\3\end{pmatrix}$, for some $\lambda \in \mathbb{R}$ F is on plane $\Rightarrow \overrightarrow{OF} \cdot \begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $\Rightarrow \lambda \begin{pmatrix}-1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $14\lambda = 12$ $\lambda = \dfrac{6}{7}$ $\overrightarrow{OO'} = \dfrac{12}{7}\begin{pmatrix}-1\\2\\3\end{pmatrix}$ $\overrightarrow{OP} = \mu \begin{pmatrix}1\\2\\1\end{pmatrix}$, for some $\mu \in \mathbb{R}$ P is on plane $\Rightarrow \overrightarrow{OP} \cdot \begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $\Rightarrow \mu \begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $6\mu = 12$ $\mu = 2$ $\overrightarrow{OP} = \begin{pmatrix}2\\4\\2\end{pmatrix}$

Qn	Solution
	$\overrightarrow{O'P} = \frac{2}{7}\begin{pmatrix}13\\2\\-11\end{pmatrix}$ Hence $l: \mathbf{r} = \begin{pmatrix}2\\4\\2\end{pmatrix} + \gamma\begin{pmatrix}13\\2\\-11\end{pmatrix}, \gamma \in \mathbb{R}$
(iii)	Let $B \equiv (3,1,0)$. Shortest distance of *B* from incident beam $= \frac{\left\lvert\begin{pmatrix}3\\1\\0\end{pmatrix}\times\begin{pmatrix}1\\2\\1\end{pmatrix}\right\rvert}{\sqrt{1+4+1}} = \frac{\left\lvert\begin{pmatrix}1\\-3\\5\end{pmatrix}\right\rvert}{\sqrt{6}} = \sqrt{\frac{35}{6}} > 2$ $\overrightarrow{PB} = \begin{pmatrix}3\\1\\0\end{pmatrix} - \begin{pmatrix}2\\4\\2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix}$ Shortest distance of *B* from reflected beam $= \frac{\left\lvert\begin{pmatrix}1\\-3\\-2\end{pmatrix}\times\begin{pmatrix}13\\2\\-11\end{pmatrix}\right\rvert}{\sqrt{169+4+121}} = \frac{\left\lvert\begin{pmatrix}37\\-15\\41\end{pmatrix}\right\rvert}{\sqrt{294}} = \sqrt{\frac{3275}{294}} > 2$ Hence sensor will not work properly

H2 Mathematics 2017 Preliminary Exam Paper 2 Question (9758)
Answer all questions [100 marks]

Section A: Pure Mathematics [40 marks]

1	**(i)** Show that if $a_r = T_r - T_{r-1}$ for r = 1, 2, 3, …, and $T_0 = 0$, then $$\sum_{r=1}^{n} a_r = T_n .$$ [1] **(ii)** Deduce that $\sum_{r=1}^{n} \pi^{-r}\left[\left(1-\pi\right)r^2 + 2\pi r - \pi\right] = n^2\pi^{-n}$. [3] **(iii)** Hence, find the exact value of $\sum_{r=4}^{20} \pi^{-r}\left[\left(1-\pi\right)r^2 + 2\pi r - \pi\right]$. [2]
2	Functions g and h are defined by $$\mathrm{g}: x \mapsto x^2 + 6x + 8,\ x \in \mathbb{R},\ x \le \alpha,$$ $$\mathrm{h}: x \mapsto -\mathrm{e}^{x},\ x \in \mathbb{R},\ x > -2.$$ **(i)** Given that the function g^{-1} exists, write down the largest value of α and define g^{-1} in similar form. State a transformation which will transform the curve $y = \mathrm{g}(x)$ onto the curve $y = \mathrm{g}^{-1}(x)$. [5] **(ii)** Given instead that $\alpha = -2$, explain why the composite function hg exists and find the exact range of hg. [2]
3	**Do not use a calculator in answering this question.** Given that $z = 1 + \mathrm{i}$ is a root of the equation $2z^4 + az^3 + 7z^2 + bz + 2 = 0$, find the values of the real numbers a and b and the other roots. [5] Deduce the roots of the equation $2z^4 + bz^3 + 7z^2 + az + 2 = 0$. [2]
4	A curve C has parametric equations $$x = t^2, \quad y = t - t^3,\ t \le 0.$$ **(i)** The point P on the curve has parameter p. Show that the equation of the tangent at P is $2py = x\left(1 - 3p^2\right) + p^2 + p^4$. [3] **(ii)** If the tangent at P passes through the point (6, 5), find the possible coordinates of P. [3] **(iii)** Find the area of the region bounded by C and the x-axis. [3]

5 The planes p_1 and p_2 have equations $x-4y+8z=4$ and $mx+ny+2z=1$ respectively, where m and n are constants.

(i) If p_1 and p_2 meet at a line that has equation $\mathbf{r} = 2\mathbf{i} - 0.5\mathbf{j} + \lambda(-4\mathbf{i} + \mathbf{j} + \mathbf{k})$, where $\lambda \in \mathbb{R}$, find the values of m and n. [3]

It is given instead that $m = 1$ and $n = 2$.

(ii) Find the acute angle between p_1 and p_2. [2]

(iii) The point $(1, b, 5)$ is equidistant from p_1 and p_2. Calculate the possible value(s) of b. [6]

Section B: Statistics [60 marks]

6 **(a)** Find the number of ways to arrange the letters of the word **TOTORO** such that

(i) all the 'O's are together, [1]

(ii) all the 'O's are separated, [2]

(iii) the last letter is a consonant. [3]

(b) Tontoro soft toys are sold in four different colours, of which each varies in three sizes, small, medium and large. Each set of Tontoro soft toys consists of a small, a medium and a large sized soft toy and exactly two are of the same colour. Find the number of different possible sets of Tontoro soft toys. [2]

7 A game is played with a set of 4 cards, each distinctly numbered 1, 2, 3 and 4. A player randomly picks a pair of cards without replacement. If the sum of the cards' numbers is an odd number, the sum is the player's score. If the sum of the two cards' number is an even number, the player randomly picks a third card from the remaining cards. The square of the third card's number is the player's score.

(i) Find the probability that a player obtains a score of 4. [2]

(ii) Find the probability distribution of a player's score, S. Hence, find the expected score of a player. [4]

(iii) Find the probability that a player obtains a score lower than 5, given that he draws three cards. [3]

8	An archaeologist examines rocks to look for fossils. On average, 10% of the rocks selected from a particular area with a large number of rocks contain fossils. The archaeologist selects a random sample of 25 rocks from this area. The number of rocks that contain fossils is denoted by X. **(i)** Find the probability that more than 4 but at most 10 rocks contain fossils. [2] **(ii)** Show that $\frac{\mathrm{P}(X=k+1)}{\mathrm{P}(X=k)}=\frac{25-k}{9(k+1)}$, for k = 0, 1, 2, 3, …, 24. Hence, by considering $\mathrm{P}(X=k+1)>\mathrm{P}(X=k)$, find the most probable value of X. [4] The archaeologist explores a new area. On average, p% (p > 10) of the rocks in the new area contain fossils. A random sample of 20 rocks from the new area is selected. Given that the probability that there are two rocks that contain fossils is 0.17, find the value of p, giving your answer correct to 2 decimal places. [3]
9	A researcher investigates the relationship between the population of a particular species of bacteria in millions (b) and the surrounding temperature in °C (t). The researcher keeps records so that she can estimate the population of the bacteria at a certain temperature. Observations at different temperatures give the data as shown in the following table. <table><tr><td>t</td><td>26.5</td><td>27.5</td><td>28.5</td><td>29.5</td><td>30.5</td><td>31.5</td><td>32.5</td></tr><tr><td>b</td><td>1.31</td><td>2.10</td><td>3.65</td><td>5.80</td><td>α</td><td>19.56</td><td>31.20</td></tr></table> **(i)** Given that the regression line of b on t is $b=-129.368+4.75214t$, show that $\alpha=12.12$, correct to 2 decimal places. [2] **(ii)** Sketch a scatter diagram for the data. [1] **(iii)** Explain which of $b=ct+d$ or $b=kt^3+l$ is the more appropriate model for the relationship between b and t and find the equation of a suitable regression line for this model. [2] **(iv)** Use the model you chose in part **(iii)** to estimate the population of the bacteria when the temperature is 33°C. Comment on the reliability of the estimate obtained. [2] **(v)** It is given that the temperature T, in °F, is related to the temperature t, in °C, by the equation $T=1.8t+32$. Rewrite your equation from part **(iii)** so that it can be used to estimate the population of bacteria when the temperature is given in °F. [2]

10	In a factory, the average time taken by a machine to assemble a smartphone is 53 minutes. A new assembly process is trialled and the time taken to assemble a smartphone, x minutes, is recorded for a random sample of 60 smartphones. The total time taken was found to be 3129 minutes and the variance of the time was 18.35 minutes^2. The engineer wants to test whether the average time taken by a machine to assemble a smartphone has decreased, by carrying out a hypothesis test. **(i)** Explain why the engineer is able to carry out a hypothesis test without assuming anything about the distribution of the times taken to assemble a smartphone. [1] **(ii)** Find unbiased estimates of the population mean and variance and carry out the test at the 10% level of significance. [6] **(iii)** Explain, in the context of the question, the meaning of 'at 10% level of significance'. [1] After several trials, the engineer claims that the average time taken by a machine to assemble a smartphone is 45 minutes using the new assembly process. The internal control manager wishes to test whether the engineer's claim is valid. The population variance of the time taken to assemble a smartphone using the new assembly process may be assumed to be 9 minutes^2. A random sample of 50 smartphones is taken. **(iv)** Find the range of values of the mean time of this sample for which the engineer's claim would be rejected at the 10% significance level. [4] **[Question 11 is printed on the next page.]**

11 In the manufacture of child car seats, a resin made up of three ingredients is used. The ingredients are polymer A, polymer B and an impact modifier. The resin is prepared in batches and each ingredient is supplied by a separate feeder. The masses, in kg, of polymer A, polymer B and the impact modifier in each batch of resin are assumed to be normally distributed with means and standard deviations as shown in the table. The three feeders are also assumed to operate independently of each other.

	Mean	Standard deviation
Polymer A	2030	44.8
Polymer B	1563	22.7
Impact modifier	μ	σ

It is known that 3% of the batches of resin have less than 1350 kg of impact modifier and 30% of the batches of resin have more than 1414 kg of impact modifier.

(i) Show that $\mu \approx 1400$ and $\sigma \approx 26.6$. [3]

(ii) Given that polymer A costs \$2.20 per kg, polymer B costs \$2.80 per kg and the impact modifier costs \$1.50 per kg, find the probability that the total cost of 2 batches of resin exceeds \$22,000. [4]

(iii) A random sample of n batches of resin is chosen. If the probability that at most 6 batches of resin has more than 1414 kg of impact modifier is less than 0.001, find the least value of n. [3]

(iv) Each batch of resin is used to make a large number of car seats. It is found that the tensile strength (N/m^2) of resin for a car seat has mean 125 and standard deviation 17. A random sample of 50 car seats is selected. Find the probability that the average tensile strength of resin for these 50 car seats is less than 130 N/m^2. [3]

~ ~ End of Paper ~ ~

YISHUN JUNIOR COLLEGE

Mathematics Department

PRELIM Solution

Subject : JC2 H2 MATHEMATICS 9758 P2 **Date** :

Qn	Solution
1(i)	$\sum_{r=1}^{n} a_r = \sum_{r=1}^{n}\left(T_r - T_{r-1}\right)$ $= \cancel{T_1} - T_0$ $+ \cancel{T_2} - \cancel{T_1}$ $+ \cancel{T_3} - \cancel{T_2}$ $\vdots$ $+ \cancel{T_{n-1}} - \cancel{T_{n-2}}$ $+ T_n - \cancel{T_{n-1}}$ $= T_n - T_0$ $= T_n$
(ii)	Let $T_r = r^2\pi^{-r}$ Note $T_0 = 0$ $T_r - T_{r-1} = r^2\pi^{-r} - (r-1)^2\pi^{-r+1}$ $= \pi^{-r}\left[r^2 - (r^2 - 2r + 1)\pi\right]$ $= \pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$ $= a_r$ $\therefore$ From (i), $\sum_{r=1}^{n}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right] = \sum_{r=1}^{n} a_r$ $= T_n = n^2\pi^{-n}$
(iii)	$\sum_{r=4}^{20}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$ $= \sum_{r=1}^{20}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right] - \sum_{r=1}^{3}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$ $= 400\pi^{-20} - 9\pi^{-3}$
2(i)	Largest $\alpha = -3$ Let $y = \mathrm{g}(x) = x^2 + 6x + 8$ $= (x+3)^2 - 1$ $(x+3)^2 = y + 1$ $x + 3 = \pm\sqrt{y+1}$

Qn	Solution
	$x = -3 \pm\sqrt{y+1}$ Since $x \le -3$, $x = -3 -\sqrt{y+1}$ $g^{-1} : x \mapsto -3 -\sqrt{x+1},\ \ x \in [-1, \infty)$ A reflection about the line $y = x$ will transform the curve $y = g(x)$ onto the curve $y = g^{-1}(x)$.
(ii)	Since $R_g = [-1, \infty) \subseteq (-2, \infty) = D_h$, the composite function hg exists. $R_{hg} = \left(-\infty, -\frac{1}{e}\right]$
3	By Conjugate Root Theorem, $z = 1 - i$ is also a root. $\left[z-(1+i)\right]\left[z-(1-i)\right] = \left[(z-1)-i\right]\left[(z-1)+i\right]$ $= (z-1)^2 - i^2$ $= z^2 - 2z + 2$ $(z^2-2z+2)(Az^2+Bz+C) = 2z^4 + az^3 + 7z^2 + bz + 2$ By observation, $A = 2$, $C = 1$. i.e. $(z^2-2z+2)(2z^2+Bz+1) = 2z^4 + az^3 + 7z^2 + bz + 2$ Coeff. of z^2 : $1 - 2B + 4 = 7 \Rightarrow B = -1$ Coeff. of z^3 : $B - 4 = a \Rightarrow a = -5$ Coeff. of z : $-2 + 2B = b \Rightarrow b = -4$ $2z^2 - z + 1 = 0$ $z = \frac{1 \pm \sqrt{1-4(2)}}{2(2)}$ $z = \frac{1 \pm \sqrt{7}i}{4}$ Hence other roots are $1 - i$, $\frac{1 \pm \sqrt{7}i}{4}$. $2z^4 + bz^3 + 7z^2 + az + 2 = 0$ $2 + b\frac{1}{z} + 7\frac{1}{z^2} + a\frac{1}{z^3} + 2\frac{1}{z^4} = 0$ Hence $z = \frac{1}{1+i},\quad \frac{1}{1-i},\quad \frac{4}{1+\sqrt{7}i},\quad \frac{4}{1-\sqrt{7}i}$ $z = \frac{1-i}{2},\quad \frac{1+i}{2},\quad \frac{1-\sqrt{7}i}{2},\quad \frac{1+\sqrt{7}i}{2}$

Qn	Solution
4(i)	$x=t^2$, $\quad y=t-t^3$. $\dfrac{dx}{dt}=2t,\quad \dfrac{dy}{dt}=1-3t^2$ $\dfrac{dy}{dx}=\dfrac{dy}{dt}\div\dfrac{dx}{dt}=\dfrac{1-3t^2}{2t}$ At P, $x=p^2$, $\quad y=p-p^3$, $\quad \dfrac{dy}{dt}=\dfrac{1-3p^2}{2p}$ Equation of tangent at P: $\dfrac{y-(p-p^3)}{x-p^2}=\dfrac{1-3p^2}{2p}$ $\Rightarrow 2py-2p(p-p^3)=\left(x-p^2\right)\left(1-3p^2\right)$ $\Rightarrow 2py-2p^2+2p^4=x\left(1-3p^2\right)-p^2+3p^4$ $\Rightarrow 2py=x\left(1-3p^2\right)+p^2+p^4$ (shown) -------(1)
(ii)	At A, substitute $x=6,\ y=5$ into eqn (1) $2p(5)=6\left(1-3p^2\right)+p^2+p^4$ $10p=6-18p^2+p^2+p^4$ $p^4-17p^2-10p+6=0$ From GC, $p=4.35$ (rejected) or $p=-3.7261$ or $p=-1$ or $p=0.370$ (rejected) Hence coordinates of P: (1,0) and (13.9, 48.0)
(iii)	Required area = $-\int_0^1 y\ \ dx$ $=-\int_0^{-1}\left(t-t^3\right)\left(2t\right)\ dt$ $=0.267\ \text{unit}^2$
5(i)	If p_1 and p_2 meet at l, then $\mathbf{m}$ is perpendicular to $\mathbf{n_2}$. $\mathbf{m}\bullet\mathbf{n_2}=0\Rightarrow\begin{pmatrix}2\\1\\1\end{pmatrix}\bullet\begin{pmatrix}m\\n\\2\end{pmatrix}=0$ $2m+n=-2$ Since $\left(2,-0.5,0\right)$ lies on p_2, $2m-0.5n=1$ $m=0$ $n=-2$
(ii)	Let θ be the acute angle between p_1 and p_2. $\cos\theta=\dfrac{\left\lvert\begin{pmatrix}1\\-4\\8\end{pmatrix}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}\right\rvert}{\sqrt{1+16+64}\sqrt{1+4+4}}$ $=\dfrac{1}{3}$ Therefore $\theta=70.5°$

Qn	Solution
(iii)	Let $B \equiv (1, b, 5)$. Observe $A_1(4, 0, 0)$ lies on p_1 $\overrightarrow{A_1B} = \begin{pmatrix} 1 \\ b \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ b \\ 5 \end{pmatrix}$ Shortest distance of B from p_1 $= \dfrac{\left\| \begin{pmatrix} -3 \\ b \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix} \right\|}{\sqrt{1+16+64}} = \dfrac{\|37-4b\|}{9}$ Observe $A_2(1, 0, 0)$ lies on p_2 $\overrightarrow{A_2B} = \begin{pmatrix} 1 \\ b \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 5 \end{pmatrix}$ Shortest distance of B from p_1 $= \dfrac{\left\| \begin{pmatrix} 0 \\ b \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\|}{\sqrt{1+4+4}} = \dfrac{\|10+2b\|}{3}$ $\dfrac{37-4b}{9} = \dfrac{2b+10}{3}$ or $\dfrac{37-4b}{9} = -\dfrac{2b+10}{3}$ $b = \dfrac{7}{10}$ $\quad$ $b = -\dfrac{67}{2}$
6(a)(i)	No. of ways = $\dfrac{4!}{2!} = 12$
(ii)	No. of ways = $\dfrac{3! \times {}^4C_3}{2!} = 12$
(iii)	Case 1: Ending with "T" No. of ways $= \dfrac{5!}{3!} = 20$ Case 2: Ending with "R" No. of ways $= \dfrac{5!}{3!2!} = 10$ Total no. of ways $= 20 + 10 = 30$
(b)	Choose the two sizes that have the same colour: ${}^3C_2 = 3$ Choose colour that is same for two sizes: ${}^4C_1 = 4$ Choose colour of remaining size: ${}^3C_1 = 3$ No. of ways = ${}^3C_2 \times {}^4C_1 \times {}^3C_1 = 36$

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>7(i)

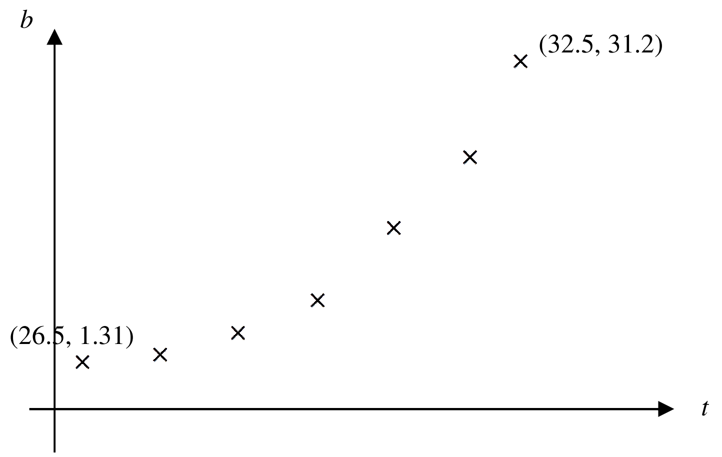
(ii)

(iii)</td><td>P(score of 4)
$=$P(obtain 1 and 3 for the first 2 cards, and obtain 2 for the third card)
$=\left(\frac{1}{4}\times\frac{1}{3}\times 2\right)\times\frac{1}{2}=\frac{1}{12}$

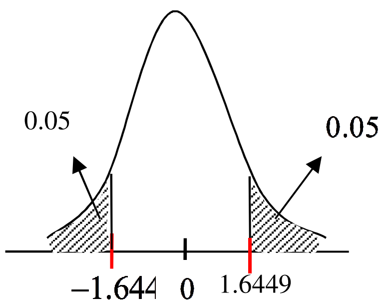
<table>
<tr><td>s</td><td>1</td><td>3</td><td>4</td><td>5</td><td>7</td><td>9</td><td>16</td></tr>
<tr><td>$\mathrm{P}(S=s)$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{12}$</td><td>$\frac{4}{12}$</td><td>$\frac{1}{6}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td></tr>
</table>
Expected score $=1\left(\frac{1}{12}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{12}\right)+5\left(\frac{4}{12}\right)+7\left(\frac{1}{6}\right)+9\left(\frac{1}{12}\right)+16\left(\frac{1}{12}\right)$
$=\frac{35}{6}$

P(score < 5 | draws three cards)
$=\frac{\text{P(score <5 and draws three cards)}}{\text{P(draws three cards)}}$
$=\frac{\text{P(score 4 and 3 cards)+P(score 1 and 3 cards)}}{\text{P(obtain 1,3 or 2,4 for first two cards)}}$
$=\frac{\frac{1}{12}+\frac{1}{12}}{\left(\frac{1}{4}\times\frac{1}{3}\times 2\right)\times 2}$
$=\frac{1}{2}$</td></tr>
<tr><td>8(i)</td><td>Let X be the random variable ‘number of rocks that contain fossils out of 25 rocks’
$X\sim\mathrm{B}(25,0.1)$
$\mathrm{P}(4<X\leq 10)=\mathrm{P}(X\leq 10)-\mathrm{P}(X\leq 4)$
≈ 0.0979819403
≈ 0.0980 (3 sig fig)</td></tr>
<tr><td>(ii)</td><td>$\frac{\mathrm{P}(X=k+1)}{\mathrm{P}(X=k)}=\frac{{}^{25}C_{k+1}(0.1)^{k+1}(0.9)^{25-k-1}}{{}^{25}C_{k}(0.1)^{k}(0.9)^{25-k}}$
$=\frac{\frac{25!}{(k+1)!(25-k-1)!}(0.1)^{k+1}(0.9)^{25-k-1}}{\frac{25!}{k!(25-k)!}(0.1)^{k}(0.9)^{25-k}}$
$=\frac{(25-k)(0.1)}{(k+1)(0.9)}=\frac{25-k}{9(k+1)}$ for k = 0, 1,2 …, 24 (shown)

$\mathrm{P}(X=k+1)>\mathrm{P}(X=k)$
$\frac{\mathrm{P}(X=k+1)}{\mathrm{P}(X=k)}=\frac{(25-k)}{9(k+1)}>1$
$25-k>9k+9$
$10k<16$</td></tr>
</table>

Qn	Solution
	$k < 1.6$ $\Rightarrow \; k = 0$ or 1 for $\mathrm{P}(X = k+1) > \mathrm{P}(X = k)$ Since $\mathrm{P}(X = 2) > \mathrm{P}(X = 1) > \mathrm{P}(X = 0)$, most probable value of $X = 2$
	Let Y be the 'number of rocks that contain fossils out of 20 rocks in the new area' $Y \sim \mathrm{B}(20, \frac{p}{100})$ $\mathrm{P}(Y = 2) = 0.17$ Using g.c. $\frac{p}{100} = 0.045473$ or $\frac{p}{100} = 0.1815827$ Since $p > 10$, $p = 18.16$ (2 d.p)
9(i)	$b = -129.39 + 4.7529t$ From GC, $\bar{t} = 29.5$ $\bar{b} = -129.39 + 4.7529\bar{t}$ $\bar{b} = -129.39 + 4.7529(29.5)$ $= 10.82055$ $\frac{1.31 + 2.1 + 3.65 + 5.8 + \alpha + 19.56 + 31.2}{7} = 10.82055$ $\alpha = 12.124 = 12.12$ (2 dp)
(ii)	
(iii)	From (ii), the scatter diagram shows that as t increases, b increases at an increasing rate which would not be the case if the data follows a linear model. Hence the model $b = kt^3 + l$ is a better model. $b = -37.370 + 0.0018516t^3$ $= -37.4 + 0.00185t^3$ (3s.f.)

Qn	Solution
(iv)	When $t=33$, $b=-37.370+0.0018516(33)^3$ $=29.171$ $=29.2(3\text{s.f.})$ The population of the bacteria is 29.2 millions. Since the estimate is obtained via extrapolation, the estimate is not reliable.
(v)	$b=-37.370+0.0018516\left(\frac{T-32}{1.8}\right)^3$ $=-37.370+\left(3.1749\times10^{-4}\right)(T-32)^3$ $=-37.4+\left(3.17\times10^{-4}\right)(T-32)^3 \quad (3\text{ s.f.})$
10(i)	Since n is large, by Central Limit Theorem, the sample mean time for 60 smartphones is approximately normal. Hence the assumption that the time taken by a machine to assembly a smartphone is not necessary.
(ii)	Unbiased estimate for population mean μ is $\bar{x}$ $=\frac{3129}{60}=52.15$ Unbiased estimate for population variance σ^2 is s^2 $=\frac{60}{59}(18.35)$ $=18.661$ $=18.7$ (3sf) $H_0: \mu=53$ $H_1: \mu<53$ Under H_0, the test statistic $Z=\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim N(0,1)$ approx. by CLT, where $\mu=53, s=\sqrt{18.661}, \bar{x}=52.15, \ n=60$. By GC, p-value $=0.0637(3\text{ s.f.})$. Since p-value < 0.1, we reject H_0 and conclude at 10% level that there is sufficient evidence that average time taken by a machine to assembly a smartphone has reduced.
(iii)	There is a probability of 0.1 of concluding that the average time taken by a machine to assembly a smartphone has decreased when the average time taken by a machine to assembly a smartphone is 53 minutes.

Qn	Solution
(iv)	$H_0 : \mu = 45$ $H_1 : \mu \neq 45$ Under H_0, the test statistic $Z = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$ approx. by CLT, where $\mu = 45, \sigma = \sqrt{9},\ n = 50$.  Since H_0 is rejected, $\dfrac{\bar{x} - 45}{\sqrt{9}/\sqrt{50}} < -1.6449$ or $\dfrac{\bar{x} - 45}{\sqrt{9}/\sqrt{50}} > 1.6449$ $\bar{x} < 44.3021$ $\bar{x} > 45.698$ $\bar{x} < 44.3(3 \text{ s.f.})$ $\bar{x} > 45.7(3 \text{ s.f.})$ Range of values of $\bar{x}$: $\bar{x} < 44.3(3 \text{ s.f.})$ or $\bar{x} > 45.7(3 \text{ s.f.})$
11(i)	Let X be the random variable 'amount (in kg) of impact modifier in a batch of resin' $X \sim N(\mu, \sigma^2)$ $P(X < 1350) = 0.03$ $P(Z < \dfrac{1350 - \mu}{\sigma}) = 0.03$ $\dfrac{1350 - \mu}{\sigma} = -1.88079361$ $\mu - 1.88079361\sigma = 1350 --(1)$ $P(X > 1414) = 0.3$ $P(Z < \dfrac{1414 - \mu}{\sigma}) = 0.7$ $\dfrac{1414 - \mu}{\sigma} = 0.5244005101$ $\mu + 0.5244005101\sigma = 1414 --- (2)$ Solve (1) and (2), $\mu = 1400.046 = 1400$ (shown) $\sigma = 26.609 = 26.6$ (shown)

Qn	Solution
(ii)	Let Y be the random variable 'amount (in kg) of Polymer A in a batch of resin' Let W be the random variable 'amount (in kg) of Polymer B in a batch of resin' $Y \sim N(2030, 44.8^2)$, $W \sim N(1563, 22.7^2)$ Total cost of a batch, $T = 2.20Y + 2.80W + 1.50X \sim N(10942.4, 15345.9572)$ Total cost of 2 batches, $T_1 + T_2 \sim N(21884.8, 30715.9144)$ $P(T_1 + T_2 > 22000) = 0.255$ (3.s.f.)
(iii)	Let H be the r.v.' number of batches of resin with more than 1414 kg of impact modifier out of n batches.' $H \sim B(n, 0.3)$ $\mathrm{P}(H \le 6) < 0.001$ Using GC. When n = 53, $\mathrm{P}(H \le 6) = 0.00120 > 0.001$ When n = 54, $\mathrm{P}(H \le 6) = 9.44 \times 10^{-4} < 0.001$ Therefore, least n = 54
(iv)	Let S be the r.v.' tensile strength (in N/m^2) of resin in a car seat' E(S) = 125, Var(S) = 17^2 $\overline{S} = \dfrac{S_1 + S_2 + \ldots + S_{50}}{50}$ $\overline{S} \sim N\left(125, \dfrac{17^2}{50}\right)$ approx by Central Limit Thm $\mathrm{P}(\overline{S} < 130) = 0.981$ (3 s.f.)