INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
**Higher 2**

CANDIDATE NAME

CIVICS GROUP | INDEX NUMBER

# Mathematics | 9758/01

Paper 1

**24 August 2018**

**3 hours**

Additional materials: Answer Paper
Cover Page
List of Formulae (MF 26)

**READ THESE INSTRUCTIONS FIRST**

**Do not open this booklet until you are told to do so.**

Write your name, class and index number on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

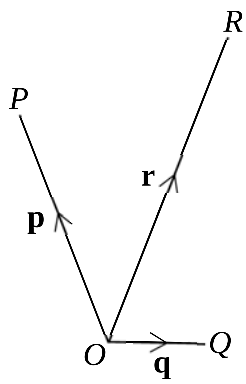
1 The variables $x$ and $y$ are related by

$$\frac{dy}{dx}=\sqrt{y^2+e^{2x}}\ .$$

It is given that the curve of $y$ passes through the point $(0,1)$. Find the Maclaurin series for $y$ in ascending powers of $x$, up to and including the term in $x^2$. Give the coefficients in exact form. [4]

2 Find $\int_0^{\pi} x^2 \sin(nx)\,dx$, where $n$ is an even integer. Leave your answer in the form $\frac{k\pi^2}{n}$, where the value of $k$ is to be determined. [4]

3




The origin $O$ and the points $P$, $Q$ and $R$ lie in the same plane, where $\overrightarrow{OP}=\mathbf{p}$, $\overrightarrow{OQ}=\mathbf{q}$ and $\overrightarrow{OR}=\mathbf{r}$ (see diagram).

**(i)** Explain why $\mathbf{r}$ can be expressed as $\mathbf{r}=\alpha\mathbf{p}+\beta\mathbf{q}$, for constants $\alpha$ and $\beta$. [1]

**(ii)** The point $X$ is on $PR$ such that $PX : XR = 2:1$. It is given that the area of triangle $OPX$ is equal to the area of triangle $OQR$, find the ratio $\alpha:\beta$ in the case where $\alpha$ and $\beta$ are positive. [4]

4 **(a)** Solve the inequality $x^2(x-5)\geq(x-5)\left(2kx-k^2\right)$, given that $k$ is a constant and $k<5$. [3]

**(b)** It is given that $f(x)=a+bx-x^2$ and $g(x)=|x-c|$ where $a$, $b$ and $c$ are constants and $2<c<4$. Given further that $f(2)=g(2)$, $f(5)=g(5)$ and $f(4)-g(4)=\frac{8}{3}$, find the values of $a$, $b$ and $c$. [3]

Hence, find the exact value of the area bounded by the graphs of $y=f(x)$ and $y=g(x)$ for $2\leq x\leq 3$. [3]

**5** **(a)** A sequence $u_1$, $u_2$, $u_3$, … is given by

$$u_n = \frac{1}{n!} \text{ and } u_{n+1} = u_n - \frac{1}{(n-1)!+n!} \text{ for } n \geq 1.$$

**(i)** Find a simplified expression for $\sum_{r=1}^{N} \frac{1}{(r-1)!+r!}$. [2]

**(ii)** Hence show that $\sum_{r=4}^{N} \frac{1}{(r-1)!+r!} < \frac{1}{24}$. [2]

**(b)** D'Alembert's ratio test states that a series of the form $\sum_{r=0}^{\infty} a_r$ converges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, and diverges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$. When $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=0}^{\infty} \frac{(-1)^r \pi^{2r}}{(2r)!}$ converges and state the sum to infinity of this series. [4]

**6** It is given that $\text{f}(x) = \frac{x}{\sqrt{2+x^2}}$.

**(i)** Find $\int_0^k \text{f}(x)\,\text{d}x$ in terms of $k$, where $k$ is a positive constant. [2]

**(ii)** It is now given that $\text{g}(x)$ is the first three terms, in ascending powers of $x$, of the series expansion of $\text{f}(x)$. Find $\text{g}(x)$ and the set of values of $x$ for the expansion to be valid. [5]

**(iii)** Given that $\left|\int_0^k \text{f}(x)\,\text{d}x - \int_0^k \text{g}(x)\,\text{d}x\right| < 0.005$, where $k > 0$, find the range of values of $k$ correct to 4 significant figures. [2]

**7** Two swimmers are training for a long distance swimming competition. They are to swim a distance of 4 km by swimming 80 laps at a swimming pool, where 1 lap covers 50 m from one end of the pool to the other end. Both swimmers aim to complete the distance in between $2\frac{1}{3}$ hours and $2\frac{5}{6}$ hours inclusive.

**(i)** Swimmer *A* swims the first lap in *T* seconds and each subsequent lap takes 1.5 seconds longer than the previous lap. Find the set of values of *T* which will enable *A* to complete the distance within the required time interval. [3]

**(ii)** Swimmer *B* swims the first lap in *t* seconds and the time for each subsequent lap is 1.5% more than the time for the previous lap. Find the set of values of *t* which will enable *B* to complete the distance within the required time interval. Leave your answers correct to 2 decimal places. [3]

**(iii)** Assuming each swimmer completes a distance of 1.8 km in exactly 50 minutes, determine which swimmer is faster in their 80th lap. Justify your answer. [4]

**8** **(a)** Without using a calculator, find the complex numbers *z* and *w* which satisfy the simultaneous equations

$$(3+\mathrm{i})z+3w=-5\mathrm{i} \quad \text{and} \quad (\mathrm{i}-2)z-6\mathrm{i}w=1-3\mathrm{i}\,. \qquad [4]$$

**(b)** **(i)** Given that $-\frac{1}{2}(1+\mathrm{i})$ is a root of the equation

$$k\omega^4-2\omega^3+5\omega^2+6\omega+4=0\,,$$

find the value of the real number *k* and the other roots in exact form. [5]

**(ii)** The roots of the equation in part **(i)** are denoted by $\omega_1$, $\omega_2$, $\omega_3$ and $\omega_4$, where $\arg\omega_1<\arg\omega_2<\arg\omega_3<\arg\omega_4$.

Find $\dfrac{\omega_3}{\omega_4}$ in polar form $r(\cos\theta+\mathrm{i}\sin\theta)$, where $r>0$ and $-\pi<\theta\leq\pi$.

Give *r* and $\theta$ in exact form. [3]

**9** The curve $C$ has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{2x^2 + kx + 8}{x - 4} \text{ and } k \text{ is a constant.}$$

**(i)** Find the equation of the tangent to $C$ at the point where $x = 1$. [4]

**(ii)** Find the range of values of $k$ for which $C$ has more than one stationary point. [2]

Let $k = -7$ for the remaining parts of the question.

**(iii)** Sketch $C$, stating the coordinates of the turning point(s) and the points of intersection with the axes and the equations of any asymptotes. [3]

**(iv)** On separate diagrams, sketch the graphs of

**(a)** $y = \mathrm{f}(|x|)$, [2]

**(b)** $y = \mathrm{f}'(x)$, [2]

stating the equations of any asymptotes and the coordinates of any points of intersection with the axes where appropriate.

**10** A curve $C_k$ has parametric equations

$$x = 1 + k\cos\theta, \qquad y = -2 + \frac{1}{2}k\sin\theta,$$

where $k$ is a positive constant.

**(i)** Find the cartesian equation of $C_k$ and show that its gradient function is $\dfrac{1-x}{4(y+2)}$. [4]

**(ii)** On the same diagram, sketch the graphs of $C_1$ and $C_4$. Label the two graphs clearly. [3]

On a map, the curves $C_1, C_2, C_3$ and $C_4$ represent the contours of a mountain. A stream flows down the mountain. Its path on the map is always at right angles to the contour it is crossing.

**(iii)** Explain why the path of the stream is modelled by the differential equation

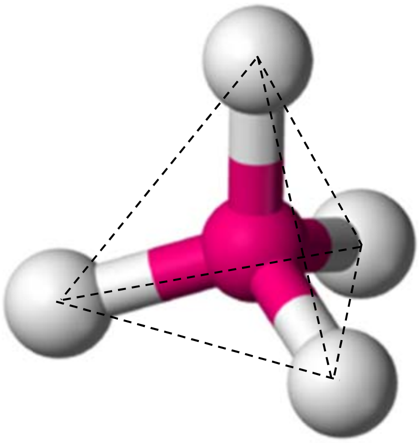
$$\left(\frac{1}{y+2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{x-1}.$$

By considering $\int \frac{1}{y+2}\,\mathrm{d}y = \int \frac{4}{x-1}\,\mathrm{d}x$, show that the path of the stream on the map is represented by the general solution $y = A(x-1)^4 - 2$, where $A$ is an arbitrary constant. [5]

**(iv)** The path of the stream on the map passes through the point $(-1, -1)$. find the equation of the path. [1]

**11** In Chemistry, the molecular structure of chemical compounds is often of interest to chemists as this will aid them in predicting the chemical properties of the compound.

In studying the molecular structure of *silicon tetrachloride*, it is found that this compound takes the form of a regular tetrahedron, that is, it consists of a *silicon* atom at the centre with four *chlorine* atoms symmetrically positioned at the corners of the regular tetrahedron (see diagram).



Suppose the centers of the *chlorine* atoms are at the points $A$, $B$, $C$ and $D$ with coordinates $(5,-2,5)$, $(5,4,-1)$, $(-1,-2,-1)$ and $(7,-4,-3)$ respectively, where $ABCD$ forms a regular tetrahedron.

**(i)** Verify that triangle $ABC$ is an equilateral triangle. [2]

**(ii)** Find a vector that is perpendicular to the plane containing triangle $ABC$. [1]

**(iii)** $\pi_1$ is a plane that is perpendicular to $\overrightarrow{AB}$ and passes through the mid-point of the line segment $AB$. Find the cartesian equation of $\pi_1$. [2]

**(iv)** $\pi_2$ is a plane that is perpendicular to $\overrightarrow{BC}$ and passes through the mid-point of the line segment $BC$. Given that $\pi_1$ and $\pi_2$ meet in the line $l$, find a vector equation for $l$. [3]

**(v)** The position of the *silicon* atom is at the point $G$, where $G$ is equidistant from $A$, $B$, $C$ and $D$. Find the coordinates of $G$. [3]

**(vi)** The angle $AGD$ is also known as the bonding angle of the compound. Find the bonding angle. Show your workings clearly. [2]

INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
**Higher 2**

| | | | |
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| CANDIDATE NAME | | | |
| CLASS | | INDEX NUMBER | |

# MATHEMATICS 9758/02

Paper 2 **11 September 2018**

**3 hours**

Additional Materials: Answer Paper
Cover Page
List of Formulae (MF 26)

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This document consists of **7** printed pages and **1** blank page.

## Section A: Pure Mathematics [40 marks]

**1** It is given that

$$f(x) = \begin{cases} 2x-8 & \text{for } 0 \le x < 2, \\ -x-2 & \text{for } 2 \le x < 6, \end{cases}$$

and that $f(x) = f(x+6)$ for all real values of $x$.

**(i)** Evaluate $f(-21) + f(49)$. [2]

**(ii)** Sketch the graph of $y = f(x)$ for $-6 \le x \le 10$. [3]

**(iii)** Find $\int_{-2}^{7} f(x)\,dx$. [3]

**2** The function f is defined by

$$f(x) = \cos x - \sqrt{3}\sin x\text{, where } -\frac{\pi}{3} \le x \le \frac{2\pi}{3}.$$

**(i)** Express $f(x)$ in the form $R\cos(x+\alpha)$ where $R$ and $\alpha$ are exact positive constants to be found. State the range of f. [3]

**(ii)** Show that $f(x)$ decreases as $x$ increases. [2]

**(iii)** Define $f^{-1}$. [2]

**(iv)** The composite function fg is defined by

$$fg(x) = x - 3\text{, where } 1 \le x \le 3.$$

Find $g(x)$. [2]

**3** Liquid is being poured into a cylindrical tank at a constant rate of 1200 $\text{cm}^3\ \text{s}^{-1}$ and is leaking out of a hole in the base at a rate proportional to the square root of the height of liquid in the tank. The tank is initially empty, and at time $t$ seconds after pouring, the liquid in the tank has volume $V$ $\text{cm}^3$ and height $h$ cm. The circular surface area of the liquid in the tank is 3000 $\text{cm}^2$.

**(i)** Write down a differential equation expressing $\frac{\mathrm{d}V}{\mathrm{d}t}$ in terms of $h$. Hence show that $\frac{\mathrm{d}h}{\mathrm{d}t} = 0.4 - k\sqrt{h}$, where $k$ is a positive constant. [3]

When $h = 36$, liquid is leaking out of the hole at 360 $\text{cm}^3\ \text{s}^{-1}$.

**(ii)** Show that $k = 0.02$. [1]

**(iii)** By using the substitution $\sqrt{h} = 20 - x$, find the particular solution of $t$ in terms of $h$. [5]

**(iv)** Hence find the time taken for the liquid to reach a height of 100 cm, giving your answer in minutes and seconds, correct to the nearest second. [2]

**4** Given that $\mathrm{f}(x) = \frac{1}{\sqrt{1+x^2}}$ and $\mathrm{g}(x) = \frac{a}{\sqrt{1+(0.5x-1)^2}}$ where $a$ is a constant greater than 2, describe fully a sequence of transformations which would transform the graph of $y = \mathrm{f}(x)$ onto the graph of $y = \mathrm{g}(x)$. [3]

The region $R$ is bounded by the curves $y = \mathrm{f}(x)$, $y = \mathrm{g}(x)$, the line $x = 2$ and the $y$-axis. A sculpture is made in the shape of the solid of revolution formed by rotating $R$ through $2\pi$ radians about the $x$-axis.

**(i)** Find the exact volume of the sculpture, giving your answer in terms of $a$ and $\pi$. [4]

**(ii)** Another region $S$ is bounded by the curve $y = \mathrm{f}(x)$, the line $x = 2$ and the $x$- and $y$- axes. A second sculpture takes the shape of the solid of revolution formed by rotating $S$ through $2\pi$ radians about the $y$-axis. Find the exact volume of the second sculpture. [3]

**(iii)** Given that the volume of the first sculpture found in part **(i)** is at least 50 times the volume of the second sculpture found in part **(ii)**, find the smallest integer value of $a$. [2]

**Section B: Probability and Statistics [60 marks]**

**5** The random variable $X$ has the distribution $\mathrm{B}(25, p)$, where $0 < p < 1$. Given that $\mathrm{P}(X \le 1) = 0.15$, write down an equation for the value of $p$ and find this value numerically. Hence find $\mathrm{Var}(X)$. [4]

**6** Find the number of ways in which the letters of the word APPRECIATE can be arranged if

**(i)** vowels (A, E, I) and consonants (P, R, C, T) must alternate, [2]

**(ii)** between the two Es, there are exactly two other letters and at least one of which must be an A. [3]

**7** The discrete random variable $X$ takes values 0, 1, 2 and 3 only. The probability distribution of $X$ is shown in the table, where $p$ is a constant and $0 < p < \frac{1}{10}$.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\mathrm{P}(X = x)$ | $1-6p$ | $3p$ | $2p$ | $p$ |

**(i)** Given that $\mathrm{Var}(X) = 0.75$, find the value of $\mathrm{E}(X)$. [3]

**(ii)** The random variable $S$ is the sum of $n$ independent observations of $X$, where $n$ is large. Given that the probability that $S$ exceeds 150 is at least 0.75, find the set of possible values of $n$. [3]

**8** For events $A$, $B$ and $C$, it is given that $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.45$ and $\mathrm{P}(C) = 0.35$.

It is further given that $\mathrm{P}(B | C) = 0.5$, $\mathrm{P}(A \cap C) = 0.15$ and $\mathrm{P}(A \cap B \cap C) = 0.1$.

**(i)** Find $\mathrm{P}(A' \cap B \cap C)$. [2]

**(ii)** Given also that events $A$ and $B$ are independent, find $\mathrm{P}(A \cup B)$. [2]

**(iii)** Given instead that events $A$ and $B$ are **not** independent, find the greatest and least possible values of $\mathrm{P}(A' \cap B' \cap C')$. [4]

**9** Bottles of tomato juice produced by a company are said to contain 250 ml, with a standard deviation of 10 ml. After receiving feedback from some consumers regarding the volume of tomato juice per bottle, the manager takes a random sample of 50 bottles to test whether the mean volume has been overstated. He measures the volume, $x$ ml of tomato juice in each bottle and the sample mean volume is found to be 247.5 ml.

**(i)** State appropriate hypotheses for the test, defining any symbols you use. [2]

**(ii)** Find the $p$-value of the test and state the meaning of this $p$-value in context. [2]

**(iii)** State, giving a reason, whether it is necessary to assume a normal distribution for this test to be valid. [1]

The company installs a new machine to produce smaller bottles of tomato juice with mean volume $\mu_0$ ml. A random sample of these smaller bottles of tomato juice is taken. The sample size is 60 and the volumes, $y$ ml, are summarised as follows.

$$\sum y = 10\,757 \qquad \sum y^2 = 1\,931\,597$$

**(iv)** Calculate unbiased estimates of the population mean and variance of the volume of smaller bottles of tomato juice. [2]

**(v)** A two-tail test is to be carried out at the 5% significance level by the manager. Find the range of values of $\mu_0$, correct to 1 decimal place, such that the null hypothesis will not be rejected. [3]

**10** A company wants to investigate the effect of using strong acid solution in reducing the weight of metal plates. Eight metal plates are randomly selected and immersed in a strong acid solution for different lengths of time, $t$ hours. The percentages of weight loss, $w$ %, are calculated and the results are shown in the table below.

| $t$ | 100 | 150 | 200 | 250 | 300 | 350 | 400 | 450 |
|---|---|---|---|---|---|---|---|---|
| $w$ | 0.80 | 1.40 | 2.00 | 2.31 | 2.53 | 2.65 | 2.71 | 2.77 |

**(i)** Calculate the product moment correlation coefficient between $t$ and $w$, and explain whether your answer suggests that a linear model is appropriate. [3]

**(ii)** Draw the scatter diagram for these values, labelling the axes clearly. Explain which of the following equations, where $a$ and $b$ are constants and $b > 0$, provides the most accurate model of the relationship between $t$ and $w$.

(A) $w = a + b\ln t$

(B) $w = a + bt^2$

(C) $w = a + \dfrac{b}{t^2}$

[2]

**(iii)** Using the model you chose in part **(ii)**, write down the equation for the relationship between $t$ and $w$, giving the numerical values of the coefficients. State the product moment correlation coefficient for this model and comment on its value. [3]

**(iv)** Given that a metal plate being immersed in the strong acid solution for $t$ hours has a weight loss of 2.4%, estimate the value of $t$. Give two reasons why this estimate is reliable. [3]

**(v)** Given that 1 day = 24 hours, re-write your equation from part **(iii)** so that it can be used to estimate the percentage weight loss of metal plates when the length of time of immersing the metal plates in the strong acid solution is measured in days. [1]

**11** **(a)** Flash Electrics is a company which specializes in installing electricity meters in new houses in Central City. The time taken, $T$ minutes, by its employees to install an electricity meter may be assumed to be normally distributed with mean $\mu$ and standard deviation $\sigma$.

**(i)** Given that $\mathrm{P}(T < 40) = \mathrm{P}(T > 50) = 0.36$, find the values of $\mu$ and $\sigma$. [3]

**(ii)** A random sample of 18 new houses in Central City with electricity meters installed by Flash Electrics is taken. Find the probability that at least 5 but fewer than 10 houses in this sample have an installation time of at most 50 minutes. [3]

**(b)** The electricity consumption, measured in kilowatt hour (kWh), of the households in Central City has a normal distribution with mean 520 and standard deviation 35.

**(i)** Find the probability that the electricity consumption of a randomly chosen household is more than 500 kWh. [1]

**(ii)** Two households in Central City are randomly chosen. Find the probability that both households each have electricity consumption of less than 500 kWh. [2]

**(iii)** The probability that the total electricity consumption of two randomly chosen households is less than 1000 kWh is denoted by $p$. Without calculating its value, explain why $p$ will be greater than your answer to part **(ii)**. [1]

The electricity consumption of the households in Star City has a normal distribution with mean 475 kWh and standard deviation 25 kWh. It is known that electricity costs $0.18 per kWh and $0.15 per kWh in Central City and Star City respectively.

Let $X$ represent the electricity bill of a randomly chosen household in Central City.

Let $Y$ represent the electricity bill of a randomly chosen household in Star City.

**(iv)** Find $\mathrm{P}(4Y - 3X > 7)$ and explain, in the context of this question, what your answer represents. [5]

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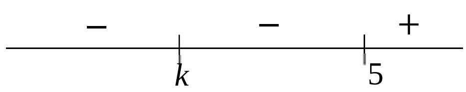
**Innova Junior College**
**H2 Mathematics**
**2018 Prelim Exam Paper 1 Solution**

| **Q1** | **Suggested Solution** |
|---|---|
| | $\dfrac{dy}{dx}=\sqrt{\left(y^2+e^{2x}\right)} \Rightarrow \left(\dfrac{dy}{dx}\right)^2 = y^2+e^{2x}$ --- (1)<br>Differentiating (1) with respect to *x*.<br>$2\left(\dfrac{dy}{dx}\right)\dfrac{d^2y}{dx^2}=2y\dfrac{dy}{dx}+2e^{2x}$ --- (2)<br><br>When $x=0$,<br>$y=1$ (given)<br>$\dfrac{dy}{dx}=\sqrt{1+e^0}=\sqrt{2}$<br>From (2): $2\sqrt{2}\dfrac{d^2y}{dx^2}=2(1)(\sqrt{2})+2e^0 \Rightarrow \dfrac{d^2y}{dx^2}=\dfrac{\sqrt{2}+1}{\sqrt{2}}$<br>Therefore the Maclaurin series for *y* is<br>$y=1+x\sqrt{2}+\left(\dfrac{1+\sqrt{2}}{2\sqrt{2}}\right)x^2+\ldots$ |
| **Q2** | **Suggested Solution** |
| | $\displaystyle\int_0^{\pi} x^2\sin(nx)\,dx$<br>$\displaystyle=\left[-x^2\frac{\cos(nx)}{n}\right]_0^{\pi}+\frac{1}{n}\int_0^{\pi}2x\cos(nx)\ dx$ --- (1)<br>$\displaystyle=-\frac{\pi^2}{n}\cos(n\pi)+\frac{1}{n}\left\{\left[(2x)\frac{\sin(nx)}{n}\right]_0^{\pi}-\int_0^{\pi}(2)\frac{\sin(nx)}{n}dx\right\}$<br>$\displaystyle=-\frac{\pi^2}{n}-\frac{2}{n^2}\int_0^{\pi}\sin(nx)\ dx$ --- (2)<br>$\displaystyle=-\frac{\pi^2}{n}-\frac{2}{n^2}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi}$<br>$\displaystyle=-\frac{\pi^2}{n}+\frac{2}{n^3}\left[\cos(n\pi)-\cos 0\right]$<br>$\displaystyle=-\frac{\pi^2}{n}$<br>$\therefore\ k=-1$ |
| **Q3** | **Suggested Solution** |
| **(i)** | Since **p** and **q** are non-parallel vectors, for some $\alpha$ and $\beta$, the sum of $\alpha$**p** and $\beta$**q** is **r** by law of parallelogram for vector addition. |

| | |
|---|---|
| **(ii)** | Using ratio theorem,<br>$\overrightarrow{OX} = \mathbf{x} = \frac{1}{3}\mathbf{p} + \frac{2}{3}\mathbf{r} = \frac{1}{3}\mathbf{p} + \frac{2}{3}\alpha\mathbf{p} + \frac{2}{3}\beta\mathbf{q}$<br>Area of triangle *OPX*<br>$= \frac{1}{2}\lvert\mathbf{p}\times\mathbf{x}\rvert$<br>$= \frac{1}{2}\left\lvert\mathbf{p}\times\left(\frac{1}{3}\mathbf{p} + \frac{2}{3}\alpha\mathbf{p} + \frac{2}{3}\beta\mathbf{q}\right)\right\rvert$<br>$= \frac{1}{3}\beta\lvert\mathbf{p}\times\mathbf{q}\rvert$<br>Area of triangle *OQR*<br>$= \frac{1}{2}\lvert\mathbf{q}\times\mathbf{r}\rvert$<br>$= \frac{1}{2}\lvert\mathbf{q}\times(\alpha\mathbf{p} + \beta\mathbf{q})\rvert$<br>$= \frac{1}{2}\alpha\lvert\mathbf{p}\times\mathbf{q}\rvert$<br>Since the area of the triangles are the same,<br>$\frac{1}{3}\beta\lvert\mathbf{p}\times\mathbf{q}\rvert = \frac{1}{2}\alpha\lvert\mathbf{p}\times\mathbf{q}\rvert$<br>$\frac{\alpha}{\beta} = \frac{2}{3}$<br>The ratio required is 2:3. |
| **Q4** | **Suggested Solution** |
| **(a)** | $x^2(x-5) \geq (x-5)(2kx-k^2)$<br>$(x-5)\left[x^2-(2kx-k^2)\right] \geq 0$<br>$(x-5)(x^2-2kx+k^2) \geq 0$<br>$(x-5)(x-k)^2 \geq 0$<br><br>$x = k$ or $x \geq 5$ |
| **(b)** | $f(x) = a + bx - x^2$ ; $g(x) = \lvert x-c\rvert$ where $2 < c < 4$<br>$2 < c \Rightarrow g(2) = \lvert 2-c\rvert = c-2$<br>$c < 4 \Rightarrow g(5) = \lvert 5-c\rvert = 5-c$<br>$f(2) = g(2) \Rightarrow a + 2b - 4 = c - 2$<br>$a + 2b - c = 2$ …… (1) |

$$\mathrm{f}(5) = \mathrm{g}(5) \quad \Rightarrow \quad a + 5b - 25 = 5 - c$$

$$a + 5b + c = 30 \ \ldots\ldots \ (2)$$

$$\mathrm{f}(4) - \mathrm{g}(4) = \frac{8}{3} \quad \Rightarrow \quad a + 4b - 16 - (4 - c) = \frac{8}{3}$$

$$a + 4b + c = \frac{68}{3} \ldots\ldots \ (3)$$

$$a = -\frac{29}{3}; \quad b = \frac{22}{3}; \quad c = 3$$

Area bounded

$$\int_2^3 -\frac{29}{3} + \frac{22}{3}x - x^2 + (x - 3)\,\mathrm{d}x$$

$$= \left[-\frac{29}{3}x + \frac{11}{3}x^2 - \frac{x^3}{3} + \frac{1}{2}x^2 - 3x\right]_2^3$$

$$= \left(-29 + 33 - 9 + \frac{9}{2} - 9\right) - \left(-\frac{58}{3} + \frac{44}{3} - \frac{8}{3} + 2 - 6\right)$$

$$= \frac{11}{6}$$

**Q5** **Suggested Solution**

**(a)**

$$\sum_{r=1}^{N} \frac{1}{(r-1)! + r!} = \sum_{r=1}^{N} (u_r - u_{r+1})$$

$$\begin{aligned} = \quad & u_1 - u_2 \\ + \quad & u_2 - u_3 \\ & \vdots \\ + \quad & u_{N-1} - u_N \\ + \quad & u_N - u_{N+1} \end{aligned}$$

$$= u_1 - u_{N+1}$$

$$= 1 - \frac{1}{(N+1)!}$$

$$\sum_{r=4}^{N} \frac{1}{(r-1)! + r!} = \sum_{r=1}^{N} \frac{1}{(r-1)! + r!} - \sum_{r=1}^{3} \frac{1}{(r-1)! + r!}$$

$$= \left(1 - \frac{1}{(N+1)!}\right) - \left(1 - \frac{1}{4!}\right)$$

$$= \frac{1}{24} - \frac{1}{(N+1)!}$$

$$< \frac{1}{24} \quad \left(\text{since } \frac{1}{(N+1)!} > 0\right)$$

| | |
|---|---|
| **(b)** | Let $a_n = \dfrac{(-1)^n \pi^{2n}}{(2n)!}$<br>$\dfrac{a_{n+1}}{a_n} = \dfrac{(-1)^{n+1}\pi^{2n+2}}{(2n+2)!} \times \dfrac{(2n)!}{(-1)^n \pi^{2n}}$<br>$= -\dfrac{\pi^2}{(2n+1)(2n+2)}$<br>$\lim_{n\to\infty}\left\|\dfrac{a_{n+1}}{a_n}\right\| = \lim_{n\to\infty}\dfrac{\pi^2}{(2n+1)(2n+2)} = 0 \quad < \quad 1$<br>Hence by ratio test, $\sum_{r=0}^{\infty}\dfrac{(-1)^r \pi^{2r}}{(2r)!}$ converges.<br><br>$\sum_{r=0}^{\infty}\dfrac{(-1)^r \pi^{2r}}{(2r)!} = \cos\pi = -1$ |
| **Q6** | **Suggested Solution** |
| **(i)** | $\int_0^k \mathrm{f}(x)\,\mathrm{d}x = \int_0^k x(2+x^2)^{-1/2}\,\mathrm{d}x$<br>$= \dfrac{1}{2}\int_0^k 2x(2+x^2)^{-1/2}\,\mathrm{d}x$<br>$= \dfrac{1}{2}\left[\dfrac{(2+x^2)^{1/2}}{1/2}\right]_0^k$<br>$= \sqrt{2+k^2} - \sqrt{2}$ |
| **(ii)** | $\mathrm{f}(x) = x(2+x^2)^{-1/2}$<br>$= x\left[2\left(1+\dfrac{x^2}{2}\right)\right]^{-1/2}$<br>$= \dfrac{1}{\sqrt{2}}x\left(1 - \dfrac{1}{2}\cdot\dfrac{x^2}{2} + \dfrac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2!}\left(\dfrac{x^2}{2}\right)^2 + \ldots\right)$<br>$= \dfrac{1}{\sqrt{2}}x\left(1 - \dfrac{1}{4}x^2 + \dfrac{3}{32}x^4 + \ldots\right)$<br><br>$\mathrm{g}(x) = \dfrac{1}{\sqrt{2}}x - \dfrac{1}{4\sqrt{2}}x^3 + \dfrac{3}{32\sqrt{2}}x^5$ |

The expansion is valid when $\left|\frac{x^2}{2}\right| < 1$ ,

$$\Rightarrow \quad x^2 < 2$$

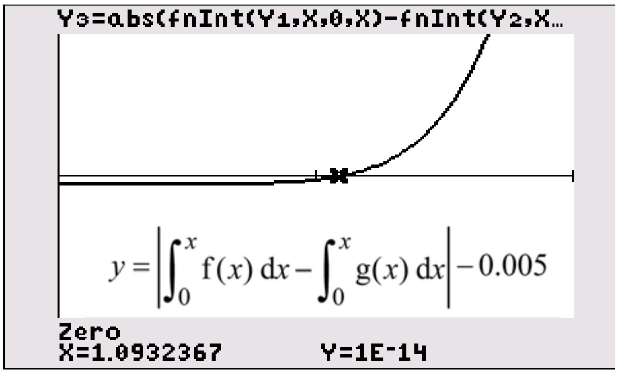
$$|x| < \sqrt{2}$$

$$\left\{x : x \in \mathbb{R}, -\sqrt{2} < x < \sqrt{2}\right\}$$

**(iii)**

$$\left|\int_0^k \mathrm{f}(x)\,\mathrm{d}x - \int_0^k \mathrm{g}(x)\,\mathrm{d}x\right| < 0.005$$

$$\left|\int_0^k \mathrm{f}(x)\,\mathrm{d}x - \int_0^k \mathrm{g}(x)\,\mathrm{d}x\right| - 0.005 < 0$$




From GC, $0 < k < 1.093$ (4 s.f.)

| **Q7** | **Suggested Solution** |
|---|---|

**(i)**

1st term $= T$ & Common difference $= 1.5$

Total time taken to complete the distance of 4 km (i.e. 80 laps)

$$= \frac{80}{2}\left[2T + (80-1)1.5\right] = 80T + 4740$$

To complete within the required time interval,

$$\left(2\tfrac{1}{3}\right)(60)(60) \le 80T + 4740 \le \left(2\tfrac{5}{6}\right)(60)(60)$$

$$8400 \le 80T + 4740 \le 10200$$

$$45.75 \le T \le 68.25$$

Set of values of $T$ is $\{T \in \mathbb{R} : 45.75 \le T \le 68.25\}$

**(ii)**

1st term $= t$ & Common ratio $= 1.015$

Total time taken to complete the distance of 4 km (i.e. 80 laps)

$$= \frac{t\left(1.015^{80} - 1\right)}{1.015 - 1} = \frac{200t}{3}\left(1.015^{80} - 1\right)$$

To complete within the required time interval,

$$8400 \le \frac{200t}{3}\left(1.015^{80} - 1\right) \le 10200$$

$$55.0059 \le t \le 66.7928$$

$$55.01 \le t \le 66.79 \text{ (2 dec pl)}$$

| | |
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| | Set of values of $t$ is $\{t \in \mathbb{R} : 55.01 \le t \le 66.79\}$ (to 2d.p.) |
| **(iii)** | Completing a distance of 1.8 km is equivalent to swimming 36 laps<br><br>For swimmer *A*:<br>$\frac{36}{2}\left[2T+(36-1)1.5\right]=(50)(60) \Rightarrow T=57.083333$<br>$\therefore$ time taken to swim the $80^{\text{th}}$ lap<br>$=57.083333+(80-1)(1.5)=175.58333$<br><br>For swimmer *B*:<br>$\frac{t\left(1.015^{36}-1\right)}{1.015-1}=(50)(60) \Rightarrow t=63.457186$<br>$\therefore$ time taken to swim the $80^{\text{th}}$ lap<br>$=63.457186(1.015)^{80-1}=205.73024$<br><br>Swimmer *A* is faster in his $80^{\text{th}}$ lap. |
| **Q8** | **Suggested Solution** |
| **(a)** | $(3+\text{i})z+3w=-5\text{i}$ --- (1)<br>$(\text{i}-2)z-6\text{i}w=1-3\text{i}$ --- (2)<br><br>$(1)\times 2\text{i}:\ 2\text{i}(3+\text{i})z+6\text{i}w=-5\text{i}(2\text{i})$ --- (3)<br><br>$(2)+(3)$:<br>$(\text{i}-2)z+2\text{i}(3+\text{i})z=1-3\text{i}-5\text{i}(2\text{i})$<br>$z(\text{i}-2+6\text{i}-2)=1-3\text{i}+10$<br>$z=\frac{11-3\text{i}}{-4+7\text{i}}\times\frac{-4-7\text{i}}{-4-7\text{i}}$<br>$=\frac{-44-77\text{i}+12\text{i}-21}{16+49}$<br>$=\frac{-65-65\text{i}}{65}$<br>$=-1-\text{i}$<br><br>Substitute $z=-1-\text{i}$ into (1):<br>$(3+\text{i})(-1-\text{i})+3w=-5\text{i}$<br>$3w=-5\text{i}-(-3-3\text{i}-\text{i}+1)$<br>$w=\frac{2-\text{i}}{3}$ |

| | |
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| **(b)(i)** | Since the coefficients of the equation are real, $-\frac{1}{2}(1-\mathrm{i})$ is another root of the equation.<br><br>Quadratic factor $=\left(\omega+\frac{1}{2}+\frac{1}{2}\mathrm{i}\right)\left(\omega+\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)$<br>$=\left(\omega+\frac{1}{2}\right)^2-\left(\frac{1}{2}\mathrm{i}\right)^2$<br>$=\omega^2+\omega+\frac{1}{4}+\frac{1}{4}$<br>$=\omega^2+\omega+\frac{1}{2}$<br>By inspection,<br>$k\omega^4-2\omega^3+5\omega^2+6\omega+4=\left(\omega^2+\omega+\frac{1}{2}\right)\left(k\omega^2+p\omega+8\right)$<br><br>Comparing $\omega$: $6=8+\frac{1}{2}p \Rightarrow p=-4$<br>Comparing $\omega^3$: $-2=p+k \Rightarrow k=-2+4=2$<br>Solving $2\omega^2-4\omega+8=0$:<br>$\omega=\frac{4\pm\sqrt{16-4(2)(8)}}{2(2)}=\frac{4\pm 4\mathrm{i}\sqrt{3}}{4}=1\pm\mathrm{i}\sqrt{3}$ |
| **(b)(ii)** | Note that $\omega_3=1+\mathrm{i}\sqrt{3}$ and $\omega_4=-\frac{1}{2}(1-\mathrm{i})$.<br>$\left\lvert\frac{\omega_3}{\omega_4}\right\rvert=\frac{\lvert\omega_3\rvert}{\lvert\omega_4\rvert}=\frac{2}{\sqrt{2}/2}=\frac{4}{\sqrt{2}}$<br>$\arg\left(\frac{\omega_3}{\omega_4}\right)=\arg\omega_3-\arg\omega_4=\frac{\pi}{3}-\frac{3\pi}{4}=-\frac{5\pi}{12}$<br>$\frac{\omega_3}{\omega_4}=\frac{4}{\sqrt{2}}\left[\cos\left(-\frac{5\pi}{12}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{12}\right)\right]$ |
| **9** | **Suggested Solution** |
| **(i)** | $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(x-4)(4x+k)-\left(2x^2+kx+8\right)}{(x-4)^2}$<br>$=\frac{4x^2+(k-16)x-4k-2x^2-kx-8}{(x-4)^2}$<br>$=\frac{2x^2-16x-4k-8}{(x-4)^2}$<br><br>When $x$ = 1, $y=\frac{10+k}{-3}$, and<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2-16-4k-8}{(-3)^2}=\frac{-22-4k}{9}$ |

| | |
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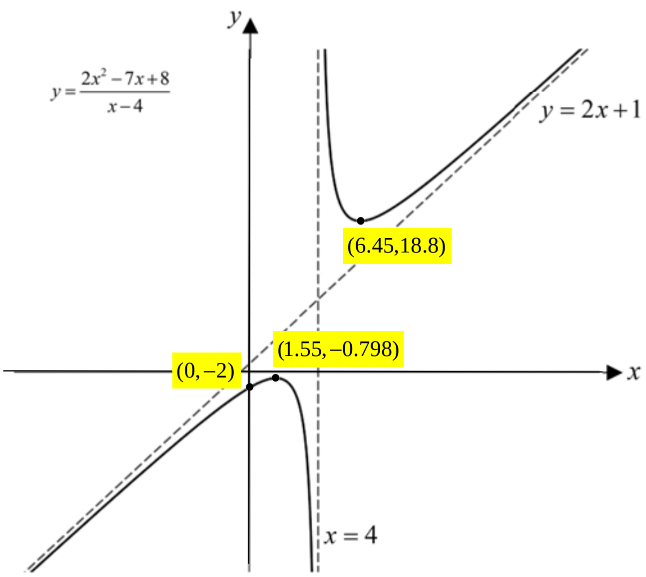
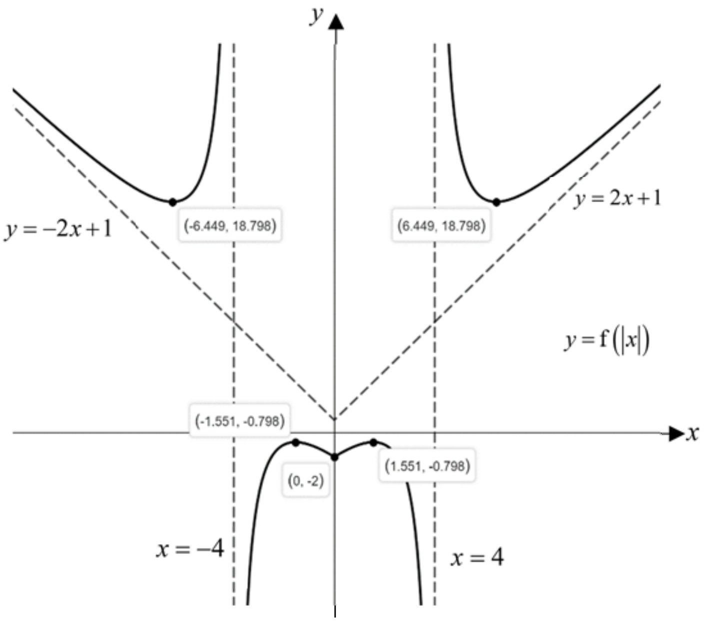
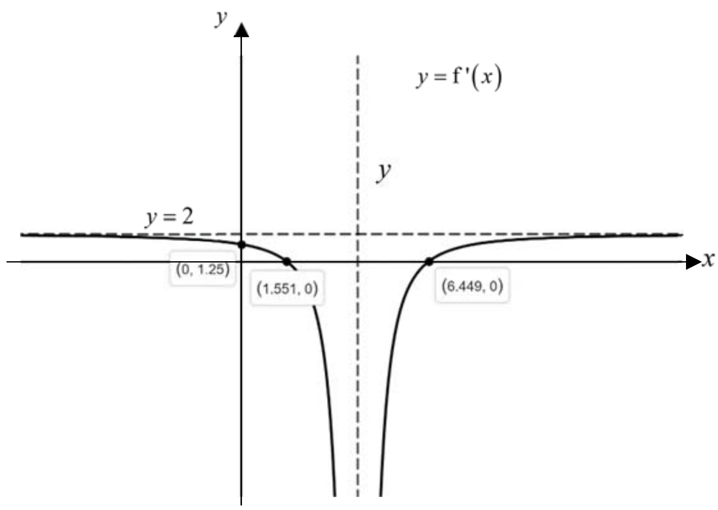
| | Equation of tangent to $C$ at $x$ = 1:<br>$y-\frac{10+k}{-3}=\frac{-22-4k}{9}(x-1)$<br>$9y+30+3k=(-22-4k)x+22+4k$<br>$9y+2(11+2k)x=k-8$<br>Or $y=-\frac{2(11+2k)}{9}x+\frac{k-8}{9}$ |
| **(ii)** | For stationary points,<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=0$<br>$\frac{2x^2-16x-4k-8}{(x-4)^2}=0$<br>$x^2-8x-2k-4=0$<br><br>For more than 1 stationary point, this equation must have real and distinct roots,<br>$(-8)^2-4(-2k-4)>0$<br>$64+8k+16>0$<br>$k>-10$ |
| **(iii)** |  |

**(iv) (a)**




**(b)**




| **Q10** | **Suggested Solution** |
|---|---|

**(i)**

$x = 1 + k\cos\theta \quad \Rightarrow \quad \cos\theta = \dfrac{(x-1)}{k}$

$y = -2 + \dfrac{1}{2}k\sin\theta \quad \Rightarrow \quad \sin\theta = \dfrac{2(y+2)}{k}$

$\cos^2\theta + \sin^2\theta = 1 \quad \Rightarrow \dfrac{(x-1)^2}{k^2} + \dfrac{4(y+2)^2}{k^2} = 1$

Cartesian eqn of $C_k$ is $(x-1)^2 + 4(y+2)^2 = k^2$

Applying implicit differentiation,

$$2(x-1) + 8(y+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(x-1)}{4(y+2)} = \frac{1-x}{4(y+2)}$$

| | |
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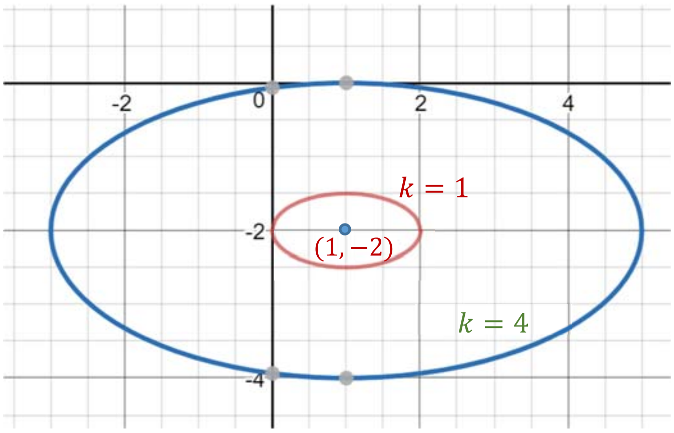
| **(ii)** |  |
| **(iii)** | The path of the stream on the map is perpendicular to the ellipses implies that (gradient of the path)×(grad of ellipse) = −1.<br><br>Since gradient of ellipse $= -\dfrac{(x-1)}{4(y+2)}$,<br>We have $\dfrac{dy}{dx}\left[-\dfrac{(x-1)}{4(y+2)}\right]=-1$<br>Thus $\dfrac{1}{y+2}\times\dfrac{dy}{dx}=\dfrac{4}{x-1}$.<br>Using the given result,<br>$\int\left(\dfrac{1}{y+2}\right)dy=\int\dfrac{4}{x-1}\,dx$<br>$\ln\lvert y+2\rvert=4\ln\lvert x-1\rvert+C$<br>$\lvert y+2\rvert=e^{4\ln\lvert x-1\rvert+C}$<br>$=\left(e^{\ln\lvert x-1\rvert^4}\right)\left(e^C\right)$<br>$y+2=\pm\left(e^C\right)(x-1)^4$<br>$y=A(x-1)^4-2$ where $A=\pm e^C$ |
| **(iv)** | The path passes through the point $(-1,-1)$,<br>$-1=A(-1-1)^4-2$<br>$A=\dfrac{1}{16}$<br>Eqn of the path is $y=\dfrac{1}{16}(x-1)^4-2$ |

| **Q11** | **Suggested Solution** |
|---|---|
| **(i)** | $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} = 6\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \left\|\overrightarrow{AB}\right\| = 6\sqrt{2}$<br>$\overrightarrow{AC} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \left\|\overrightarrow{AC}\right\| = 6\sqrt{2}$<br>$\overrightarrow{BC} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \left\|\overrightarrow{AC}\right\| = 6\sqrt{2}$<br>Since *AB* = *BC* = *CA*, triangle *ABC* is an equilateral triangle. |
| **(ii)** | $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ |
| **(iii)** | Since $\pi_1$ is perpendicular to $\overrightarrow{AB}$, the normal vector of $\pi_1$ is $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.<br>By symmetry, $\pi_1$ will pass through *C*.<br>$\pi_1 : \underset{\sim}{r} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = -1$<br>Cartesian equation of $\pi_1$ is: $y - z = -1$ |
| **(iv)** | Since $\pi_2$ is perpendicular to $\overrightarrow{BC}$, the normal vector of $\pi_2$ is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.<br>By symmetry, $\pi_2$ will pass through *A*.<br>$\pi_2 : \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 3$<br>Cartesian equation of $\pi_2$ is: $x + y = 3$<br>Using GC, the equation of the line of intersection of the two planes is<br>$l: \underset{\sim}{r} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, where $\lambda \in \mathbb{R}$. |

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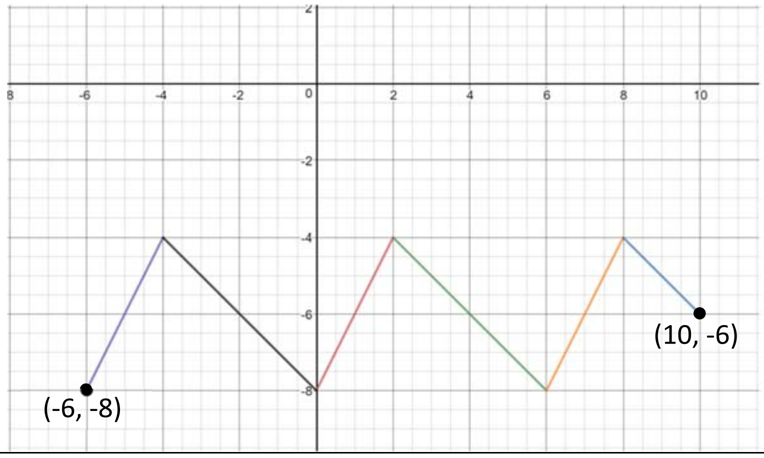
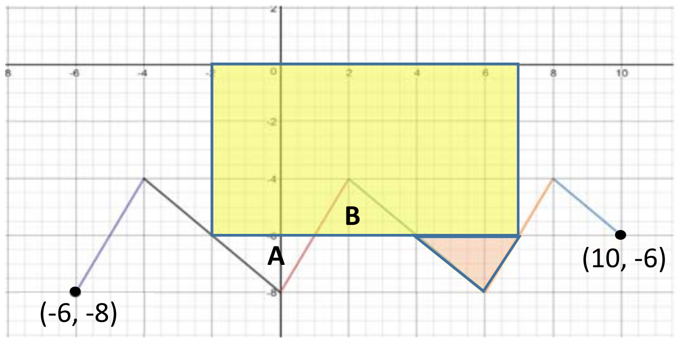
| **(v)** | Note that point *G* lies on the line *l* found in part **(iv)**.<br>Since *G* lies on *l*, $\overrightarrow{OG}=\begin{pmatrix}4\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$<br>$\overrightarrow{AG}=\begin{pmatrix}-1-\lambda\\1+\lambda\\-5+\lambda\end{pmatrix}$ & $\overrightarrow{DG}=\begin{pmatrix}-3-\lambda\\3+\lambda\\\lambda+3\end{pmatrix}$<br>Given that $\left\|\overrightarrow{DG}\right\|=\left\|\overrightarrow{AG}\right\|$ ,<br>$2(1+\lambda)^2+(\lambda-5)^2=3(\lambda+3)^2$<br>$2(\lambda^2+2\lambda+1)+(\lambda^2-10\lambda+25)=3(\lambda^2+6\lambda+9)$<br>$4\lambda+2-10\lambda+25=18\lambda+27$<br>$\lambda=0$<br>$\overrightarrow{OG}=\begin{pmatrix}4\\-1\\0\end{pmatrix}$<br>Thus, coordinates of *G* are (4, -1, 0). |
| **(vi)** | $\cos\angle AGD=\dfrac{\overrightarrow{DG}\cdot\overrightarrow{AG}}{\left\|\overrightarrow{DG}\right\|\left\|\overrightarrow{AG}\right\|}=\dfrac{3\begin{pmatrix}-1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\-5\end{pmatrix}}{\left(3\sqrt{3}\right)\left(\sqrt{27}\right)}=-\dfrac{1}{3}$<br>$\angle AGD=109.5^{\circ}$ |
| | **<u>Alternative:</u>**<br>Let angle *AGX* be $\alpha$.<br>$\cos\alpha=\dfrac{1}{3}\Rightarrow\alpha=70.52^{\circ}\ \Rightarrow\ \theta=180^{\circ}-\alpha=109.5^{\circ}$ |

**2018 IJC H2 Math Prelim Paper 2 Solution**

| Qn | Solution |
|---|---|
| **1(i)** | $\mathrm{f}(-21)+\mathrm{f}(49)=\mathrm{f}(-21+24)+\mathrm{f}(49-48)$<br>$=\mathrm{f}(3)+\mathrm{f}(1)$<br>$=(-3-2)+(2-8)$<br>$=-11$ |
| **1(ii)** | <br> |
| **1(iii)** | **<u>Method 1</u>**<br>$\int_{-2}^{7} \mathrm{f}(x)\ \mathrm{d}x$<br>$=-\{\text{sum of areas of 4 trapezia}\}$<br>$=-\left[\frac{1}{2}(6+8)2\right]-\left[\frac{1}{2}(8+4)2\right]-\left[\frac{1}{2}(4+8)4\right]-\left[\frac{1}{2}(8+6)\right]$<br>$=-14-12-24-7$<br>$=-57$<br>**<u>Method 2</u>**<br>$\int_{-2}^{7} \mathrm{f}(x)\ \mathrm{d}x$<br>$=\int_{4}^{6}(-x-2)\ \mathrm{d}x+\int_{0}^{2}(2x-8)\ \mathrm{d}x+\int_{2}^{6}(-x-2)\ \mathrm{d}x+\int_{0}^{1}(2x-8)\ \mathrm{d}x$<br>$=-57$<br>**<u>Method 3</u>**<br>Since areas of triangles A and B are identical,<br>$\int_{-2}^{7} \mathrm{f}(x)\ \mathrm{d}x=-[\text{Area of rectangle + area of triangle}]$<br>$=-\left[9\times 6+\frac{1}{2}\times 2\times 3\right]$<br>$=-57$<br>Note: Areas of triangles A and B are identical.<br><br> |

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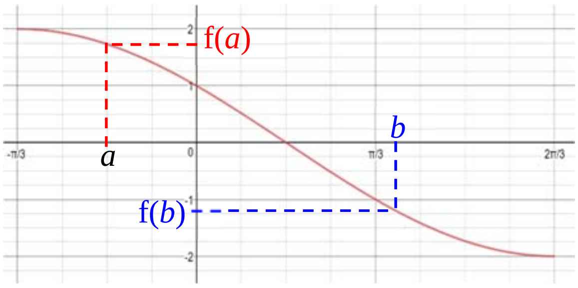
| **2(i)** | $R=\sqrt{1+3}=2$<br>$\alpha=\tan^{-1}\sqrt{3}=\dfrac{\pi}{3}$<br>$\cos x-\sqrt{3}\sin x=2\cos\left(x+\dfrac{\pi}{3}\right)$<br>$R_{\mathrm{f}}=[-2,2]$ |
| **2(ii)** | $\mathrm{f}: x \mapsto \cos x-\sqrt{3}\sin x,\quad -\dfrac{\pi}{3}\le x\le\dfrac{2\pi}{3}$<br>**<u>Method 1:</u>**<br>$\mathrm{f}(x)=2\cos\left(x+\dfrac{\pi}{3}\right) \quad\Rightarrow\quad \mathrm{f}'(x)=-2\sin\left(x+\dfrac{\pi}{3}\right)$<br>$-\dfrac{\pi}{3}\le x\le\dfrac{2\pi}{3} \quad\Rightarrow\quad 0\le x+\dfrac{\pi}{3}\le\pi$<br>For $0< x+\dfrac{\pi}{3}<\pi \quad\Rightarrow\quad \sin\left(x+\dfrac{\pi}{3}\right)>0$<br>$\mathrm{f}'(x)<0$<br>When $x+\dfrac{\pi}{3}=0$ *or* $\pi$, $\quad \sin\left(x+\dfrac{\pi}{3}\right)=0$<br>$\Rightarrow \quad \mathrm{f}'(x)=0$<br>The end-points are stationary points.<br>Thus f($x$) decreases as $x$ increases.<br><br>**<u>Method 2:</u>**<br>Sketch the graph of $y=\mathrm{f}(x)$.<br><br>From the graph, for $-\dfrac{\pi}{3}\le a<b\le\dfrac{2\pi}{3}$,<br>$\mathrm{f}(a)>\mathrm{f}(b)$.<br>Thus f($x$) decreases as $x$ increases<br>**<u>Method 3:</u>**<br>For $-\dfrac{\pi}{3}\le a<b\le\dfrac{2\pi}{3}$<br>$\mathrm{f}(a)-\mathrm{f}(b)=2\cos\left(a+\dfrac{\pi}{3}\right)-2\cos\left(b+\dfrac{\pi}{3}\right)$<br>$=-4\sin\left[\dfrac{1}{2}\left(a+b+\dfrac{2\pi}{3}\right)\right]\sin\left[\dfrac{1}{2}(a-b)\right]$<br>$=-4\sin\left(\dfrac{a+b}{2}+\dfrac{\pi}{3}\right)\sin\left[\dfrac{1}{2}(a-b)\right]$<br>Since $-\dfrac{\pi}{3}<\dfrac{a+b}{2}<\dfrac{2\pi}{3}, \quad\Rightarrow\quad 0<\dfrac{a+b}{2}+\dfrac{\pi}{3}<\pi$<br>$\sin\left(\dfrac{a+b}{2}+\dfrac{\pi}{3}\right)>0$ |

| | |
|---|---|
| | Since $-\frac{\pi}{3} \le a < b \le \frac{2\pi}{3}$, $\Rightarrow$ $-\frac{\pi}{3} - \frac{2\pi}{3} \le a - b < 0$<br>$-\frac{\pi}{2} \le \frac{a-b}{2} < 0$<br>$-1 \le \sin\left(\frac{a-b}{2}\right) < 0$<br>Thus $\mathrm{f}(a) - \mathrm{f}(b) > 0$<br>i.e. For $a < b$, $\mathrm{f}(a) > \mathrm{f}(b)$<br>Therefore, f($x$) decreases as $x$ increases. |
| **2(iii)** | Let<br>$y = 2\cos\left(x + \frac{\pi}{3}\right)$<br>$\cos\left(x + \frac{\pi}{3}\right) = \frac{y}{2}$<br>$x = \cos^{-1}\left(\frac{y}{2}\right) - \frac{\pi}{3}$<br>$\mathrm{f}^{-1}(x) = \cos^{-1}\left(\frac{x}{2}\right) - \frac{\pi}{3}$, $-2 \le x \le 2$ |
| **2(iv)** | $\mathrm{fg}(x) = x - 3$, $1 \le x \le 3$<br>$\mathrm{f}^{-1}\mathrm{fg}(x) = \mathrm{f}^{-1}(x-3)$<br>$\mathrm{g}(x) = \cos^{-1}\left(\frac{x-3}{2}\right) - \frac{\pi}{3}$<br><br>**<u>Alternatively</u>**<br>$\mathrm{f}(\mathrm{g}(x)) = x - 3$, $1 \le x \le 3$<br>$2\cos\left(\mathrm{g}(x) + \frac{\pi}{3}\right) = x - 3$<br>$\mathrm{g}(x) + \frac{\pi}{3} = \cos^{-1}\left(\frac{x-3}{2}\right)$<br>$\mathrm{g}(x) = \cos^{-1}\left(\frac{x-3}{2}\right) - \frac{\pi}{3}$ |
| **3(i)** | $\frac{\mathrm{d}V}{\mathrm{d}t} = 1200 - A\sqrt{h}$, where $A$ is a positive constant.<br><br>$V = \pi r^2 h = 3000h$ $\Rightarrow$ $\frac{\mathrm{d}V}{\mathrm{d}t} = 3000\frac{\mathrm{d}h}{\mathrm{d}t}$<br>$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{3000}\left(1200 - A\sqrt{h}\right)$<br>$= \frac{1200}{3000} - \frac{A}{3000}\sqrt{h}$<br>$= 0.4 - k\sqrt{h}$ where $k = \frac{A}{3000} > 0$ |

**3(ii)**

When $h=36$, $A\sqrt{36}=360 \quad \Rightarrow \quad A=60$

$$k=\frac{60}{3000}=0.02$$

**3(iii)**

$h=(20-x)^2$ L L (1)

$\Rightarrow \quad \frac{\mathrm{d}h}{\mathrm{d}t}=-2(20-x)\frac{\mathrm{d}x}{\mathrm{d}t}$ L L (2)

Substituting (1) & (2) into $\frac{\mathrm{d}h}{\mathrm{d}t}=0.4-0.02\sqrt{h}$,

$$-2(20-x)\frac{\mathrm{d}x}{\mathrm{d}t}=0.4-0.02(20-x)$$

$$(x-20)\frac{\mathrm{d}x}{\mathrm{d}t}=0.01x \text{ L L } (*)$$

$$\int \frac{x-20}{x}\,\mathrm{d}x=\int 0.01\,\mathrm{d}t$$

$$\int 1-\frac{20}{x}\,\mathrm{d}x=\int 0.01\,\mathrm{d}t$$

$$x-20\ln|x|=0.01t+C$$

$$t=100\left[x-20\ln|x|-C\right]$$

$$=100\left[\left(20-\sqrt{h}\right)-20\ln\left|20-\sqrt{h}\right|-C\right]$$

$$=100\left[-\sqrt{h}-20\ln\left|20-\sqrt{h}\right|+C'\right] \quad \text{where } C'=20-C$$

When $t=0, h=0 \quad \Rightarrow \quad C'=20\ln 20$

$$\frac{\mathrm{d}h}{\mathrm{d}t}>0 \Rightarrow 0.4-0.02\sqrt{h}>0$$

$$0.02\left(20-\sqrt{h}\right)>0$$

$$20-\sqrt{h}>0$$

Thus $t=100\left[-\sqrt{h}+20\ln\frac{20}{20-\sqrt{h}}\right]$

**3(iv)**

When $h=100$, $t=100\left[-10+20\ln 2\right]=386.294$

It takes 6 mins and 26 seconds for the height to reach 100 cm.

**4**

**<u>Method 1:</u>**

$\mathrm{g}(x)=a\mathrm{f}(0.5x-1)$

Sequence of transformations involved:

A: A scaling of the graph of $y=\mathrm{f}(x)$ parallel to the $y$-axis with scale factor $a$

B: A translation of 1 unit in the positive $x$-direction

C: A scaling parallel to the $x$-axis with scale factor 2.

**<u>Method 2:</u>**

$\mathrm{g}(x)=a\mathrm{f}\left(0.5(x-2)\right)$

Sequence of transformations involved:

A: A scaling of the graph of $y=\mathrm{f}(x)$ parallel to the $y$-axis with scale factor $a$

B: A scaling parallel to the $x$-axis with scale factor 2.

C: A translation of 2 units in the positive $x$-direction

Acceptable alternative order of sequence: BCA.

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| **4(i)** | Volume<br>$=\pi\int_0^2 \frac{a^2}{1+(0.5x-1)^2}-\frac{1}{1+x^2}\ \mathrm{d}x$<br>$=\pi\left[\frac{a^2}{0.5}\tan^{-1}(0.5x-1)-\tan^{-1}x\right]_0^2$<br>$=\pi\left[\left(2a^2\tan^{-1}0-\tan^{-1}2\right)-\left(2a^2\tan^{-1}(-1)-\tan^{-1}0\right)\right]$<br>$=\pi\left(\frac{\pi}{2}a^2-\tan^{-1}2\right)$ |
| **4(ii)** | $y=\frac{1}{\sqrt{1+x^2}} \Rightarrow x^2=\frac{1}{y^2}-1$<br>Volume of 2nd sculpture<br>$=Vol\ of\ cylinder\ +\ \pi\int_{\frac{1}{\sqrt{5}}}^{1}\frac{1}{y^2}-1\ \mathrm{d}y$<br>$=\pi\left(2^2\right)\frac{1}{\sqrt{5}}+\pi\left[-\frac{1}{y}-y\right]_{\frac{1}{\sqrt{5}}}^{1}$<br>$=\pi\left(\frac{4}{\sqrt{5}}\right)+\pi\left[(-2)-\left(-\sqrt{5}-\frac{1}{\sqrt{5}}\right)\right]$<br>$=2\pi\left(\sqrt{5}-1\right)$ |
| **4(iii)** | $\pi\left(\frac{\pi}{2}a^2-\tan^{-1}2\right)\geq 50\left[2\pi\left(\sqrt{5}-1\right)\right]$<br>$a^2\geq\frac{2}{\pi}\left[100\left(\sqrt{5}-1\right)+\tan^{-1}2\right]$<br>$a\geq 8.91$ since $a>2$<br>Smallest integer value of $a$ is 9 |
| **5** | $\mathrm{P}(X\leq 1)=0.15$<br>$\binom{25}{0}(1-p)^{25}+\binom{25}{1}p(1-p)^{24}=0.15$<br>$(1-p)^{25}+25p(1-p)^{24}=0.15$<br>$\left[\text{or } (1-p)^{24}(1+24p)=0.15\right]$<br><br>Using GC, $p=0.12865$<br>$=0.129$ (to 3 s.f.)<br><br>$\mathrm{Var}(X)=np(1-p)$<br>$=25(0.12865)(1-0.12865)$<br>$=2.80$ (to 3 s.f.) |

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| **6(i)** | AA EE I<br>PP R C T<br><br>c_c_c_c_c_<br>_c_c_c_c_c<br><br>Number of ways = $\left(\frac{5!}{2!2!}\right)\left(\frac{5!}{2!}\right)\times 2! = 3600$ |
| **6(ii)** | E_ _ E _ _ _ _ _ _<br><br>Case 1 :2 As<br>No of ways $= \frac{7!}{2!} = 2520$<br><br>Case 2 : 1A with 1P<br>No of ways= $2!7! = 10080$<br><br>Case 3 : 1A without P<br>No of ways= $\left({}^4C_1 \times 2!\right)\times\frac{7!}{2!} = 20160$<br>Total no of ways = 32760 |
| **7(i)** | Given $\text{Var}(X) = 0.75$,<br>$\text{E}\left(X^2\right) = (1)^2(3p) + (2)^2(2p) + (3)^2(p) = 20p$<br>$\text{E}(X) = 1(3p) + (2)(2p) + (3)(p) = 10p$<br>$20p - (10p)^2 = 0.75$<br>$100p^2 - 20p + 0.75 = 0$<br>$p = \frac{1}{20}$ *or* $p = \frac{3}{20}$ (Reject since $0 < p < \frac{1}{10}$)<br>$\therefore p = \frac{1}{20}$ (*or* 0.05), $\text{E}(X) = \frac{1}{2}$ *or* 0.5 |
| **7(ii)** | $S = X_1 + X_2 + X_3 + ... + X_n$<br>Since $n$ is large, **by Central Limit Theorem,**<br>$S : N(0.5n, 0.75n)$ approximately.<br>**<u>Method 1: Algebraic method</u>**<br>$\text{P}(S > 150) \geq 0.75$<br>$\text{P}\left(Z < \frac{150 - 0.5n}{\sqrt{0.75n}}\right) \leq 0.25$<br>$\frac{150 - 0.5n}{\sqrt{0.75n}} \leq -0.6744897$<br>$150 - 0.5n \leq -0.6744897\sqrt{0.75n}$<br>$0.5n - 0.6744897\sqrt{0.75n} - 150 \geq 0$<br>$\sqrt{n} \leq -16.746$ (reject since $\sqrt{n} > 0$) *or* $\sqrt{n} \geq 17.914$<br>$n \geq 320.93$<br><br>Thus $\{n : n \in ¢^{+}, \ n \geq 321\}$ |

**Method 2: Using GC (table)**

$\mathrm{P}(S > 150) \geq 0.75$

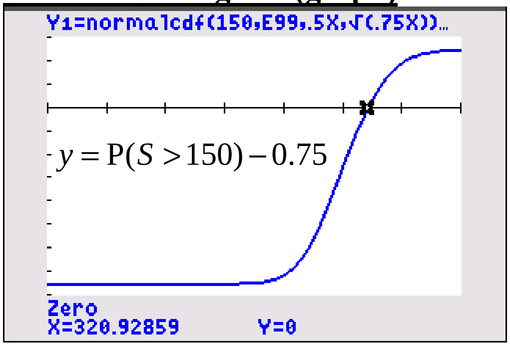
When $n = 320$, $\mathrm{P}(S > 150) = 0.7407 < 0.75$

When $n = 321$, $\mathrm{P}(S > 150) = 0.7507 > 0.75$

When $n = 322$, $\mathrm{P}(S > 150) = 0.7605 > 0.75$

Thus $\{n : n \in ¢^{+}, \ n \geq 321\}$

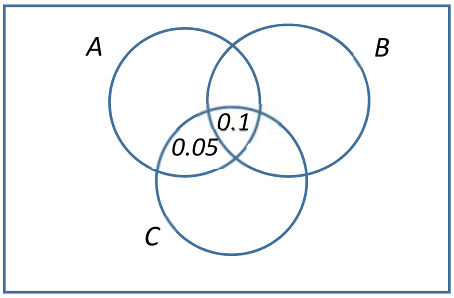
**Method 3: Using GC (graph)**




From the graph, $n \geq 320.92859$

Thus $\{n : n \in ¢^{+}, \ n \geq 321\}$

**8(i)**




Given $\mathrm{P}(B \mid C) = 0.5$,

$$\mathrm{P}(B \mid C) = \frac{\mathrm{P}(B \cap C)}{\mathrm{P}(C)}$$

$$0.5 = \frac{\mathrm{P}(B \cap C)}{0.35}$$

$\mathrm{P}(B \cap C) = 0.175$

$\mathrm{P}(A' \cap B \cap C) = 0.175 - 0.1$

$= 0.075$

**8(ii)**

Since A and B are independent events,

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A) \times \mathrm{P}(B)$

$= 0.5 + 0.45 - 0.5(0.45)$

$= 0.725$

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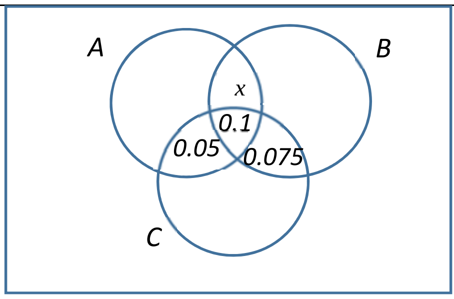
| **8(iii)** | <br>$\text{P}(A'\cap B'\cap C')$<br>$=1-\text{P}(A\cup B\cup C)$<br>$=1-0.35-(0.5-0.15-x)-(0.45-0.175-x)-x$<br>$=0.025+x$<br><br>Max $x$ occurs when $x+0.1+0.075=0.45$<br>$x=0.275$,<br>greatest possible value of $\text{P}(A'\cap B'\cap C')=0.3$<br>when $x=0$,<br>least possible value of $\text{P}(A'\cap B'\cap C')=0.025$ |
| **9(i)** | $H_0:\ \mu=250$<br>$H_1:\ \mu<250$<br>where $\mu$ represents the population mean volume of tomato juice per bottle. |
| **9(ii)** | Assume that $H_0$ is true. Since $n=50$ is large, by CLT, $\overline{X}\sim N\left(250,\dfrac{10^2}{50}\right)$ approximately.<br>Using GC, $p$-value = 0.0385498886 $\approx 0.0385$ (3 s.f.)<br><br>There is 0.0385 probability of drawing a random sample of 50 bottles of tomato juice and obtaining a sample mean volume of 247.5 ml or less, assuming that the population mean volume is 250 ml. |
| **9(iii)** | No, it is not necessary to assume a normal distribution for the test to be valid, since $n=50$ is large, Central Limit Theorem can be applied for $\overline{X}$ to be normally distributed. |
| **9(iv)** | Unbiased estimate of population mean of smaller bottles of tomato juice<br>$=\dfrac{10757}{60}=179.283\approx 179$ (3 s.f.)<br>Unbiased estimate of population variance<br>$=\dfrac{1}{59}\left[1931597-\dfrac{10757^2}{60}\right]$<br>$=51.63022599$<br>$\approx 51.6$ (3 s.f.) |
| **9(iv)** | $H_0:\mu=\mu_0$<br>$H_1:\mu\neq\mu_0$<br>Test at 5% significance level.<br>Assuming $H_0$ is true, since $n=60$ is large, by CLT,<br>$\overline{Y}\sim N\left(\mu_0,\dfrac{51.6302}{60}\right)$ approximately |

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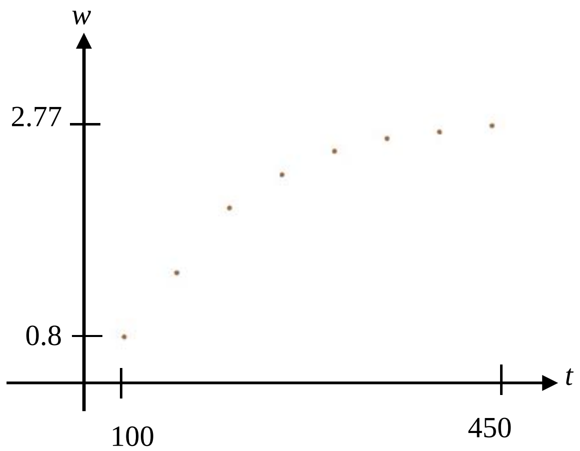
| | Test statistic $Z=\dfrac{\overline{Y}-\mu_0}{s/\sqrt{60}} \sim N(0,1)$<br>$H_0$ is not rejected $\Rightarrow$ The test statistic lies outside the critical region.<br>$-1.959963986<\dfrac{179.283-\mu_0}{\sqrt{\dfrac{51.630226}{60}}}<1.959963986 \quad$ L L (*)<br>$-1.959963986\left(\sqrt{\dfrac{51.630226}{60}}\right)<179.283-\mu_0<1.959963986\left(\sqrt{\dfrac{51.630226}{60}}\right)$<br>$-1.8181281<179.283-\mu_0<1.8181281$<br>$-1.8181281-179.283<-\mu_0<1.8181281-179.283$<br>$177.4652052<\mu_0<181.1014614$<br>$177.5<\mu_0<181.1$ (1 d.p.) |
| **10(i)** | $r$ = 0.925 (3 s.f.)<br>Acceptable answers:<br>• As the pmcc value is close to 1, indicating a **strong positive <u>linear</u>** correlation, it suggests that a linear model is appropriate.<br>• A linear model with positive linear correlation would suggest that the weight loss may exceed 100%, which is impossible. Thus a linear model is not appropriate. |
| **10(ii)** | <br><br>(A) $w=a+b\ln t$, from the scatter diagram, as length of time increases, percentage of weight loss also **increases at a decreasing rate**. |
| **10(iii)** | $w=-5.31+1.35\ln t$<br>The product moment correlation coefficient between ln*t* and *w* is $r=0.9828402622$<br>$=0.983 \quad (3s.f.)$<br>This pmcc value of 0.983 is closer to 1 than the earlier pmcc value of 0.925, indicating stronger positive linear correlation between *w* and ln*t* compared to the linear model. |

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| **10(iv)** | $2.4 = -5.3074 + 1.3522 \ln t$<br>$\ln t = \dfrac{2.4 + 5.3074}{1.3522}$<br>$t = e^{5.6999}$<br>$t = 298.84 \approx 299$ (3 s.f.)<br><br>This estimate is reliable since<br>(1) The estimate is an interpolation, because *w* = 2.4 is within the data range of *w*.<br>(2) the product moment correlation coefficient between ln *t* and *w* is $r = 0.983$ which is very close to 1, showing a very strong positive linear correlation between ln *t* and *w*. |
| **10(v)** | 1 day = 24 hours<br><br>$w = -5.31 + 1.35 \ln(24t)$ |
| **11 (a)(i)** | $\mu = \dfrac{40+50}{2} = 45$<br>$P(T < 40) = 0.36$<br>$P\left(Z < \dfrac{40-45}{\sigma}\right) = 0.36$<br>$\dfrac{-5}{\sigma} = -0.3584588$<br>$\sigma = 13.9486 = 13.9$ (to 3 s.f.) |
| **11 (a)(ii)** | $P(T \le 50) = 1 - 0.36 = 0.64$<br><br>Let *C* be the number of houses, out of 18, for which it takes at most 50 min to install an electricity meter.<br>$C \sim B(18, 0.64)$<br><br>$P(5 \le C < 10) = P(C \le 9) - P(C \le 4)$<br>$= 0.160$ (3 s.f.) |
| **(b)(i)** | Let *E* be the electricity consumption of the households in Central City<br>$E \sim N(520,\ 35^2)$<br>$P(E > 500) = 0.7161454588 = 0.716$ (to 3 s.f.) |
| **(b)(ii)** | Required probability<br>$= [P(E < 500)]^2$<br>$= [1 - 0.7161454588]^2$<br>$= 0.0806$ (to 3 s.f.) |
| **(b)(iii)** | Part (ii) is a subset of the event where the total electricity consumption for two randomly chosen households is less than 1000 kWh. |

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| **(b)(iv)** | Let *S* be the electricity consumption of the households in Star City.<br>$S \sim \text{N}(475,\ 25^2)$<br>$X = 0.18E \quad \Rightarrow \quad \text{E}(X) = 0.18 \times 520 = 93.6$<br>$\text{Var}(X) = 0.18^2 \times 35^2 = 39.69$<br>$Y = 0.15S \quad \Rightarrow \quad \text{E}(Y) = 0.15 \times 475 = 71.25$<br>$\text{Var}(Y) = 0.15^2 \times 25^2 = 14.0625$<br><br>$\text{E}(4Y - 3X) = 4 \times 71.25 - 3 \times 93.6 = 4.2$<br>$\text{Var}(4Y - 3X) = 4^2 \times 14.0625 + 3^2 \times 39.69$<br>$= 582.21$<br>$4Y - 3X \sim \text{N}(4.2\ ,\ 582.21)$<br>$\text{P}(4Y - 3X > 7) = 0.454$ (3 s.f.)<br><br>It means that there is 0.454 probability that 4 times the electricity bill of a randomly chosen household in Star City exceeds 3 times the electricity bill of a randomly chosen household in Central City by more than $7. |