MERIDIAN JUNIOR COLLEGE
JC2 Preliminary Examination
Higher 2

H2 Mathematics 9758/01

Paper 1 **14 September 2018**

3 Hours

Additional Materials: Writing paper
Graph Paper
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 Express $\dfrac{-x^2+11x-11}{x^2-4x+4}+3$ as a single simplified fraction.

Hence, without using a calculator, solve the inequality

$$\frac{x^2-11x+11}{x^2-4x+4}<3. \qquad [5]$$

2 **(a)** Interpret geometrically what $\mathbf{a}\times\mathbf{b}$ means, given that $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. [1]

(b) Show that a formula for the area of triangle OAB can be given as $k\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|$, where k is a constant to be determined. [2]

Hence give the geometrical meaning of $\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|$ in relation to an appropriate quadrilateral. [1]

3 **(i)** Show that $1-\mathrm{e}^{in\theta}=\mathrm{e}^{\frac{1}{2}in\theta}\left(-2\mathrm{i}\sin\dfrac{n\theta}{2}\right)$, and $1+\mathrm{e}^{in\theta}=\mathrm{e}^{\frac{1}{2}in\theta}\left(2\cos\dfrac{n\theta}{2}\right)$ where $n\in$ ¡ .

[2]

(ii) It is given that $z=\mathrm{e}^{\mathrm{i}\theta}$, where $0\le\theta\le\dfrac{\pi}{2}$, using **(i)**, show that

$$\left|1-z+z^2-z^3\right|=4\sin\frac{\theta}{2}\cos\theta. \qquad [4]$$

4 Find

(a) $\displaystyle\int x\sqrt{5-x^2}\ \mathrm{d}x$, [2]

(b) $\displaystyle\int \sin(\ln x)\ \mathrm{d}x$ where $x>0$, [3]

(c) the exact value of $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{12}}\cos x\left|\sin x\right|\ \mathrm{d}x$. [4]

5 It is given that

$$\mathrm{f}: x \mapsto \left|\frac{1}{x-3}\right|, \text{ where } x \in \mathbb{R}, x \neq 3,$$
$$\mathrm{g}: x \mapsto \ln x, \text{ where } x \in \mathbb{R}, x > 0.$$

(i) Explain why the composite function gf exists and find gf in a similar form. [3]

(ii) Explain why f does not have an inverse. [1]

(iii) If the domain of f is further restricted to $x < k$, state the maximum value of k such that f^{-1} will exist. Hence find f^{-1} in a similar form. [4]

(iv) State the geometrical relationship between f and f^{-1}. [1]

6 **(a)** **(i)** Using standard series from the List of Formulae (MF26), find the first three non-zero terms of the Maclaurin's series for $y = \dfrac{1}{\sqrt{1+\ln(1+3x)}}$. [3]

(ii) Deduce the approximate value of $\int_0^1 y \, \mathrm{d}x$. Explain why the approximation is not good. [2]

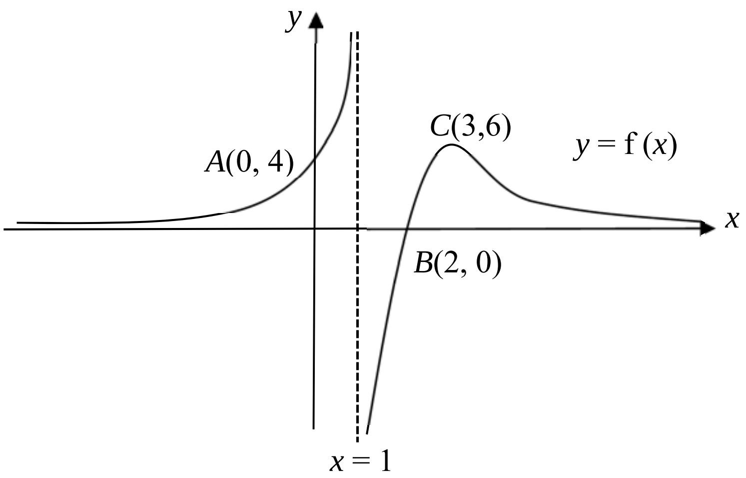
(iii) State the equation of the tangent to the curve $y = \dfrac{1}{\sqrt{1+\ln(1+3x)}}$ at $x = 0$. [1]

(b) Show that, when x is sufficiently small for x^3 and higher powers of x to be neglected,

$$\frac{\cos 2x}{1-\sin x} \approx a + bx + cx^2,$$

where a, b and c are constants to be determined. [3]

7 **(a)**



The diagram above shows the curve $y = \mathrm{f}(x)$ with a maximum point at $C(3, 6)$. The curve crosses the axes at the points $A(0, 4)$ and $B(2, 0)$. The lines $x = 1$ and $y = 0$ are the asymptotes of the curve.

Sketch on separate diagrams, the graphs of

(i) $y = \mathrm{f}'(x)$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

stating clearly, where applicable, the equations of the asymptotes, the axial intercepts and the coordinates of the points corresponding to A, B and C.

(b) Show that the equation $y = \dfrac{3x^2 - 6x}{x-1}$ can be written as $y = A(x-1) + \dfrac{B}{(x-1)}$, where A and B are constants to be found. Hence state a sequence of transformations that will transform the graph of $y = \dfrac{1}{x} - x$ to the graph of $y = \dfrac{3x^2 - 6x}{x-1}$. [4]

8 The points A and B have position vectors $-2\mathbf{i}-7\mathbf{j}+3\mathbf{k}$ and $\mathbf{j}+\mathbf{k}$ respectively. The plane p has equation $x-y=5$.

(i) Find a vector equation of line l passing through the points A and B. [2]

(ii) Find the acute angle between l and p. [2]

(iii) Verify that the point A lies on the plane p. Given that the point C is the reflection of the point B in the plane p, describe the shape formed by the points A, B and C. [2]

(iv) Find a vector equation of the line which is a reflection of the line l in the plane p. [4]

9 An analyst is studying how the population of bluegill fish in a lake changes over time. By considering their natural birth and death rates, he found that the rate of change of the bluegill fish population is proportional to $6x-x^2$, where x is the population, in thousands, of bluegill fish in the lake at time t years. It is given that the initial population of the bluegill fish in the lake is 12000 and there were 8600 bluegill fish after 1 month.

(i) Show that $x=\dfrac{12\mathrm{e}^{12t\ln\frac{43}{26}}}{2\mathrm{e}^{12t\ln\frac{43}{26}}-1}$. [6]

(ii) Find the time taken for the bluegill fish population to decrease to 75% of its initial population. Leave your answer in years to 3 significant figures. [2]

(iii) Deduce the long term implication on the population of bluegill fish in a lake following this model and state an assumption for this model to hold in the long term. [3]

10 **(a)** The sum of the first n terms of a sequence $\{u_n\}$ is given by $S_n = kn^2 - 3n$, where k is a non-zero real constant.

(i) Prove that the sequence $\{u_n\}$ is an arithmetic sequence. [3]

(ii) Given that u_2, u_3 and u_6 are consecutive terms in a geometric sequence, find the value of k. [3]

(b) A zoology student observes jaguars preying on white-tailed deer in the wild. He observes that when a jaguar spots its prey from a distance of d m away, it starts its chase. At the same time, the white-tailed deer senses danger and starts escaping.

He models the predator-prey movements as follows:

The jaguar starts its chase with a leap distance of 6 m. Subsequently, each leap covers a distance of 0.1 m less than its preceding leap.

The white-tailed deer starts its escape with a leap distance of 9 m. Subsequently, each leap covers a distance of 5% less than its preceding leap.

(i) Find the total distance travelled by a white-tailed deer after n leaps. Deduce the maximum distance travelled by a white-tailed deer. [3]

(ii) Assume that both predator and prey complete the same number of leaps in the same duration of time. Given that $d = 11$ m, find the least value of n for a jaguar to catch a white-tailed deer within n leaps. [3]

11 A curve C has parametric equations

$$x = 1 - \cos^3\theta, \qquad y = 1 - 3\sin\theta\cos^2\theta, \qquad \text{for } 0 \le \theta \le \frac{1}{2}\pi.$$

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\tan\theta - \cot\theta$. [3]

(ii) Show that the exact coordinates of the turning point is at $\left(1-\left(\sqrt{\dfrac{2}{3}}\right)^3,\ 1-\dfrac{2}{\sqrt{3}}\right)$ and explain why it is a minimum. [6]

(iii) Show that the equation of the normal to the curve at $\theta = \dfrac{\pi}{6}$ is $y = 1.73205x - 0.73205$ (rounded off to 5 decimal places) and hence evaluate the area of the region bounded by C, the y-axis and the normal to the curve at $\theta = \dfrac{\pi}{6}$. [6]

End of Paper

MERIDIAN JUNIOR COLLEGE
JC2 Preliminary Examination
Higher 2

H2 Mathematics 9758/02

Paper 2 **19 September 2018**

3 Hours

Additional Materials: Writing paper
Graph Paper
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

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Section A: Pure Mathematics [40 marks]

1 **(a)** Consider the polynomial $\mathrm{P}(z) = a_0 + a_1 z + a_2 z^2$, where $a_0, a_1, a_2 \in \mathbb{R}, a_2 \neq 0$. Show that if the equation $\mathrm{P}(z) = 0$ has a complex root *w*, then its complex conjugate w^* must also be a root to the equation. [3]

(b) The polynomial $\mathrm{P}(z)$ has real coefficients. The equation $\mathrm{P}(z) = 0$ has a root $z_1 = -a + a\mathrm{i}$, where *a* is a positive real number.

(i) Write down a second root, z_2, in terms of *a*. [1]

(ii) Hence sketch the points Z_1 and Z_2, representing z_1 and z_2 respectively, on an Argand diagram. [2]

(iii) Deduce the area of OZ_1Z_2, in terms of *a*. [1]

2 **(i)** Using the formulae for $\cos(A \pm B)$, prove that

$$\cos\left(r+\frac{1}{2}\right)\theta - \cos\left(r-\frac{1}{2}\right)\theta \equiv -2\sin r\theta \sin\frac{1}{2}\theta .$$ [2]

(ii) Hence show that $\sum_{r=1}^{n} \sin r\theta = -\frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta\left[\cos\left(n+\frac{1}{2}\right)\theta - \cos\frac{1}{2}\theta\right]$ [3]

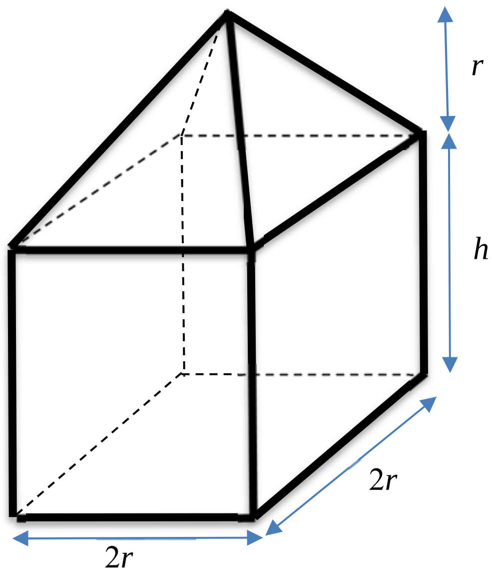
(iii) Using the result in **(ii)**, find the exact value for $\sum_{r=5}^{62} \sin(r-2)\theta$ when $\theta = \frac{\pi}{3}$. [4]

3 **(a)** A semicircle has radius r cm, perimeter P cm and area A cm^2. Show that

$$\frac{\mathrm{d}P}{\mathrm{d}A} = \frac{2+\pi}{\pi r}.$$

Determine the exact value of the radius when the area of the semicircle is increasing at a constant rate of 3 cm^2/s and the perimeter is increasing at a constant rate of $\frac{3}{5}$ cm/s. [4]

(b) An architectural firm wants to make a model of a greenhouse as shown.



The model is to be made up of three parts.

- The roof is modelled by a pyramid with a square base $2r$ cm by $2r$ cm and whose apex is r cm directly above the center of its base.
- The walls are modelled by four rectangles measuring $2r$ cm by h cm.
- The floor is modelled by a square measuring $2r$ cm by $2r$ cm.

The three parts are joined together as shown in the diagram. The model is made of material of negligible thickness.

It is given that the external surface area of the model has a fixed value of 160 cm^2. Show that $V = 80r - \left(\frac{2}{3} + 2\sqrt{2}\right)r^3$. Hence, using differentiation, find the value of r which gives the maximum value of V. (You do not need to verify that the volume is a maximum for this value of r.) [Volume of Pyramid $= \frac{1}{3} \times$ base area $\times$ height] [7]

4 **(a)** **(i)** Given that f is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{f}\left(\frac{1}{n}\right)+\mathrm{f}\left(\frac{2}{n}\right)+\ldots+\mathrm{f}\left(\frac{n}{n}\right)\right\}$$

is $\int_0^1 \mathrm{f}(x)\mathrm{d}x$. [2]

(ii) Hence evaluate $\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{e}^{\frac{3}{n}}+\mathrm{e}^{\frac{6}{n}}+\ldots+\mathrm{e}^{3}\right\}$, leaving your answer in exact form. [3]

(b) The function f is defined by

$$\mathrm{f}(x)=\begin{cases}(x+1)^2+a & \text{for } -1<x\le 1,\\ (4+a)(2-x) & \text{for } 1<x\le 2,\end{cases}$$

where a is a positive real constant and that $\mathrm{f}(x)=\mathrm{f}(x+3)$ for all real values of x.

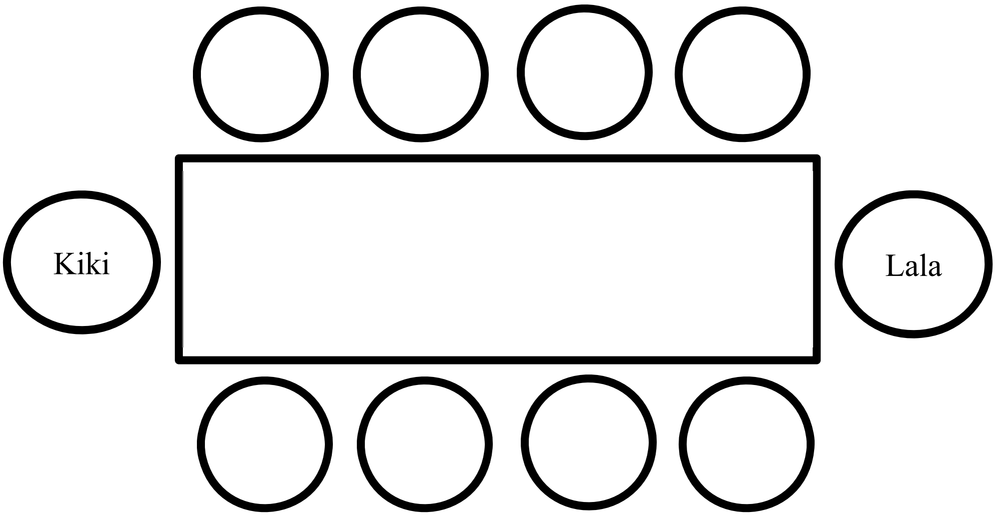
(i) Evaluate $\mathrm{f}(-41)$ and $\mathrm{f}(2018)$. [2]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $-4<x\le 6$. [3]

(iii) Hence find the value of $\int_{-4}^{4}\mathrm{f}(x)\,\mathrm{d}x$ in terms of a. [3]

Section B: Statistics [60 marks]

5 Kiki and Lala host a dinner for four other married couples. They sit at a rectangular table with Kiki and Lala at the left and right ends of the table respectively as shown in the diagram below.



Find the number of ways to seat the four couples such that

(i) there are no restrictions. [1]

(ii) each married couple is seated directly facing each other on opposite sides. [2]

(iii) each married couple is seated directly facing each other on opposite sides and two particular ladies cannot be seated next to each other on the same side. [3]

6 A bag contains five balls numbered 1 to 5. A game consists of a player taking two balls from the bag consecutively, with replacement. The number on each ball is noted down. His score X is found by taking the absolute value of the difference between the two numbers.

(i) Obtain the probability distribution of X. [2]

(ii) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [2]

A player pays \$ a to play this game. He wins an amount (in dollars) corresponding to twice his score. Determine the range of values of a if the player is expected to make a loss. [2]

7 During the National Day Rally in 2017, PM Lee cited that an average of one in nine Singaporeans has diabetes. Suppose we examine 24 patients who are randomly selected from a polyclinic. Let X be the number of these patients who have diabetes.

(i) State, in the context of this question, two assumptions needed to model X by a binomial distribution. [2]

(ii) Explain why one of the assumptions stated in part **(i)** may not hold in this context. [1]

Assume now that these assumptions do in fact hold.

(iii) Find the probability that in a sample of 25 randomly selected patients, the 25^{th} patient is the third patient who has diabetes. [2]

(iv) Find the least value of n such that the probability of having at most n diabetic patients in the random sample of 24 patients is greater than 0.9. [3]

8 A circular disc is divided into ten equal sectors where five sectors are coloured blue, three sectors are coloured green and two sectors are coloured red.

In a game, a player is given a maximum of three times to spin a pointer pivoted at the centre of the disc.

- If the pointer lands on a blue sector, the player wins \$2 and the game continues.
- If the pointer lands on a green sector, the game ends and the player loses all of his winnings.
- If the pointer lands on a red sector in the first spin, the game ends and the player wins the grand prize of \$20.
- If the pointer lands on a red sector in the second or third spin, the player wins double the amount of his total winnings from all his previous spins, and the game continues. For example, if the pointer lands on a blue sector in the first spin and on a red sector in the second spin, the total winnings for the first two spins will be \$6.

(i) Construct a probability tree showing this information. [3]

(ii) Find the exact probability that the player wins \$12 or more when the game ends. [2]

(iii) Find the exact probability that the pointer in the second spin lands on a blue sector given that the player wins \$12 or more when the game ends. [2]

(iv) Suppose there is no limit to the number of times a player can spin, find the exact probability that the player has no spins in which the pointer lands on a red sector and wins nothing when the game ends. [2]

9 Based on past records, shoppers spend an average of 1.5 hours at MJ mall per visit. After a revamp and expansion to include more stores, the mall management wishes to conduct a hypothesis test to check if the duration that shoppers spend in the mall per visit has increased. A random sample of 50 shoppers were surveyed and the duration of time, x, in hours per visit is recorded. The sample sum is 82.5 hours and the sample variance is 0.425 hours^2.

(i) State appropriate hypotheses for the test and find the unbiased estimates of the population mean and variance. [3]

(ii) Explain why the mall management is able to carry out a hypothesis test without making any assumptions about the distribution of the duration shoppers spend at MJ mall per visit. [1]

(iii) Calculate the p-value and state its meaning in context of the question. Hence state the set of values of the level of significance α% for which the management's hypothesis is valid. [4]

(iv) The mall management believes that the mean spending per shopper per visit is \$150. A test at 5% level of significance found that there is significant evidence that the population mean spending is less than \$150. Using only this information, explain if the following statements are necessarily true, necessarily false, or neither necessarily true nor necessarily false.

(a) There is significant evidence at the 10% level of significance that the population mean spending is less than \$150.

(b) There is significant evidence at the 5% level of significance that the population mean spending differs from \$150. [2]

10 Biologists monitor the population of wild rabbits in a dry grassland region of the Australian outback. The population of wild rabbits, y, in hundreds, in month x are as follows.

Month x	1	3	6	9	11	13	15
Population (in hundreds) y	40.3	35.2	36.3	40.8	39.7	41.5	42.1

(i) Draw a scatter diagram showing these data. [1]

(ii) Suggest a possible reason, in context, why one of the data points does not seem to follow the trend. [1]

(iii) It is desired to predict the population of rabbits in the future months. Explain why, in this context, a linear model is not appropriate. [1]

After removing the outlier, the biologists decided to fit a model of the form $\ln(M-y)=a+bx$, where M is a suitable constant for the remaining data points. The product moment correlation coefficient between x and $\ln(M-y)$ is denoted by r. The following table gives values of r for some possible values of M.

M	44	45	46
r	− 0.945 943		− 0.945 994

(iv) Calculate the value of r for $M = 45$, giving your answer correct to 6 decimal places. [1]

(v) Use the table and your answer in part **(iv)** to suggest with a reason which of 44, 45 or 46 is the most appropriate value for M. [1]

(vi) Using the value for M, calculate the values of a and b, and use them to predict the population of the wild rabbits after 2 years, to the nearest whole number. [4]

(vii) Give an interpretation, in context, of the value of M. [1]

11 **(a)** The time T (in minutes) taken by Kathy to drive from her house to the nearest supermarket has a mean of 5 and a variance of 8.

Explain why T is unlikely to be normally distributed. [2]

(b) In this question you should state clearly the values of the parameters of any normal distribution you use.

In a supermarket, the masses in kilograms of chickens have the distribution $\mathrm{N}\left(2.4, 0.5^2\right)$ and the masses in kilograms of ducks have the distribution $\mathrm{N}\left(4.3, 1.8^2\right)$.

(i) Find the probability that the total mass of 2 randomly chosen chickens is more than 5kg. [2]

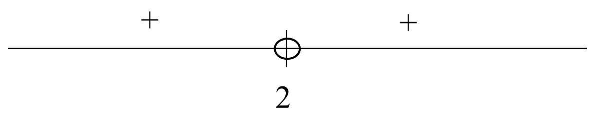
(ii) Find the probability that the mean mass of a randomly chosen duck and 2 randomly chosen chickens is more than 3.20kg. [3]

Chickens are sold at \$7 per kilogram and ducks are sold at \$13 per kilogram. The supermarket is conducting their annual sale and has a discount of 10% and 25% off the prices of chickens and ducks respectively.

(iii) Find the probability that the total cost of 2 randomly chosen chickens and a randomly chosen duck is less than \$80. [4]

End of Paper

2018 H2 MATH (9758/01) JC 2 PRELIM
SUGGESTED SOLUTIONS

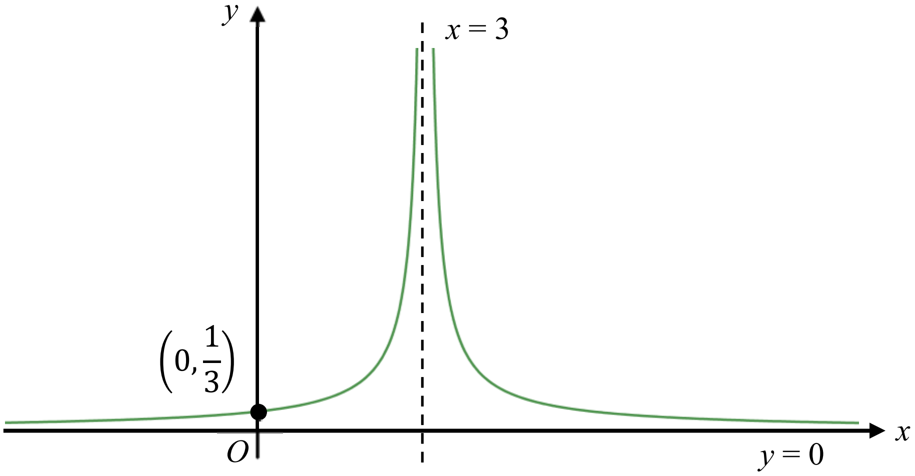
Qn	Solution
1	**Equations and Inequalities**
	$\dfrac{-x^2+11x-11}{x^2-4x+4}+3$ $=\dfrac{-x^2+11x-11+3\left(x^2-4x+4\right)}{x^2-4x+4}$ $=\dfrac{-x^2+11x-11+3x^2-12x+12}{x^2-4x+4}$ $=\dfrac{2x^2-x+1}{x^2-4x+4}$
	$\dfrac{x^2-11x+11}{x^2-4x+4}<3$ $\dfrac{-x^2+11x-11}{x^2-4x+4}+3>0$ $\dfrac{2x^2-x+1}{\left(x-2\right)^2}>0$ **Method 1** Since the discriminant of $2x^2-x+1$ is $\left(-1\right)^2-4\left(2\right)\left(1\right)=-7<0$, and the coefficient of $x^2=2$ is positive, $2x^2-x+1>0$ for all $x\in\mathbb{R}$. **Method 2** $2x^2-x+1=2\left[\left(x^2-\frac{1}{2}x+\frac{1}{2}\right)\right]=2\left(x-\frac{1}{4}\right)^2+\frac{7}{8}>0$ for all $x\in\mathbb{R}$. Therefore, solving $\dfrac{1}{\left(x-2\right)^2}>0$, we have $x\in\mathbb{R}$, $x\neq 2$. 

Qn	Solution
2	**Vectors**
(a)	$\mathbf{a}\times\mathbf{b}$ is a vector that is perpendicular to <u>both</u> **a** and **b.** **Other possible answers:** Normal vector of a plane that is <u>parallel</u> to both **a** and **b (**no marks for normal vector of a plane containing vectors **a** and **b)**
(b)	Area of triangle OAB $= 0.5\left(\left\|\overrightarrow{OA}\right\|\right)\left(\left\|\overrightarrow{OB}\right\|\right)\sin \angle AOB$ $= 0.5\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$
	$\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$ is the area of a parallelogram with <u>OA and OB as its adjacent sides</u>. **Alternative:** $\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$ is area of parallelogram $OACB$ (and include a diagram with O,A, a fourth point C, and B, with the 2 parallel sides also indicated clearly in diagram). Cannot accept $\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$ is area of sum of two triangles, because question asked for geometrical meaning in relation to an appropriate quadrilateral.

Qn	Solution
3	**Complex Numbers**
(i)	$1-\mathrm{e}^{\mathrm{i}n\theta}$ $=\mathrm{e}^{\frac{1}{2}\mathrm{i}n\theta}\left(\mathrm{e}^{-\frac{1}{2}\mathrm{i}n\theta}-\mathrm{e}^{\frac{1}{2}\mathrm{i}n\theta}\right)$ $=\mathrm{e}^{\frac{1}{2}\mathrm{i}n\theta}\left(\cos\frac{n\theta}{2}-\mathrm{i}\sin\frac{n\theta}{2}-\left(\cos\frac{n\theta}{2}+\mathrm{i}\sin\frac{n\theta}{2}\right)\right)$ $=\mathrm{e}^{\frac{1}{2}\mathrm{i}n\theta}\left(-2\mathrm{i}\sin\frac{n\theta}{2}\right)$ $1+\mathrm{e}^{\mathrm{i}n\theta}$ $=\mathrm{e}^{\frac{1}{2}\mathrm{i}n\theta}\left(\mathrm{e}^{-\frac{1}{2}\mathrm{i}n\theta}+\mathrm{e}^{\frac{1}{2}\mathrm{i}n\theta}\right)$ $=\mathrm{e}^{\frac{1}{2}\mathrm{i}n\theta}\left(\cos\frac{n\theta}{2}-\mathrm{i}\sin\frac{n\theta}{2}+\cos\frac{n\theta}{2}+\mathrm{i}\sin\frac{n\theta}{2}\right)$ $=\mathrm{e}^{\frac{1}{2}\mathrm{i}n\theta}\left(2\cos\frac{n\theta}{2}\right)$
(ii)	**Method 1** $\left\|1-z+z^2-z^3\right\|$ $=\left\|(1-z)+(1-z)z^2\right\|$ $=\left\|(1-z)(1+z^2)\right\|$ $=\left\|1-z\right\|\left\|1+z^2\right\|$ $=\left\|\mathrm{e}^{\frac{1}{2}\mathrm{i}\theta}\left(-2\mathrm{i}\sin\frac{\theta}{2}\right)\right\|\left\|\mathrm{e}^{\mathrm{i}\theta}(2\cos\theta)\right\|$ $=2\sin\frac{\theta}{2}(2\cos\theta)\quad\left(\because\left\|\mathrm{e}^{\frac{1}{2}\mathrm{i}\theta}\right\|=1,\ \left\|\mathrm{e}^{\mathrm{i}\theta}\right\|=1, \text{ and } 0\le\theta\le\frac{\pi}{2}\Rightarrow\sin\frac{\theta}{2}>0 \text{ and } \cos\theta>0\right)$ $=4\sin\frac{\theta}{2}\cos\theta$

	Method 2 $1-z+z^2-z^3=\dfrac{1-(-z)^4}{1-(-z)}=\dfrac{1-z^4}{1+z}=\dfrac{1-e^{i4\theta}}{1+e^{i\theta}}$ Using **(i)**, $1-z+z^2-z^3$ $=\dfrac{1-e^{i4\theta}}{1+e^{i\theta}}$ $=\dfrac{e^{i2\theta}\left(-2i\sin 2\theta\right)}{e^{i\frac{\theta}{2}}\left(2\cos\dfrac{\theta}{2}\right)}$ $=\dfrac{e^{i\frac{3\theta}{2}}\left(-2i\sin\theta\cos\theta\right)}{\left(\cos\dfrac{\theta}{2}\right)}$ $=\dfrac{e^{i\frac{3\theta}{2}}\left(-4i\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\cos\theta\right)}{\left(\cos\dfrac{\theta}{2}\right)}$ $=e^{i\frac{3\theta}{2}}\left(-4i\sin\dfrac{\theta}{2}\cos\theta\right)$ Since $\left\|e^{i\frac{3\theta}{2}}\right\|=1$, and $0\le\theta\le\dfrac{\pi}{2}\Rightarrow\sin\dfrac{\theta}{2}>0$ and $\cos\theta>0$ $\left\|1-z+z^2-z^3\right\|=4\sin\dfrac{\theta}{2}\cos\theta$ (shown).

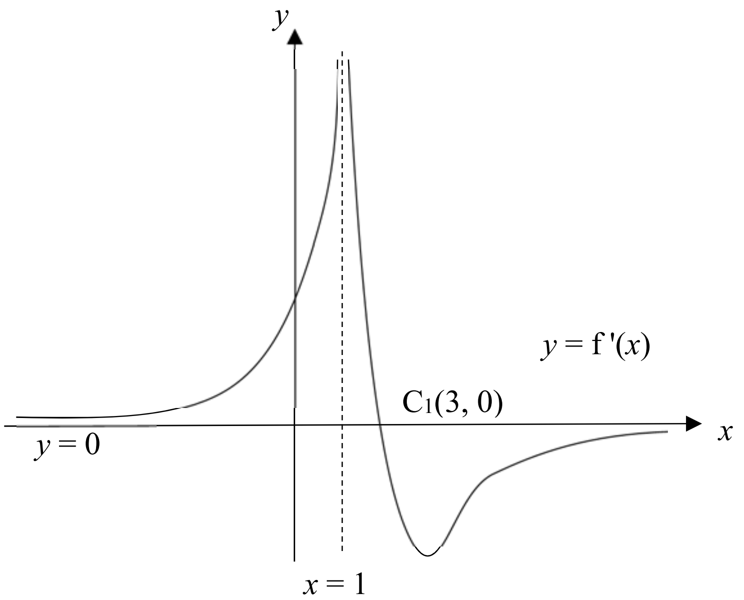
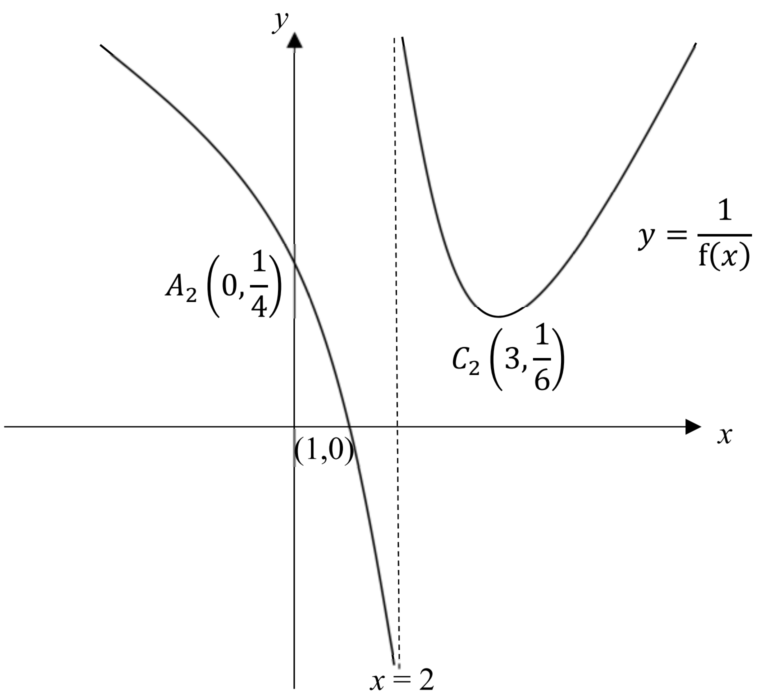
Qn	Solution
4	**Techniques of integration**
(a)	$\int x\sqrt{5-x^2}\ \mathrm{d}x = -\frac{1}{2}\int -2x\sqrt{5-x^2}\ \mathrm{d}x$ $= -\frac{1}{2}\frac{\left(5-x^2\right)^{\frac{3}{2}}}{\frac{3}{2}} + c$ $= -\frac{1}{3}\left(5-x^2\right)^{\frac{3}{2}} + c$
(b)	$\int \sin(\ln x)\ \mathrm{d}x$ $= \int (1)(\sin(\ln x))\ \mathrm{d}x$ $= x\sin(\ln x) - \int x\cos(\ln x)\frac{1}{x}\ \mathrm{d}x$ $= x\sin(\ln x) - \int \cos(\ln x)\ \mathrm{d}x$ $= x\sin(\ln x) - \left[x\cos(\ln x) + \int x\sin(\ln x)\frac{1}{x}\ \mathrm{d}x\right]$ $= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\ \mathrm{d}x$ $\Rightarrow 2\int \sin(\ln x)\ \mathrm{d}x = x\sin(\ln x) - x\cos(\ln x)$ $\Rightarrow \int \sin(\ln x)\ \mathrm{d}x = \frac{1}{2}\left[x\sin(\ln x) - x\cos(\ln x)\right] + c$
(c)	$\int_{-\frac{\pi}{4}}^{\frac{\pi}{12}} \cos x\lvert\sin x\rvert \mathrm{d}x$ $= \int_{-\frac{\pi}{4}}^{0} \cos x(-\sin x)\mathrm{d}x + \int_{0}^{\frac{\pi}{12}} \cos x(\sin x)\mathrm{d}x$ $= -\frac{1}{2}\int_{-\frac{\pi}{4}}^{0} 2\sin x\cos x\mathrm{d}x + \frac{1}{2}\int_{0}^{\frac{\pi}{12}} 2\sin x\cos x\mathrm{d}x$ $= -\frac{1}{2}\int_{-\frac{\pi}{4}}^{0} \sin 2x\mathrm{d}x + \frac{1}{2}\int_{0}^{\frac{\pi}{12}} \sin 2x\mathrm{d}x$ $= -\frac{1}{2}\left[\frac{-\cos 2x}{2}\right]_{-\frac{\pi}{4}}^{0} + \frac{1}{2}\left[\frac{-\cos 2x}{2}\right]_{0}^{\frac{\pi}{12}}$ $= -\frac{1}{4}(-1-0) + \frac{1}{4}\left(-\frac{\sqrt{3}}{2}+1\right)$ $= \frac{1}{2} - \frac{\sqrt{3}}{8}$ Recall that $\lvert f(x)\rvert = \begin{cases} -f(x) & \text{when } f(x) < 0 \\ f(x) & \text{when } f(x) \geq 0 \end{cases}$ Hence $\lvert\sin x\rvert = \begin{cases} -\sin x & \text{when } -\frac{\pi}{4} \leq x < 0 \\ \sin x & \text{when } 0 \leq x \leq \frac{\pi}{12} \end{cases}$

Qn	Solution
5	**Function**
(i)	Since $\mathrm{R_f} = (0,\infty) \subseteq \mathrm{D_g} = (0,\infty)$, gf exists. $\mathrm{gf} : x \mapsto \ln\left\|\dfrac{1}{x-3}\right\|$, where $x \in \mathbb{R}, x \neq 3$
(ii)	 Since $y = 3$ cuts the graph of f twice, f is not one-one. Hence f^{-1} does not exist.

(iii)	Maximum $k = 3$ For $\frac{1}{x-3} < 0$, Let $y = -\dfrac{1}{x-3}$ $x - 3 = -\dfrac{1}{y}$ $x = 3 - \dfrac{1}{y}$ $\therefore \mathrm{f}^{-1}(x) = 3 - \dfrac{1}{x}$ $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = (0,\infty)$ $\mathrm{f}^{-1} : x \mapsto 3 - \dfrac{1}{x},\ x \in \mathbb{R}, x > 0$
(iv)	The graph of $y = \mathrm{f}(x)$ is a reflection of the graph of $y = \mathrm{f}^{-1}(x)$ in the line $y = x$.

Qn	Solution
6	**Binomial Theorem and Maclaurin's Series**

(a) (i)	$y=\left[1+\ln(1+3x)\right]^{-\frac{1}{2}}$ $=\left[1+(3x)-\frac{(3x)^2}{2}+...\right]^{-\frac{1}{2}}$ $\approx\left(1+(3x-\frac{9}{2}x^2)\right)^{-\frac{1}{2}}$ $=1+\left(-\frac{1}{2}\right)\left(3x-\frac{9}{2}x^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(3x-\frac{9}{2}x^2\right)^2+...$ $=1-\frac{3}{2}x+\frac{9}{4}x^2+\frac{3}{8}\left(9x^2\right)+...$ $=1-\frac{3}{2}x+\frac{45}{8}x^2+...$
(a) (ii)	$\int_0^1 y\ \mathrm{d}x\approx\int_0^1 1-\frac{3}{2}x+\frac{45}{8}x^2\ \mathrm{d}x$ $=\frac{17}{8}$ Approximation is not good as $x=1$ is not close to 0 **OR** the two graphs deviate significantly in the interval [0,1].
(a) (iii)	$y=1-\frac{3}{2}x$
(b)	$\frac{\cos(2x)}{1-\sin x}$ $\approx\frac{1-\frac{1}{2}(2x)^2}{1-x}$ $=\frac{1-2x^2}{1-x}$ $=(1-2x^2)(1-x)^{-1}$ $=(1-2x^2)(1+x+x^2)+...$ $=1+x-x^2+...$ $a=1,b=1,\ c=-1$

Qn	Solution
7	**Graphing Techniques - Transformations**
(a) (i)	
(a) (ii)	

(b)	$y=\dfrac{3x^2-6x}{x-1}=3(x-1)-\dfrac{3}{x-1}$ $\therefore A=3,\ B=-3$ A sequence of transformation from the graph of $y=\dfrac{1}{x}-x$ to the graph of $y=\dfrac{3x^2-6x}{x-1}$ is: 1. a. **Reflect** graph **abou**t the x-axis (Replace y with $(-y)$, get $y=x-\dfrac{1}{x}$) OR b. **reflect** graph **about** the y-axis (Replace x with $(-x)$, get $y=x-\dfrac{1}{x}$) 2. **Translate** graph 1 unit in the positive x-direction (Replace x with $(x-1)$, get $y=(x-1)-\dfrac{1}{x-1}$) 3. **Scaling** parallel to y-axis by **factor** of 3 (Replace y with $\dfrac{y}{3}$, get $y=3(x-1)-\dfrac{3}{x-1}$) Other possible transformations: 2, 1a, 3 or 3,1,2 or 2, 3, 1a

Qn	Solution
8	**Vectors 3**
(i)	$\overrightarrow{OA}=\begin{pmatrix}-2\\-7\\3\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\overrightarrow{AB}=\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}-2\\-7\\3\end{pmatrix}=\begin{pmatrix}2\\8\\-2\end{pmatrix}=2\begin{pmatrix}1\\4\\-1\end{pmatrix}$ $l:\quad \mathbf{r}=\begin{pmatrix}-2\\-7\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\4\\-1\end{pmatrix}, \lambda\in\mathbb{R}$
(ii)	Let θ be the acute angle between l and p. $\sin\theta=\dfrac{\left\|\begin{pmatrix}1\\4\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\|}{\sqrt{1^2+4^2+(-1)^2}\sqrt{1^2+(-1)^2}}$ $=\dfrac{\|1-4\|}{\sqrt{18}\sqrt{2}}=\dfrac{3}{6}$ $\Rightarrow\theta=30°$
(iii)	$p:\quad \mathbf{r}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}=5$ Subst. $\overrightarrow{OA}$ into p, $\text{LHS}=\begin{pmatrix}-2\\-7\\3\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}=-2+7=5=\text{RHS}$ Hence A lies on p. Note: $\angle BAC=2(30°)=60°$ and $BA=CA$ (since reflection) ABC is an <u>equilateral</u> triangle.
(iv)	Let F be the foot of perpendicular from B to p $\overrightarrow{OF}=\begin{pmatrix}0\\1\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}\mu\\1-\mu\\1\end{pmatrix}$, for some $\mu\in\mathbb{R}$ $\begin{pmatrix}\mu\\1-\mu\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}=5$ $\Rightarrow\mu-1+\mu=5$ $\Rightarrow\mu=3$ $\therefore\overrightarrow{OF}=\begin{pmatrix}3\\1-3\\1\end{pmatrix}=\begin{pmatrix}3\\-2\\1\end{pmatrix}$

	$\overrightarrow{AF} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ -7 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ -2 \end{pmatrix}$ By ratio theorem, $\overrightarrow{AC} = 2\overrightarrow{AF} - \overrightarrow{AB}$ $= 2\begin{pmatrix} 5 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 8 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$ $\therefore$ Equation of reflection of line l in p is $\mathbf{r} = \begin{pmatrix} -2 \\ -7 \\ 3 \end{pmatrix} + k\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}, k \in \mathbb{R}$

Qn	Solution
9	**Differential Equations**
(i)	$\frac{dx}{dt} = k\left(6x - x^2\right)$ $= k\left[-(x^2 - 6x)\right]$ $= k\left[-(x-3)^2 + 9\right]$ $= k\left[9 - (x-3)^2\right]$ $\int \frac{1}{3^2 - (x-3)^2}\ dx = \int k\ dt$ $\frac{1}{2(3)} \ln\left\|\frac{3+(x-3)}{3-(x-3)}\right\| = kt + c$ $\ln\left\|\frac{x}{6-x}\right\| = 6kt + 6c$ $\frac{x}{6-x} = Ae^{6kt}$ where $A = \pm e^{6c}$ When $t = 0$, $x = 12$ $\frac{12}{6-12} = Ae^{6(0)}$ $A = -2$ $\frac{x}{6-x} = -2e^{6kt}$ When $t = \frac{1}{12}$, $x = 8.6$ $\frac{8.6}{6-8.6} = -2e^{6k\left(\frac{1}{12}\right)}$ $-\frac{43}{13} = -2e^{0.5k}$ $\ln\frac{43}{26} = 0.5k$ $k = 2\ln\frac{43}{26}$ $\frac{x}{6-x} = -2e^{12t\ln\frac{43}{26}}$ $x = -2e^{12t\ln\frac{43}{26}}\left(6-x\right)$ $x - 2e^{12t\ln\frac{43}{26}}x = -12e^{12t\ln\frac{43}{26}}$ $x = \frac{-12e^{12t\ln\frac{43}{26}}}{1-2e^{12t\ln\frac{43}{26}}}$ $= \frac{12e^{12t\ln\frac{43}{26}}}{2e^{12t\ln\frac{43}{26}} - 1}$

(ii)	When $x=0.75(12)=9$, $\frac{9}{6-9}=-2e^{12t\ln\frac{43}{26}}$ $\frac{3}{2}=e^{12t\ln\frac{43}{26}}$ $\ln\frac{3}{2}=12t\ln\frac{43}{26}$ $t=0.0672$ years (to 3s.f.)
(iii)	$x=\frac{-12e^{6.0372t}}{1-2e^{6.0372t}}$ $=\frac{-12}{\frac{1}{e^{6.0372t}}-2}$ As $t\to\infty,\frac{1}{e^{6.0372t}}\to 0, x\to 6$ In the long run, the population of the bluegill fish in the lake will **decrease and stabilise** at 6000. The assumption is that there are no other external factors such as pollution or diseases that may affect the population of bluegill fish in the lake.

Qn	Solution
10	**APGP**
(a)(i)	$u_n = S_n - S_{n-1}$ $= kn^2 - 3n - \left[k(n-1)^2 - 3(n-1)\right]$ $= 2kn - k - 3$ $u_n - u_{n-1} = 2kn - k - 3 - \left[2k(n-1) - k - 3\right]$ $= 2k$, a constant $\Rightarrow \{u_n\}$ is in AP.
(ii)	$\dfrac{u_3}{u_2} = \dfrac{u_6}{u_3}$ $(u_3)^2 = (u_2)(u_6)$ $\left[2k(3) - k - 3\right]^2 = \left[2k(2) - k - 3\right]\left[2k(6) - k - 3\right]$ $(5k-3)^2 = (3k-3)(11k-3)$ $8k^2 - 12k = 0$ $4k(2k-3) = 0$ $k = 0$ (rej since $k \neq 0$) or $k = \dfrac{3}{2}$
(b)(i)	Distance travelled by white-tailed deer, $S_n = \dfrac{9\left(1-0.95^n\right)}{1-0.95}$ $= 180\left(1-0.95^n\right)$ As $n \to \infty$, $S_n \to 180$. $\therefore$ max distance is 180 m.
(ii)	Distance travelled by jaguar after n leaps, $S_n = \dfrac{n}{2}\left[2(6) + (n-1)(-0.1)\right]$ For jaguar to catch white-tailed deer within n leaps, Let $D = \dfrac{n}{2}\left[12 + (n-1)(-0.1)\right] - 180\left(1-0.95^n\right) - 11 \geq 0$ When $n = 49, D = -0.21 < 0$ When $n = 50, D = 0.3501 > 0$ $\therefore$ min $n = 50$

Qn	Solutions
11	**Differentiation + Integration**

(i)

$x = 1 - \cos^3\theta, \qquad y = 1 - 3\sin\theta\cos^2\theta$

$$\frac{dx}{d\theta} = -3\cos^2\theta(-\sin\theta) = 3\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = -\left[\sin\theta 6\cos\theta(-\sin\theta) + 3\cos^2\theta(\cos\theta)\right]$$

$$= 6\sin^2\theta\cos\theta - 3\cos^3\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{6\sin^2\theta\cos\theta - 3\cos^3\theta}{3\cos^2\theta\sin\theta}$$

$= 2\tan\theta - \cot\theta$ (shown)

(ii)

$2\tan\theta - \cot\theta = 0$

$$2\tan\theta = \frac{1}{\tan\theta}$$

$2\tan^2\theta = 1$

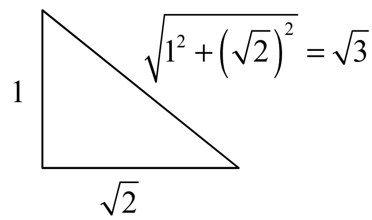
$$\tan\theta = \pm\frac{1}{\sqrt{2}}$$

$$\tan\theta = \frac{1}{\sqrt{2}} \left(\because \tan\theta \text{ is positive as } 0 \le \theta \le \frac{\pi}{2}\right)$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

Since $\tan\theta = \dfrac{1}{\sqrt{2}}$ and $0 \le \theta \le \dfrac{\pi}{2}$,

$$\cos\theta = \frac{\sqrt{2}}{\sqrt{3}} \text{ and } \sin\theta = \frac{1}{\sqrt{3}}$$



$\tan\theta = \dfrac{1}{\sqrt{2}}$, $\therefore \cos\theta = \dfrac{\sqrt{2}}{\sqrt{3}}$ and $\sin\theta = \dfrac{1}{\sqrt{3}}$

$$\therefore x = 1 - \cos^3\theta = 1 - \left(\sqrt{\frac{2}{3}}\right)^3,$$

$$y = 1 - 3\sin\theta\cos^2\theta = 1 - 3\left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 = 1 - \frac{2}{\sqrt{3}}$$

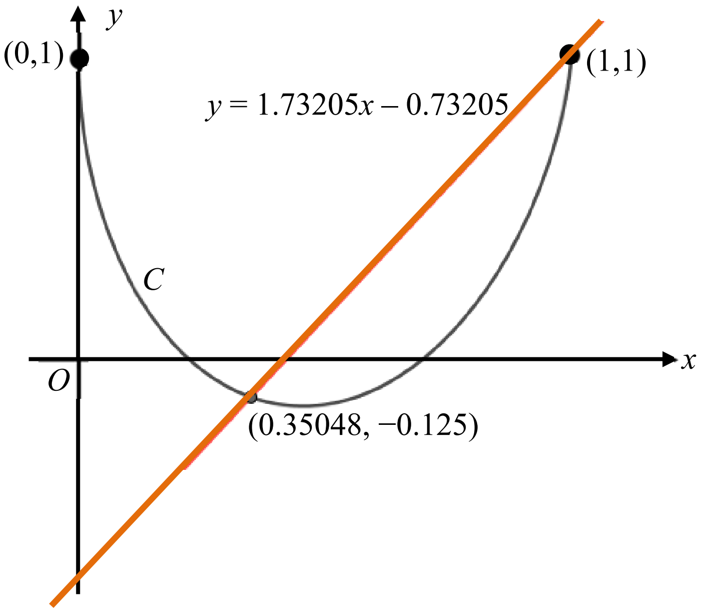
To show minimum point

x	$\left(1 - \left(\sqrt{\frac{2}{3}}\right)^3\right)^-$	$1 - \left(\sqrt{\frac{2}{3}}\right)^3$	$\left(1 - \left(\sqrt{\frac{2}{3}}\right)^3\right)^+$
$\frac{dy}{dx}$	$-$	0	$+$

$$\therefore \text{ At } \left(1 - \left(\sqrt{\frac{2}{3}}\right)^3,\ 1 - \frac{2}{\sqrt{3}}\right), \text{ it is a minimum point.}$$

Alternative:

	$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $=\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right]\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)$ $=\frac{\mathrm{d}}{\mathrm{d}\theta}\left(2\tan\theta-\cot\theta\right)\frac{1}{3\cos^2\theta\sin\theta}$ $=\left(2\sec^2\theta+\operatorname{cosec}^2\theta\right)\frac{1}{3\cos^2\theta\sin\theta}$ Using GC, $\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(2\tan\theta-\cot\theta\right)\Big\|_{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}\right]\left[\frac{1}{3\cos^2\theta\sin\theta}\Big\|_{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}\right]$ $=5.19616>0$ $\therefore$ At $\left(1-\left(\sqrt{\frac{2}{3}}\right)^3,\ 1-\frac{2}{\sqrt{3}}\right)$, it is a minimum point.
(iii)	The gradient of normal is $-\frac{1}{2\tan\theta-\cot\theta}\Big\|_{\frac{\pi}{6}}=-\frac{1}{0.577351}=1.73205$ Equation of normal is $y-(-0.125)=1.73205\left(x-(0.35048)\right)$ $y=1.73205x-0.73205$

	 Coordinates of point on C at $\theta = \frac{\pi}{6}$ is $(0.35048, -0.125)$ Area of area of the region bounded by C, the y-axis and the equation of the normal to the curve at $\theta = \frac{\pi}{6}$ $= \int_{-0.125}^{1} x \, dy + \text{area of triangle}$ $= \int_{\frac{\pi}{6}}^{0} (1-\cos^3\theta)(6\sin^2\theta\cos\theta - 3\cos^3\theta)\,d\theta$ $+ \frac{1}{2}(0.73205 - 0.125)(0.35048)$ $= 0.0997667 + 0.106379$ $= 0.206$
	Alternative: $= \int_{0}^{\frac{\pi}{6}} (1-3\sin\theta\cos^2\theta)(3\cos^2\theta\sin\theta)\,d\theta$ $- \int_{0}^{0.35048} (1.73205x - 0.73205)\,dx$ $= 0.206$

2018 H2 MATH (9758/02) JC 2 PRELIM SUGGESTED SOLUTIONS

Qn	Solution
1	**Complex No.**
(a)	If w is a complex root to the equation $\mathrm{P}(z)=0$, then $\mathrm{P}(w)=0$.

$$a_0+a_1w+a_2w^2=0$$

Taking conjugate on both sides of the equation (*):

$$\left(a_0+a_1w+a_2w^2\right)^*=0$$

$$\left(a_0\right)^*+\left(a_1w\right)^*+\left(a_2w^2\right)^*=0$$

Since $a_0, a_1, a_2 \in \mathbb{R}$,

$$a_0+a_1w^*+a_2\left(w^*\right)^2=0$$

$$\mathrm{P}\left(w^*\right)=0$$

Therefore, its complex conjugate w^* is also a root of $\mathrm{P}(z)=0$.

<u>Alternative method</u>

Let $w=x+y\mathrm{i}$ where $x, y \in \mathbb{R}$

then $w^*=x-y\mathrm{i}$.

$$a_0+a_1w+a_2w^2=0$$

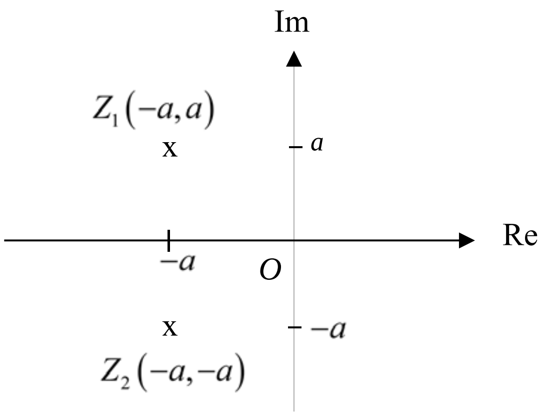
$$a_0+a_1\left(x+y\mathrm{i}\right)+a_2\left(x+y\mathrm{i}\right)^2=0$$

$$a_0+a_1\left(x+y\mathrm{i}\right)+a_2\left(x^2+2xy\mathrm{i}-y^2\right)=0$$

Comparing real and imaginary parts:

$$a_0+a_1x+a_2\left(x^2-y^2\right)=0$$

$$a_1y+a_2\left(2xy\right)=0$$

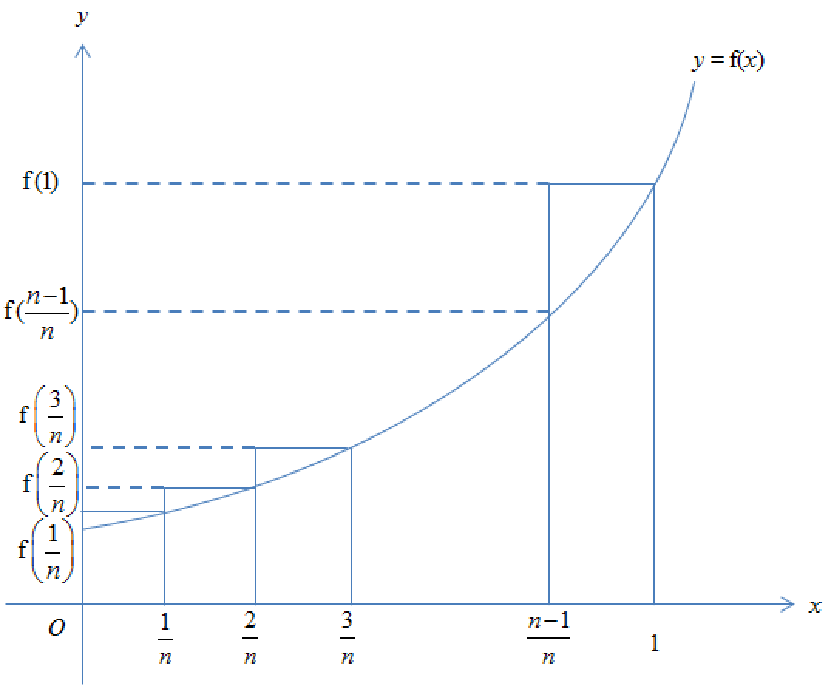
	Subst $w^* = x - y\mathrm{i}$ into $\mathrm{P}(z)$: $a_0 + a_1 w^* + a_2 \left(w^*\right)^2$ $= a_0 + a_1\left(x - y\mathrm{i}\right) + a_2\left(x - y\mathrm{i}\right)^2$ $= a_0 + a_1\left(x - y\mathrm{i}\right) + a_2\left(x^2 - 2xy\mathrm{i} - y^2\right)$ $= a_0 + a_1 x + a_2\left(x^2 - y^2\right) - \left[a_1 y + a_2\left(2xy\right)\right]\mathrm{i}$ Since $a_0 + a_1 x + a_2\left(x^2 - y^2\right) = 0$ and $a_1 y + a_2\left(2xy\right) = 0$ $a_0 + a_1 w^* + a_2 \left(w^*\right)^2$ $= 0 - 0\mathrm{i}$ $=0$ $\therefore \mathrm{P}\left(w^*\right) = 0$
(b)(i)	$z_2 = -a - a\mathrm{i}$
(b)(ii)	
(b)(iii)	Area of $OZ_1Z_2 = \frac{1}{2} \times (2a) \times a$ $= a^2$

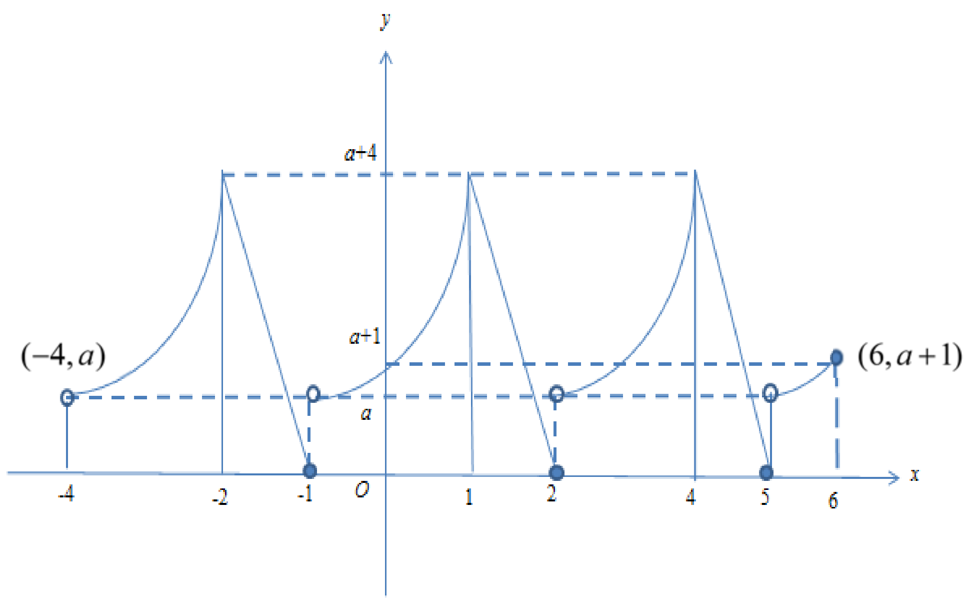
Qn	Solution
2	**Sequences & Series**
(i)	$\text{LHS}=\cos\left(r+\frac{1}{2}\right)\theta-\cos\left(r-\frac{1}{2}\right)\theta$ $=\left[\cos r\theta\cos\frac{1}{2}\theta-\sin r\theta\sin\frac{1}{2}\theta\right]-\left[\cos r\theta\cos\frac{1}{2}\theta+\sin r\theta\sin\frac{1}{2}\theta\right]$ $=-2\sin r\theta\sin\frac{1}{2}\theta=\text{RHS}\quad(\text{shown})$
(ii)	$\sum_{r=1}^{n}\sin r\theta$ $=\sum_{r=1}^{n}\frac{\cos\left(r+\frac{1}{2}\right)\theta-\cos\left(r-\frac{1}{2}\right)\theta}{-2\sin\frac{1}{2}\theta}$ $=\frac{1}{-2\sin\frac{1}{2}\theta}\sum_{r=1}^{n}\left[\cos\left(r+\frac{1}{2}\right)\theta-\cos\left(r-\frac{1}{2}\right)\theta\right]$ $=\frac{1}{-2\sin\frac{1}{2}\theta}\left[\cancel{\cos\frac{3}{2}\theta}-\cos\frac{1}{2}\theta\right.$ $+\cancel{\cos\frac{5}{2}\theta}-\cancel{\cos\frac{3}{2}\theta}$ $+\cancel{\cos\frac{7}{2}\theta}-\cancel{\cos\frac{5}{2}\theta}$ $+\ldots$ $+\cancel{\cos\left(n-\frac{1}{2}\right)\theta}-\cancel{\cos\left(n-\frac{3}{2}\right)\theta}$ $\left.+\cos\left(n+\frac{1}{2}\right)\theta-\cancel{\cos\left(n-\frac{1}{2}\right)\theta}\right]$ $=\frac{1}{-2\sin\frac{1}{2}\theta}\left[\cos\left(n+\frac{1}{2}\right)\theta-\cos\frac{1}{2}\theta\right]$ $=-\frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta\left[\cos\left(n+\frac{1}{2}\right)\theta-\cos\frac{1}{2}\theta\right]$
(iii)	$\sum_{r=5}^{62}\sin(r-2)\theta$ $=\sum_{r=3}^{60}\sin r\theta\qquad(\text{replace } r \text{ by } r+2)$ $=\sum_{r=1}^{60}\sin r\theta-\sin\theta-\sin 2\theta$ $=\frac{1}{-2\sin\frac{1}{2}\theta}\left[\cos\left(60+\frac{1}{2}\right)\theta-\cos\frac{1}{2}\theta\right]-\sin\theta-\sin 2\theta$ When $\theta=\frac{\pi}{3}$,

	$\sum_{r=5}^{62}\sin\left[(r-2)\left(\frac{\pi}{3}\right)\right]=\frac{1}{-2\sin\frac{\pi}{6}}\left[\cos\left(20\pi+\frac{\pi}{6}\right)-\cos\frac{\pi}{6}\right]-\sin\frac{\pi}{3}-\sin\frac{2\pi}{3}$ $=-1\left[\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right]-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}=-\sqrt{3}$

Qn	Solutions
3	**Applications of Differentiation: Rate of Change, Maxima/Minima**
(a)	$P=2r+\pi r \qquad \Rightarrow \dfrac{\mathrm{d}P}{\mathrm{d}r}=2+\pi$ $A=\dfrac{\pi r^2}{2} \qquad \Rightarrow \dfrac{\mathrm{d}A}{\mathrm{d}r}=\pi r$ $\therefore \dfrac{\mathrm{d}P}{\mathrm{d}A}=\dfrac{\frac{\mathrm{d}P}{\mathrm{d}r}}{\frac{\mathrm{d}A}{\mathrm{d}r}}=\dfrac{2+\pi}{\pi r} \quad (\text{Shown})$
	Given that $\dfrac{\mathrm{d}P}{\mathrm{d}t}=\dfrac{3}{5}$ cm/s and $\dfrac{\mathrm{d}A}{\mathrm{d}t}=3$ cm^2/s $\dfrac{\mathrm{d}P}{\mathrm{d}A}=\left(\dfrac{\mathrm{d}P}{\mathrm{d}t}\right)\left(\dfrac{\mathrm{d}t}{\mathrm{d}A}\right)$ $\dfrac{2+\pi}{\pi r}=\left(\dfrac{3}{5}\right)\left(\dfrac{1}{3}\right)$ $5(2+\pi)=\pi r$ $r=\dfrac{5(2+\pi)}{\pi}$ cm
(b)	Let S be the external surface area of the model. $S=4\left(\dfrac{1}{2}\right)(2r)\left(\sqrt{2}r\right)+4(2r)h+(2r)(2r)$ $=4\left(\sqrt{2}\right)r^2+8rh+4r^2$ $=4\left(\sqrt{2}+1\right)r^2+8rh=160$ $h=\dfrac{160-4\left(\sqrt{2}+1\right)r^2}{8r}$ $=\dfrac{40-\left(\sqrt{2}+1\right)r^2}{2r}$ Volume of model $V=\dfrac{1}{3}(2r)(2r)r+(2r)(2r)h$ $=\dfrac{4}{3}r^3+4r^2\left(\dfrac{40-\left(\sqrt{2}+1\right)r^2}{2r}\right)$ $=\dfrac{4}{3}r^3+2r\left(40-\left(\sqrt{2}+1\right)r^2\right)$ $=\dfrac{4}{3}r^3+80r-2\left(\sqrt{2}+1\right)r^3$ $=80r-\left(\dfrac{2}{3}+2\sqrt{2}\right)r^3$ For V to be maximum,

	$$\frac{\mathrm{d}V}{\mathrm{d}r}=80-3\left(\frac{2}{3}+2\sqrt{2}\right)r^2=0$$ $$\left(4-6\left(\sqrt{2}+1\right)\right)r^2+80=0$$ Since $r>0$, $$r=\sqrt{\frac{-80}{4-6\left(\sqrt{2}+1\right)}}$$ $$=2.7622$$ $$\approx 2.76\ \text{cm}$$

Qn	Solution
4	**Applications of Integration**
(a) (i)	 $\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{f}\left(\frac{1}{n}\right)+\mathrm{f}\left(\frac{2}{n}\right)+\ldots+\mathrm{f}\left(\frac{n}{n}\right)\right\}$ $=\lim_{n\to\infty}$ (Total area of the n rectangles) = Area under the curve $y = \mathrm{f}(x)$ from $x = 0$ to $x = 1$ $=\int_0^1 \mathrm{f}(x)\mathrm{d}x$
(a) (ii)	Let $\mathrm{f}(x)=\mathrm{e}^{3x}$ $\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{e}^{\frac{3}{n}}+\mathrm{e}^{\frac{6}{n}}+\ldots+\mathrm{e}^{3}\right\}$ $=\int_0^1 \mathrm{e}^{3x}\mathrm{d}x$ $=\frac{1}{3}\left[\mathrm{e}^{3x}\right]_0^1$ $=\frac{1}{3}(\mathrm{e}^3-1)$
(b)(i)	$\mathrm{f}(-41)=\mathrm{f}(1)=4+a$ $\mathrm{f}\left(2018\right)=\mathrm{f}(2)=0$

(ii)	
(iii)	$\int_{-4}^{4} \mathrm{f}(x)\mathrm{d}x$ $=3\int_{-1}^{1}\left((x+1)^2+a\right)\mathrm{d}x+2\left(\frac{1}{2}(1)(4+a)\right)$ $=\left[(x+1)^3+3ax\right]_{-1}^{1}+4+a$ $=(8+3a)-(0-3a)+4+a$ $=12+7a$

Qn	Solution
5	**Permutations and Combinations**
(i)	Number of ways = 8! = 40320
(ii)	Number of ways = $^{8}C_1 \times 1 \times {}^{6}C_1 \times 1 \times {}^{4}C_1 \times 1 \times {}^{2}C_1 \times 1 = 384$ Alternative: $4! \times (2!)^4 = 384$
(iii)	Number of ways = $384 - {}^{6}C_1 \times 2! \times 4 \times 1 \times 2 \times 1 = 288$

<table>
<tr><th>Qn</th><th>Suggested Solutions</th></tr>
<tr><td>6</td><td>D.R.V.</td></tr>
<tr><td>(i)</td><td>

X	**1**	**2**	**3**	**4**	**5**
1	0	1	2	3	4
2	1	0	1	2	3
3	2	1	0	1	2
4	3	2	1	0	1
5	4	3	2	1	0

Probability Distribution of X

x	0	1	2	3	4
$\mathrm{P}(X=x)$	$\frac{5}{25}=\frac{1}{5}$	$\frac{8}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{2}{25}$

</td></tr>
<tr><td>(ii)</td><td>

Using GC, $\mathrm{E}(X)=1.6, \mathrm{Var}(X)=1.44$

OR

$\mathrm{E}(X)=0\left(\frac{1}{5}\right)+1\left(\frac{8}{25}\right)+2\left(\frac{6}{25}\right)+3\left(\frac{4}{25}\right)+4\left(\frac{2}{25}\right)=1.6$

$\mathrm{E}(X^2)=0^2\left(\frac{1}{5}\right)+1^2\left(\frac{8}{25}\right)+2^2\left(\frac{6}{25}\right)+3^2\left(\frac{4}{25}\right)+4^2\left(\frac{2}{25}\right)=4$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-\left[\mathrm{E}(X)\right]^2=4-1.6^2=1.44$

</td></tr>
<tr><td>(iii)</td><td>

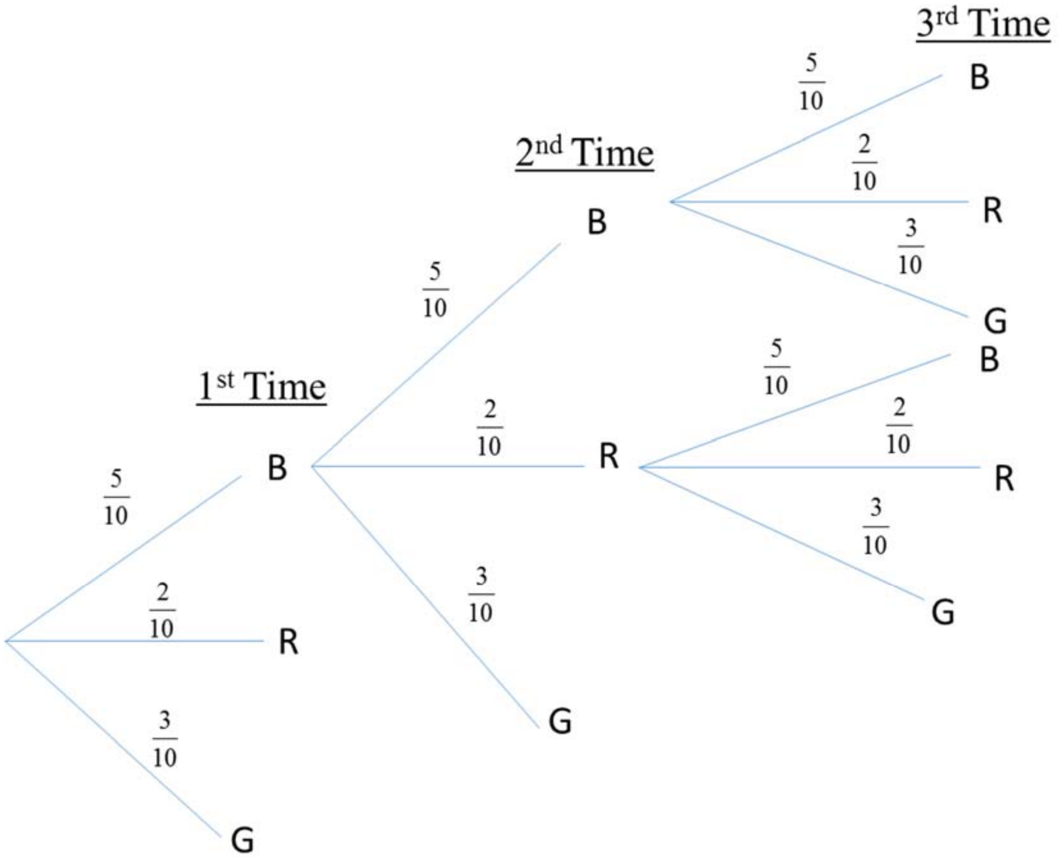
$\mathrm{E}(2X-a)<0$

$2(1.6)-a<0$

$\therefore a>3.2$

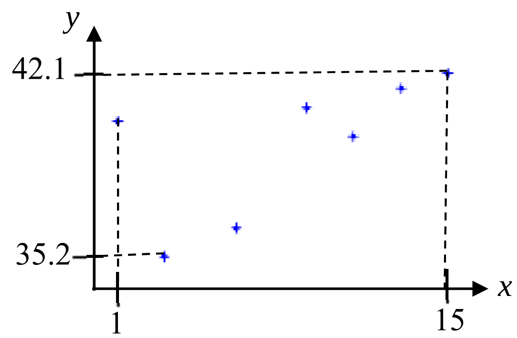
</td></tr>
</table>

Qn	Suggested Solutions
7	**Binomial Distribution**
(i)	(I) Whether a patient has diabetes is independent of any other patient who has diabetes. (S) The probability that a patient has diabetes is constant for all patients.
(ii)	(I) Patients who are selected may be family members hence each of them having diabetes are not independent of each other. <u>**OR**</u> (S) The probability of a patient having diabetes is not a constant as the elderly have a higher probability of having diabetes. <u>**OR**</u> any reasonable factors e.g. lifestyle, diet, race, age
(iii)	Let X be the number of patients, out of 24, who have diabetes. $X \sim \mathrm{B}\left(24, \frac{1}{9}\right)$ Probability that in a sample of 25, the 25$^{\text{th}}$ patient is the third patient who has diabetes $= \mathrm{P}(X=2)\,\mathrm{P}(\text{the 25th patient is the third patient who has diabetes})$ $= 0.25531 \times \frac{1}{9} = 0.0284$
(iv)	$\mathrm{P}(X \le n) > 0.9$ Using GC, When $n=4$, $\mathrm{P}(X \le 4) = 0.87974 < 0.9$ When $n=5$, $\mathrm{P}(X \le 5) = 0.95653 > 0.9$ Therefore least $n=5$

Qn	Solution
8	**Probability (Tree Diagram)**
(i)	
(ii)	P(player wins \$12 or more when the game ends) = P(BRR) + P(BBR) +P(R) $=\frac{5}{10}\left(\frac{2}{10}\right)\left(\frac{2}{10}\right)+\frac{5}{10}\left(\frac{5}{10}\right)\left(\frac{2}{10}\right)+\left(\frac{2}{10}\right)$ $=\frac{27}{100}$
(iii)	P (second spin lands on a blue sector | the player wins \$12 or more when the game ends) $=\frac{\left(\frac{5}{10}\right)\left(\frac{5}{10}\right)\left(\frac{2}{10}\right)}{0.27}=\frac{5}{27}$
(iv)	P (no spins result on a red sector and wins nothing when game ends) $=\frac{3}{10}+\left(\frac{5}{10}\right)\left(\frac{3}{10}\right)+\left(\frac{5}{10}\right)^2\left(\frac{3}{10}\right)+\left(\frac{5}{10}\right)^3\left(\frac{3}{10}\right)+\ldots$ $=\frac{\frac{3}{10}}{1-\frac{5}{10}}$ $=\frac{3}{5}$

Qn	Solution
9	**Hypothesis Testing**
(i)	Let X be the duration a randomly chosen shopper spends at MJ mall per visit (in hours). Let μ denote the population mean duration spent by shoppers at MJ mall per visit (in hours). H_0: $\mu = 1.5$ H_1: $\mu > 1.5$ $\bar{x} = \frac{\sum x}{n} = \frac{82.5}{50} = 1.65$ $s^2 = \frac{50}{49}(0.425) = 0.43367$ (to 5 s.f.) $= 0.434$ (to 3 s.f.)
(ii)	Since sample size is large, by Central Limit Theorem, the sample mean duration spent by 50 shoppers is approximately normal. $\bar{X} \sim N\left(1.5, \frac{0.43367}{50}\right)$ approximately. Hence there is no need to assume that the distribution of the duration shoppers spend at MJ mall per visit is normal.
(iii)	Using GC, Under H_0, $p\text{-value} = 0.053630$ (to 5 s.f.) $= 0.0536$ (to 3 s.f.) 0.0536 is the probability of obtaining a sample mean more than or equal to 1.65, assuming that the mean duration spent by shoppers at MJ mall is 1.5 hours. For management's hypothesis to be valid, H_0 is to be rejected. $p\text{-value} = 0.053630 < \frac{\alpha}{100}$ $\Rightarrow \alpha > 5.3630$ $\therefore$ Required set of values is $\{\alpha \in \mathbb{R} : 5.36 < \alpha \leq 100\}$ (to 3 s.f.)
(iv)	H_0: $\mu = 150$ H_1: $\mu < 150$ Given: p-value<0.05 **(a)** This is necessarily true because p-value< 0.05 < 0.1 **(b)** This is not necessarily true nor necessarily false. Now, H_0: $\mu = 150$ H_1: $\mu \neq 150$ The new p-value is double of the original p-value. Case 1: If the original p-value was more than 0.025, for example 0.03, then the new p-value would be 0.06 >0.05, hence in such a case there would not be significant evidence at the 5% level of significance that the population mean spending differs from \$150.

	Case 2: If the original p-value was less than 0.025, for example 0.02, then the new p-value would be $0.04 < 0.05$, hence in such a case there would be significant evidence at the 5% level of significance that the population mean spending differs from \$150.

Qn	Solution
10	**Correlation & Relation**
(i)	
(ii)	The outlier is (1, 40.3). Any appropriate reason, some possible reasons: • A fire in the dry grassland killed the wild rabbits • Drought/Famine in the region causes wild rabbits to die • Disease/Epidermic in the region killed the wild rabbits • Introduction of a new predator like coyote or dogs that hunts the wild rabbits, thus leading to a sudden decrease in the population. • Hunting season lead to a sudden decrease in the wild rabbit population.
(iii)	The population of wild rabbits <u>cannot increase or decrease indefinitely</u> in a region, hence a linear model is not appropriate or the population of wild rabbits should plateau due to limited resources.
(iv)	Using G.C., When $M = 45$, $r = -0.946\ 203$ (to 6 decimal places)
(v)	Since $M = 45$ gives a value of r that is closer to -1, it would be the most appropriate value for M.
(vi)	Using G.C., $\ln(45 - y) = 2.6356 - 0.10435x$ $\Rightarrow \ln(45 - y) = 2.64 - 0.104x \quad (3 \text{ s.f.})$ $\Rightarrow a = 2.64, \quad b = -0.104$ When $x = 24$, $y = 45 - e^{2.6356 - 0.10435(24)} = 43.8598$ The population of the wild rabbits after 2 years is 4386.
(vii)	$M = 45$ i.e. 4500 rabbits is the <u>maximum population</u> of wild rabbits which this region can support in the long run, assuming all factors stay the same.

Qn	Solution
11	**Normal Distribution & Sampling**
(a)	If $T \sim \mathrm{N}(5,8)$, then P ($T < 0$) = 0.0385, which means that there is a non-zero probability that the time taken could be negative. However, this is impossible. Hence it is not suitable. OR If $T \sim \mathrm{N}(5,8)$, then 99.7% of the values would lie within $5 \pm 3\sqrt{8}$, which contains a significant range of negative values.
(b) (i)	Let X and Y be the masses of a randomly chosen chicken and a randomly chosen duck in kilograms respectively. $X \sim \mathrm{N}\left(2.4, 0.5^2\right)$ $Y \sim \mathrm{N}\left(4.3, 1.8^2\right)$ $X_1 + X_2 \sim \mathrm{N}(4.8, 0.5)$ $\mathrm{P}\left(X_1 + X_2 > 5\right) = 0.38865 \approx 0.389$ (3s.f.)
(b) (ii)	$\dfrac{X_1 + X_2 + Y}{3} \sim \mathrm{N}\left(\dfrac{91}{30}, \dfrac{187}{450}\right)$ $\mathrm{P}\left(\dfrac{X_1 + X_2 + Y}{3} > 3.2\right) = \mathrm{P}\left(\dfrac{X_1 + X_2 + Y}{3} > 3.2\right)$ $= 0.39799 \approx 0.398$ (3s.f.) <u>Alternatively (Consider sum)</u> $X_1 + X_2 + Y \sim \mathrm{N}\left(9.1, \dfrac{187}{50}\right)$ $\mathrm{P}\left(\dfrac{X_1 + X_2 + Y}{3} > 3.2\right) = \mathrm{P}\left(X_1 + X_2 + Y > 9.6\right) = 0.39799 \approx 0.398$ (3s.f.)
(b) (iii)	$T = 0.9(7)\left(X_1 + X_2\right) + 0.75(13)Y \sim \mathrm{N}(72.165, 327.8475)$ $\mathrm{P}(T < 80) \approx 0.667$ (3s.f.)