EUNOIA JUNIOR COLLEGE
JC2 Preliminary Examination 2021
General Certificate of Education Advanced Level
Higher 2

1 The *Jiuzhang Suanshu* (Nine Chapters on the Mathematical Art) is a Chinese mathematical manuscript written somewhere in the middle of the 3rd century. It demonstrated that the Chinese had an understanding of negative numbers way before the Europeans did. The following problem is translated and modified from the original text found in the *Jiuzhang Suanshu*:

"Sell 2 cows and 5 sheep to buy 13 pigs: there is a surplus of 1000 coins.

Sell 3 cows and 3 pigs to buy 9 sheep: there are exactly enough coins.

Sell 6 sheep and 8 pigs, then buy 5 cows: there is a deficit of 600 coins.

Tell me: what is the surplus or deficit from buying 3 cows and selling 4 pigs and 3 sheep?"

Solve this problem, giving your answer in context. [4]

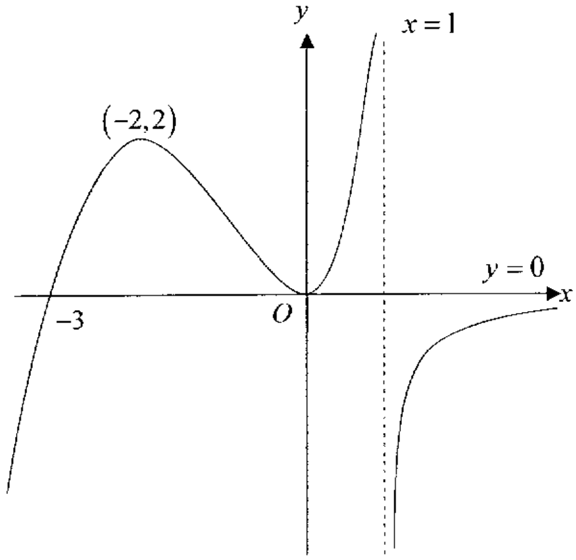
2 Using the substitution $x = \frac{\cos^2\theta}{k}$, find the exact value of $\int_0^{\frac{1}{2k}} \sqrt{\frac{kx}{1-kx}}\,\mathrm{d}x$ in terms of k, where k is a positive constant. [5]

3 **(i)** Let k be a positive constant. Sketch the curve with equation $y = \frac{kx+2}{1-x}$, stating the equations of the asymptotes. On the same diagram, sketch the line with equation $y = -x+2$. [3]

(ii) Hence, solve, in terms of k, the inequality $\frac{kx+2}{1-x} < -x+2$. [2]

(iii) Solve, in terms of k, $\frac{k\mathrm{e}^x+2}{1-\mathrm{e}^x} < -\mathrm{e}^x+2$. [2]

4 **(a)** The graph of $y = f(x)$ is given below. It has one vertical asymptote at $x = 1$ and one horizontal asymptote at $y = 0$. The graph cuts the x-axis at $x = -3$ and has turning points at $(-2, 2)$ and the origin.



Sketch the graph of $y = f'(x)$, stating the equations of any asymptotes, and indicating any axial intercepts. [3]

(b) It is given that $g(x) = x^3 - 3x + p$, where p is an unknown constant.

(i) Given that the graph of $y = \frac{1}{g(x)}$ has a vertical asymptote at $x = 2$, find the value of p. [1]

(ii) Given instead that the graph of $y = \frac{1}{g(x)}$ has a minimum point at $y = \frac{1}{5}$, find the value of p. [3]

5 **(a)** **(i)** It is given that $y\frac{dy}{dx} + x = \sqrt{x^2 + y^2}$. Using the substitution $w = x^2 + y^2$, show that the differential equation can be transformed to $\frac{dw}{dx} = f(w)$, where the function $f(w)$ is to be found. [2]

(ii) Hence, given that $y = 4$ when $x = 3$, solve the differential equation $y\frac{dy}{dx} + x = \sqrt{x^2 + y^2}$. [3]

(b) Solve the differential equation $\frac{d^2y}{dx^2} = \frac{1}{x}$, where $x > 0$. [4]

6 It is given that $\ln(y+1) = 1 - \tan x$.

(i) Show that $\frac{dy}{dx} = -(y+1)\sec^2 x$. [1]

(ii) By further differentiation of the above result, find the Maclaurin series for y, up to and including the term in x^2. [4]

(iii) Verify the correctness of your result in part **(ii)** by using small angle approximation and standard series from the List of Formulae (MF26). [3]

(iv) Use your series from part **(ii)** to estimate $\int_{-0.1}^{0} e^{1-\tan x}\,dx$, correct to 5 decimal places. [2]

7 (i) Show that $\frac{4r-1}{3^{r+2}}$ can be expressed as $\frac{Ar}{3^r} - \frac{B(r+2)}{3^{r+2}}$, where A and B are constants to be determined. [2]

(ii) Hence, find an expression for $\sum_{r=1}^{n} \frac{4r-1}{3^{r+2}}$ in terms of n. You need not express your answer as a single fraction. [3]

(iii) Give a reason why the series $\sum_{r=1}^{n} \frac{4r-1}{3^{r+2}}$ converges and write down the value of the sum to infinity. [2]

(iv) Hence, by finding $\sum_{r=1}^{\infty} \frac{1}{3^{r+2}}$, find the value of $\sum_{r=1}^{\infty} \frac{r}{3^r}$. [3]

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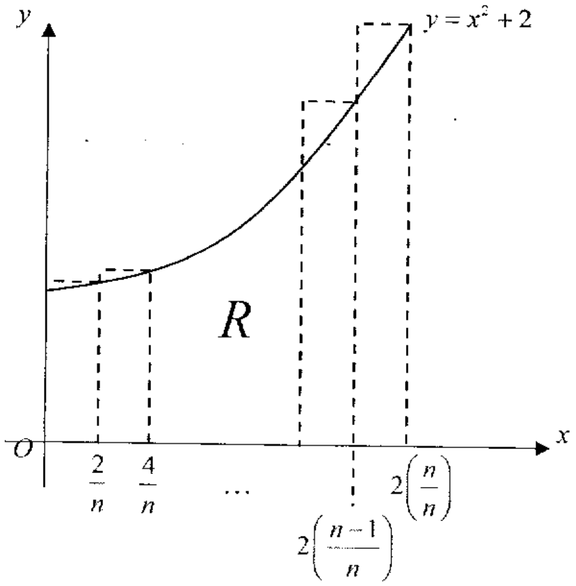


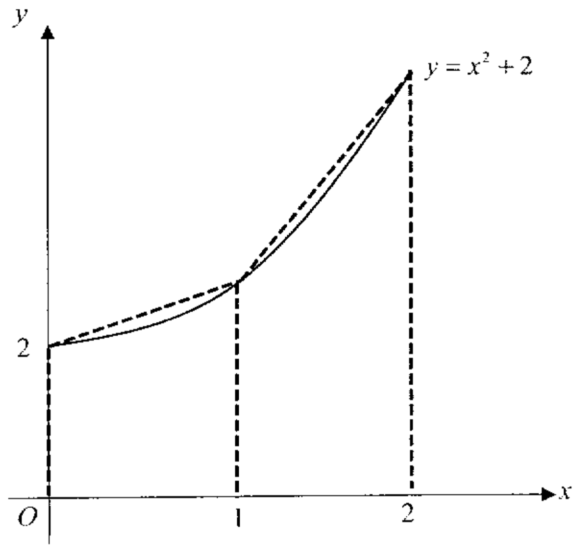
The graph of $y = f(x)$, where $f(x) = x^2 + 2, 0 \le x \le 2$, is shown. Let R be the region bounded by the x-axis, the curve $y = x^2 + 2$, and the lines $x = 0$ and $x = 2$. Let n rectangles of equal width $\frac{2}{n}$ be used to estimate the area of R, as shown in the diagram above.

(i) Given that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, show that the total area of n rectangles is $\frac{4(n+1)(2n+1)}{3n^2} + 4$. [4]

(ii) Hence find the least value of n such that the total area of n rectangles differs from the area of R by less than 0.1. [2]

Instead, n trapeziums of equal width are used to estimate the area of R.

For example, when $n = 2$, the trapeziums look like this:



(iii) Find the difference between the total area of the trapeziums and the area of R when 4 trapeziums are used. [3]

(iv) Using your answers to part **(ii)** and part **(iii)**, comment on the use of trapeziums over rectangles to estimate the value of R. [1]

9 A curve C has parametric equations $x = t + e^t$, $y = 3t + e^{-t}$.

(i) Find the exact coordinates of the stationary point of curve C. [4]

(ii) Find the coordinates of the point P on curve C for which both x and y have the same rate of change with respect to t. Hence, find the equation of the normal to the curve C at this point P. [4]

(iii) The normal at P meets the curve C again at point Q. Find the coordinates of point Q. [2]

(iv) Determine, with justification, if triangle OPQ is a right-angled triangle, where O is the origin. [2]

10 Kai is prescribed a drug called 'Aixocra' to manage a chronic long-term illness. Kai's body is able to remove half the amount of Aixocra remaining in his bloodstream every 8 hours. One pill of Aixocra contains 120mg, and one pill is to be taken every 24 hours.

(i) Show that the amount of Aixocra in Kai's bloodstream immediately after he takes the second pill is 135mg. [1]

(ii) Find, in terms of n, the amount of Aixocra in Kai's bloodstream immediately after he has taken the n^{th} pill. [4]

(iii) What is the long-term amount of Aixocra in Kai's bloodstream? [1]

Nya is also prescribed Aixocra for the same illness. However, unknown to her, her body is only able to remove 40% of Aixocra remaining in her bloodstream every 24 hours.

(iv) Studies have shown that patients with this illness will experience heart palpitations if there is more than 280mg of Aixocra in their bloodstream. Find the least number of pills it will take for Nya to experience heart palpitations. [3]

(v) On the day she experiences heart palpitations, Nya stops her intake of Aixocra. She is considered to have completely cleared Aixocra from her system when she has less than 1mg of Aixocra remaining in her bloodstream. Find the number of complete days for Aixocra to be completely cleared from her bloodstream. [2]

(vi) Nya's doctor starts her on an alternative drug called "Nefurb". She is to take 3mg of Nefurb on the first day, and increase her dosage by 2mg every day until she reaches 25mg. She is to then take 25mg every day subsequently. The doctor stops this treatment after a total of 28 days on Nefurb. What is the total amount of Nefurb taken by Nya at the end of treatment? [3]

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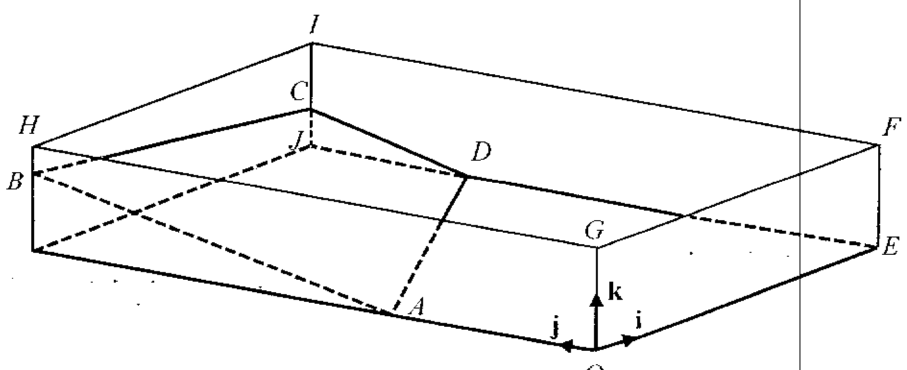


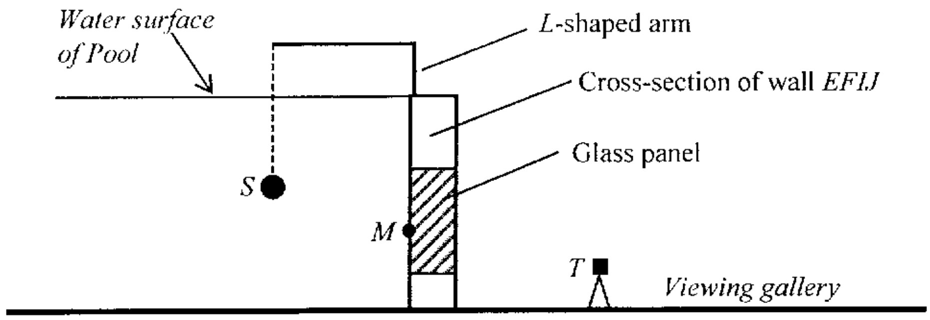
A scuba diving training pool is modelled by a rectangular cuboid (see diagram above). The floor of the pool is made up of the horizontal surface $OADE$ and the inclined surface $ABCD$. The surface of the water in the pool is the horizontal plane $FGHI$. The point O is the origin and vectors **i**, **j**, **k** each of length 1 m, are taken along OE, OA and OG respectively.

$OADE$ is part of the plane p_1 and $ABCD$ is part of the plane p_2 where p_2 has equation $-3x+6y-20z=90$.

(i) Find the acute angle between p_1 and p_2. [2]

(ii) Find a vector equation for the line passing through A and D. [2]

One important aspect of scuba diving is neutral buoyancy. Trainees learn to descend to a certain depth indicated by a spherical weight that is suspended in the water. The spherical weight hangs at the end of a chain attached to a L-shaped arm such that the centre of the spherical weight is at point $S(18, 2.5, 2)$ (see diagram below).



(iii) The diagram shows a glass panel built into the wall $EFIJ$. A video camera is located at point T in the viewing gallery behind the glass panel and it captures the underwater activity surrounding S through the glass. It is given that $M(20, 2, 1.5)$ is a point on the glass panel such that M is the mid-point of the line segment ST. Find the coordinates of T. [2]

(iv) When trainees are being assessed for their neutral buoyancy skills, a diving instructor stands along the line AD to monitor the safety of the trainees diving around the spherical weight. The instructor will proceed to swim towards S if he senses any irregularity. To ensure that the instructor is standing at a spot along AD that is nearest to S, the location where he should stand is marked with an "X". Determine the coordinates of the point marked with "X". [3]

The L-shaped arm is subsequently mounted along a different edge of the pool such that the spherical weight is now suspended in the water at point $Q(2, 15, k)$. Two objects are secured onto the floor of the pool, one object on p_1 and another on p_2. Based on a signal given by the instructor, trainees will swim from Q to either p_1 or p_2 to retrieve the object on the specified plane.

(v) Find the value of k such that Q is equidistant from p_1 and p_2. [3]

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1 Let C, P, and S denote the price of a cow, pig, and sheep respectively.

$2C-13P+5S=1000$

$3C+3P-9S=0$

$-5C+8P+6S=-600$

Solving, $C=1200, P=300, S=500$

$-3C+4P+3S=-900$ i.e. there is a deficit of 900 coins.

Alternative

$-2C+13P-5S=1000$

$-3C-3P+9S=0$

$5C-8P-6S=-600$

Solving, $C=-1200, P=-300, S=-500$

$3C-4P-3S=-900$ i.e. there is a deficit of 900 coins.

2 $x=\dfrac{\cos^2\theta}{k} \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=-\dfrac{2}{k}\cos\theta\sin\theta$

When $x=0$, $\dfrac{\cos^2\theta}{k}=0 \quad\Rightarrow\quad \theta=\dfrac{\pi}{2}$

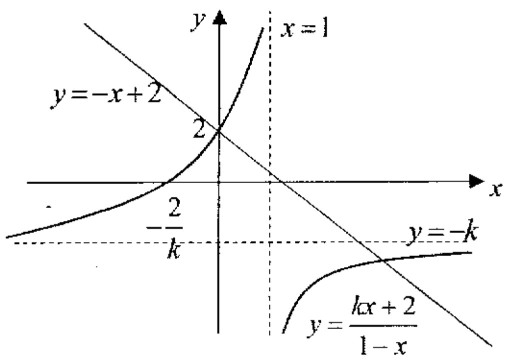
When $x=\dfrac{1}{2k}$, $\dfrac{\cos^2\theta}{k}=\dfrac{1}{2k} \quad\Rightarrow\quad \theta=\dfrac{\pi}{4}$

$$\int_0^{\frac{1}{2k}}\sqrt{\frac{kx}{1-kx}}\,\mathrm{d}x=\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\sqrt{\frac{\cos^2\theta}{1-\cos^2\theta}}\left(-\frac{2}{k}\cos\theta\sin\theta\right)\mathrm{d}\theta=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta}\left(\frac{2}{k}\cos\theta\sin\theta\right)\mathrm{d}\theta$$

$$=\frac{1}{k}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}2\cos^2\theta\,\mathrm{d}\theta=\frac{1}{k}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos2\theta+1\,\mathrm{d}\theta=\frac{1}{k}\left[\frac{\sin2\theta}{2}+\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=\frac{1}{k}\left[\frac{\sin\pi}{2}+\frac{\pi}{2}-\left(\frac{\sin\frac{\pi}{2}}{2}+\frac{\pi}{4}\right)\right]=\frac{1}{k}\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

3	**(i)**



(ii) To find the points of intersection,

$$\frac{kx+2}{1-x} = -x+2$$

$kx + 2 = (-x+2)(1-x) = x^2 - 3x + 2$.

$x[x-(3+k)] = 0$
$x = 0$ or $x = 3+k$

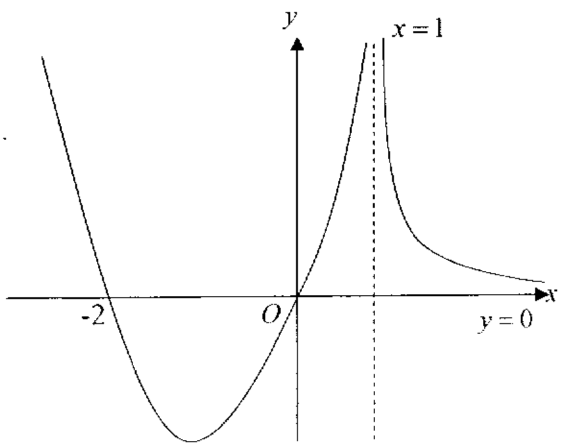
So, by interpreting the graph, $x < 0$ or $1 < x < 3+k$

(iii) Replacing x with e^x :

$e^x < 0$ (no soln $\because e^x > 0$ for all x) or $1 < e^x < 3+k$

$\therefore 0 < x < \ln(3+k)$

4	**(a)**



(bi) If $y = \frac{1}{g(x)}$ has $x = 2$ as a vertical asymptote, then $y = g(x)$ has an x-intercept at $x = 2$. So $g(2) = 0 \Rightarrow p = -2$.

(bii) If $y=\frac{1}{g(x)}$ has a minimum point at $y=\frac{1}{5}$, then $y=g(x)$ has a maximum point at $y=5$.

$g'(x)=3x^2-3=3(x+1)(x-1)$ so turning points of $y=g(x)$ are at $x=\pm 1$.

$g''(x)=6x$ so, by second derivative test, $x=1$ is a minimum point while $x=-1$ is a maximum point. So turning point is at $(-1,5)$.

$(-1)^3-3(-1)+p=5 \Rightarrow p=3$.

Alternatively,

If $y=\frac{1}{g(x)}$ has a minimum point at $y=\frac{1}{5}$, then $y=g(x)$ has a maximum point at $y=5$.

From GC, observe that $y=x^3-3x$ has a maximum point at $y=2$. Since we need to move the maximum point to $y=5$, $p=3$

5 (a)(i) $w=x^2+y^2$

Differentiating implicitly w.r.t. *x*,

$$\frac{dw}{dx}=2x+2y\frac{dy}{dx}=2\left(x+y\frac{dy}{dx}\right)$$

Thus, we have $x+y\frac{dy}{dx}=\frac{1}{2}\frac{dw}{dx}$

Substitute into the given DE: $\frac{1}{2}\frac{dw}{dx}=\sqrt{w} \Rightarrow \frac{dw}{dx}=2\sqrt{w}$

(a)(ii) Separating variables: $\frac{1}{\sqrt{w}}\frac{dw}{dx}=2$

Integrating w.r.t *x*, $\int \frac{1}{\sqrt{w}}\frac{dw}{dx}dx=\int 2\,dx$

$\Rightarrow \int w^{-\frac{1}{2}}dw=\int 2\,dx \Rightarrow 2w^{\frac{1}{2}}=2x+C$

Substituting back, $2\sqrt{x^2+y^2}=2x+C$

When $x=3$, $y=4$: $2\sqrt{3^2+4^2}=2(3)+C \Rightarrow C=4$

The particular solution is $\sqrt{x^2+y^2}=x+2$ (or $x^2+y^2=(x+2)^2$)

(b) $\frac{d^2y}{dx^2}=\frac{1}{x}$

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Integrate w.r.t. x, $\frac{dy}{dx} = \ln x + C$ (note: no need $|x|$ because $x > 0$)

Integrate w.r.t. x, $y = x\ln x - \int x \cdot \frac{1}{x}\, dx + Cx$

$= x\ln x - x + Cx + D$

$= x\ln x + C_1 x + D$ (where $C_1 = C - 1$)

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6 (i) Differentiating implicitly w.r.t. x, $\frac{1}{y+1}\frac{dy}{dx} = -\sec^2 x$.

So $\frac{dy}{dx} = -(y+1)\sec^2 x$ --(1).

(ii) Differentiating again w.r.t. x, $\frac{d^2y}{dx^2} = -\frac{dy}{dx}\sec^2 x - 2(y+1)\sec^2 x \tan x$ --(2).

When $x = 0$,

$\ln(y+1) = 1 - \tan 0$

$y = e - 1$

Substituting $x = 0, y = e - 1$ into (1):

$\frac{dy}{dx} = \frac{-(e-1+1)}{\cos^2 0} = -e$.

Substituting into (2):

$\frac{d^2y}{dx^2} = \frac{-(-e)}{\cos^2 0} - \frac{2(e)\tan 0}{\cos^2 0} = e$.

So $y = (e-1) - ex + \frac{e}{2}x^2 + \ldots$

(iii) Rearranging, $y = e^{1-\tan x} - 1$. When x is small, $\tan x \approx x$.

$y \approx e(e^{-x}) - 1$

$\approx e(1 - x + \frac{x^2}{2}) - 1$

$= (e-1) - ex + \frac{e}{2}x^2$

(iv) $e^{1-\tan x} \approx e - ex + \frac{e}{2}x^2$ so $\int_{-0.1}^{0} e^{1-\tan x}\, dx \approx \int_{-0.1}^{0} e - ex + \frac{e}{2}x^2\, dx = 0.28587$ (to 5 dp)

7 **(i)** Let $\frac{4r-1}{3^{r+2}}=\frac{Ar}{3^r}-\frac{B(r+2)}{3^{r+2}}=\frac{(9A-B)r-2B}{3^{r+2}}$

Comparing constants: $B=0.5$

Comparing coefficients of r: $4=9A-0.5 \Rightarrow A=0.5$

$\therefore \frac{4r-1}{3^{r+2}}=\frac{1}{2}\left[\frac{r}{3^r}-\frac{r+2}{3^{r+2}}\right]$

(ii) $\sum_{r=1}^{n}\frac{4r-1}{3^{r+2}}=\frac{1}{2}\sum_{r=1}^{n}\left[\frac{r}{3^r}-\frac{r+2}{3^{r+2}}\right]$.

$$=\frac{1}{2}\left[\frac{1}{3}-\frac{3}{3^3}\right.$$
$$+\frac{2}{3^2}-\frac{4}{3^4}$$
$$+\frac{3}{3^3}-\frac{5}{3^5}$$
$$\vdots$$
$$+\frac{n-2}{3^{n-2}}-\frac{n}{3^n}$$
$$+\frac{n-1}{3^{n-1}}-\frac{n+1}{3^{n+1}}$$
$$\left.+\frac{n}{3^n}-\frac{n+2}{3^{n+2}}\right]$$
$$=\frac{1}{2}\left(\frac{5}{9}-\frac{n+1}{3^{n+1}}-\frac{n+2}{3^{n+2}}\right)$$ (or equivalent)

(iii) Series converges as $\lim_{n\to\infty}\frac{n+1}{3^{n+1}}=\lim_{n\to\infty}\frac{n+2}{3^{n+2}}=0$. $\sum_{r=1}^{\infty}\frac{4r-1}{3^{r+2}}=\frac{5}{18}$.

(iv) $\sum_{r=1}^{\infty}\frac{1}{3^{r+2}}=\frac{1}{3^3}+\frac{1}{3^4}+...=\frac{\frac{1}{27}}{1-\frac{1}{3}}=\frac{1}{18}$.

Hence,

$$\sum_{r=1}^{\infty}\frac{4r-1}{3^{r+2}}=\sum_{r=1}^{\infty}\left(\frac{4r}{3^{r+2}}-\frac{1}{3^{r+2}}\right)$$

$$\sum_{r=1}^{\infty}\frac{4r-1}{3^{r+2}}=\sum_{r=1}^{\infty}\frac{4r}{3^{r+2}}-\sum_{r=1}^{\infty}\frac{1}{3^{r+2}}$$

From part **(iii)** and the sum just found,

$$\frac{5}{18}=\sum_{r=1}^{\infty}\frac{4r}{3^{r+2}}-\frac{1}{18}$$

$$\sum_{r=1}^{\infty}\frac{4r}{3^{r+2}}=\frac{5}{18}+\frac{1}{18}=\frac{1}{3}$$

$$\sum_{r=1}^{\infty}\frac{4r}{3^2\cdot 3^r}=\frac{1}{3}$$

$$\frac{4}{9}\sum_{r=1}^{\infty}\frac{r}{3^r}=\frac{1}{3}$$

$$\therefore \sum_{r=1}^{\infty}\frac{r}{3^r}=\frac{1}{3}\times\frac{9}{4}=\frac{3}{4}.$$

8 (i) total area $=\frac{2}{n}\left\{\left[\left(\frac{2}{n}\right)^2+2\right]+\left[\left(\frac{4}{n}\right)^2+2\right]+\ldots+\left[\left(\frac{2(n-1)}{n}\right)^2+2\right]+\left[\left(\frac{2n}{n}\right)^2+2\right]\right\}$

$$=\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2+2\right]$$

$$=\frac{2}{n}\sum_{i=1}^{n}\left(\frac{4}{n^2}i^2+2\right)$$

$$=\frac{2}{n}\left[\left(\frac{4}{n^2}\right)\sum_{i=1}^{n}i^2+\sum_{i=1}^{n}2\right]$$

$$=\frac{2}{n}\left(\frac{4}{n^2}\frac{n(n+1)(2n+1)}{6}+2n\right)$$

$$=\frac{4(n+1)(2n+1)}{3n^2}+4 \quad \text{(shown)}$$

(ii) Area of $R=\int_0^2 x^2+2\,\mathrm{d}x=\left[\frac{x^3}{3}+2x\right]_0^2=\frac{20}{3}$

Let A_n denote the total area when n rectangles are used.

So we need $A_n-R=\frac{4(n+1)(2n+1)}{3n^2}+4-\frac{20}{3}<0.1$

From GC,

$n=40$: $A_{40}-R=0.1008$

$n=41$: $A_{41}-R=0.0984$

So least $n=41$

(iii) [Note that area of a trapezium $=\frac{1}{2}$(sum of parallel sides)$\times$height.]

For n trapeziums, width $=\frac{2}{n}$. So for $n=4$, width $=\frac{2}{4}=\frac{1}{2}$

Let B_n denote the total area when n trapeziums are used.

$$B_4=\frac{1}{2}\left(\mathrm{f}(0)+\mathrm{f}\left(\frac{1}{2}\right)\right)\times\frac{1}{2}+\frac{1}{2}\left(\mathrm{f}\left(\frac{1}{2}\right)+\mathrm{f}\left(\frac{2}{2}\right)\right)\times\frac{1}{2}+\frac{1}{2}\left(\mathrm{f}\left(\frac{2}{2}\right)+\mathrm{f}\left(\frac{3}{2}\right)\right)\times\frac{1}{2}+\frac{1}{2}\left(\mathrm{f}\left(\frac{3}{2}\right)+\mathrm{f}(2)\right)\times\frac{1}{2}$$

$$=\frac{1}{2}\left[\left(\mathrm{f}(0)+\mathrm{f}\left(\frac{1}{2}\right)\right)+\left(\mathrm{f}\left(\frac{1}{2}\right)+\mathrm{f}\left(\frac{2}{2}\right)\right)+\left(\mathrm{f}\left(\frac{2}{2}\right)+\mathrm{f}\left(\frac{3}{2}\right)\right)+\left(\mathrm{f}\left(\frac{3}{2}\right)+\mathrm{f}(2)\right)\right]\times\frac{1}{2}$$

$$=\frac{1}{4}\left[\mathrm{f}(0)+2\mathrm{f}\left(\frac{1}{2}\right)+2\mathrm{f}\left(\frac{2}{2}\right)+2\mathrm{f}\left(\frac{3}{2}\right)+\mathrm{f}(2)\right]$$

$$=6.75 \text{ units}^2$$

$B_4-R=0.0833$

(iv) Note that $B_4-R=0.0833<0.1$.

In (ii), it takes 41 rectangles to be less than 0.1 units^2 of the actual area, whereas in (iii) it only takes 4 trapeziums to achieve that. So using trapeziums to estimate the area is more efficient.

[Possible way to organise information from the two parts to help identify the more significant aspect to comment on:

	Shape	n	Difference
(ii)	rectangles	41	<0.1
(iii)	trapeziums	4	0.0833
Comparison		~ 10 times	~ equal

9 (i) $\frac{dx}{dt}=1+e^t$, $\frac{dy}{dt}=3-e^{-t}$

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{3-e^{-t}}{1+e^t}$$

$$\frac{3-e^{-t}}{1+e^t}=0$$

$$3-e^{-t}=0$$

$$t=-\ln 3$$

$$x=-\ln 3+e^{-\ln 3}=-\ln 3+\frac{1}{3}\quad,\quad y=-3\ln 3+e^{\ln 3}=-3\ln 3+3$$

coordinates of the stationary point is $\left(-\ln 3+\frac{1}{3},-3\ln 3+3\right)$

(ii) $\frac{dx}{dt}=1+e^t$, $\frac{dy}{dt}=3-e^{-t}$

$$\frac{dy}{dt}=\frac{dx}{dt}$$

$3-e^{-t}=1+e^{t}$

$e^{2t}-2e^{t}+1=0$

$\left(e^{t}-1\right)^{2}=0$

$e^{t}=1$

$t=0$

$x=0+e^{0}=1$; $y=3(0)+e^{0}=1$

coordinates of point P is $(1,1)$

gradient of normal at $P=-\dfrac{1+e^{0}}{3-e^{-0}}=-1$

Equation of normal at P: $y-1=-1(x-1)\Rightarrow y=-x+2$

(iii) Sub $x=t+e^{t}$, $y=3t+e^{-t}$ into $y=-x-2$, $3t+e^{-t}=-\left(t+e^{t}\right)+2$

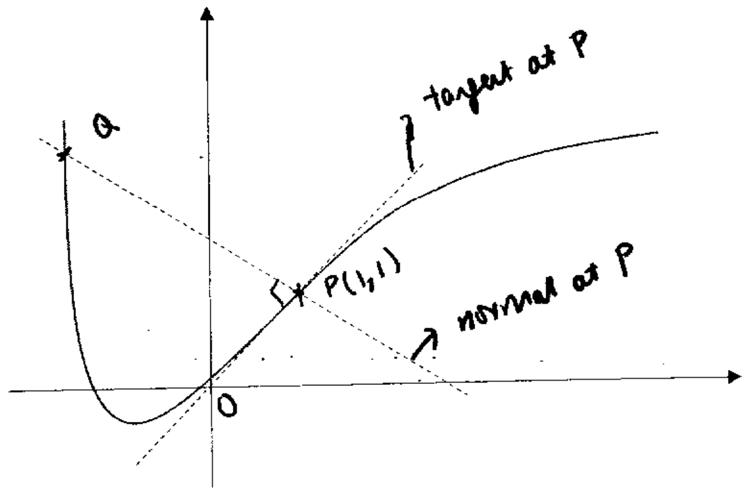
Using GC, $t=0$ or $t=-2.466497$

Point Q:

$\left(-2.466497+e^{-2.466497}, 3(-2.466497)+e^{2.466497}\right)$

$(-2.38, 4.38)$

(iv) Gradient of normal at P is -1. Thus gradient of QP is -1. Gradient of line OP is 1 since P is point $(1,1)$. Since the product of their gradients is -1, OP is perpendicular to QP and thus triangle OPQ is a right-angled triangle.



Alternative

$OQ^{2}=24.871$, $OP^{2}=2$, $PQ^{2}=22.871$

Since $OQ^{2}=OP^{2}+PQ^{2}$, by Pythagoras' Theorem, $OP\perp PQ$ and so triangle OPQ is a right-angled triangle.

$\overrightarrow{OP} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\overrightarrow{PQ} = \begin{pmatrix} -2.38-1 \\ 4.38-1 \end{pmatrix} = \begin{pmatrix} -3.38 \\ 3.38 \end{pmatrix}$

Since $\overrightarrow{OP} \cdot \overrightarrow{PQ} = 0$, then $OP \perp PQ$ and so triangle OPQ is a right-angled triangle.

10 **(i)** Amt immediately after 2nd dose $= 120 \times \left(\frac{1}{2}\right)^3 + 120 = 135\text{mg (shown)}$.

(ii) amt of drug $= 120 + 120 \times \left(\frac{1}{8}\right) + ... + 120 \times \left(\frac{1}{8}\right)^{n-1} = \dfrac{120\left(1-\left(\frac{1}{8}\right)^n\right)}{1-\frac{1}{8}} = \dfrac{960}{7}\left(1-\left(\frac{1}{8}\right)^n\right)$

(iii) As $n \to \infty$, $\left(\frac{1}{8}\right)^n \to 0$ so long-term amt = 137mg (3s.f.) (or $\frac{960}{7}$ mg)

(iv) Amt in Nya's bloodstream after k^{th} dose $= \dfrac{120(1-0.6^k)}{1-0.6} = 300(1-0.6^k)$

We want $300(1-0.6^k) > 280$

Method 1 (from GC)

k	$300(1-0.6^k)$
5	276.67
6	286

$\therefore$ least $k = 6$, i.e. Nya will experience heart palpitations after the 6th dose.

Method 2 (algebraic)

$$300(1-0.6^k) > 280 \Rightarrow 0.6^k < \frac{1}{15} \Rightarrow k > \frac{\ln\left(\frac{1}{15}\right)}{\ln 0.6} = 5.3013$$

$\therefore$ least $k = 6$, i.e. Nya will experience heart palpitations after the 6th dose.

(v) Let m be the number of complete days taken from the onset of palpitations till completely cleared.

We want $286 \times 0.6^m < 1$

m	286×0.6^m
11	1.0376
12	0.6226

$\therefore$ least $m = 12$

(vi) Let n be the number of days taken to reach 25mg. Then $3 + (n-1)(2) = 25 \Rightarrow n = 12$.

Total mg $= \frac{12}{2}[2(3) + 11(2)] + (28-12) \times 25 = 568\text{mg}$

11	**(i)**	$p_1 : \mathbf{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0 \quad p_2 : \mathbf{r}\cdot\begin{pmatrix}-3\\6\\-20\end{pmatrix}=90$ Let θ be the required acute angle. $\cos\theta = \dfrac{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-3\\6\\-20\end{pmatrix}\right\|}{\sqrt{9+36+400}} = \dfrac{20}{\sqrt{445}}$ $\Rightarrow \theta = 18.542^\circ = 18.5^\circ$ (1 d.p.)
	(ii)	The line AD is the line of intersection between p_1 and p_2. $p_1 : z=0$ $p_2 : -3x+6y-20z=90.$ **Method 1 (use GC PlySmlt2)** Solving $z=0$ and $-3x+6y-20z=90$ $\left.\begin{array}{r}x=-30+2y\\ y=y\\ z=0\end{array}\right\\} \; \mathbf{r}=\begin{pmatrix}-30\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}, \lambda\in\mathbb{R}$ **Method 2 (find direction of line AD)** A direction of line AD is $\mathbf{n}_1\times\mathbf{n}_2=\begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}-3\\6\\-20\end{pmatrix}=\begin{pmatrix}6\\3\\0\end{pmatrix}=3\begin{pmatrix}2\\1\\0\end{pmatrix}$ Also, at A, $x=0$ and $z=0$ $\Rightarrow -3(0)+6y-20(0)=90 \Rightarrow 6y=90 \Rightarrow y=15$ So $A(0,15,0)$. So eqn of $l_{AD} : \mathbf{r}=\begin{pmatrix}0\\15\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}, \lambda\in\mathbb{R}$ **Method 3 (solve 2 eqns manually)** Sub $z=0$ into $-3x+6y-20z=90$: $-3x+6y=90 \Rightarrow y=15+\dfrac{1}{2}x$ OR $x=-30+2y$

Let $x=\alpha,\ \alpha\in\mathbb{R}$ Then $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\alpha\\15+\frac{1}{2}\alpha\\0\end{pmatrix}$ $l_{AD}:\mathbf{r}=\begin{pmatrix}0\\15\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\\frac{1}{2}\\0\end{pmatrix},\ \alpha\in\mathbb{R}$ OR $\mathbf{r}=\begin{pmatrix}0\\15\\0\end{pmatrix}+\alpha\begin{pmatrix}2\\1\\0\end{pmatrix},\ \alpha\in\mathbb{R}$	Let $y=\beta,\quad \beta\in\mathrm{R}$ Then $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-30+2\beta\\\beta\\0\end{pmatrix}$ $l_{AD}:\mathbf{r}=\begin{pmatrix}-30\\0\\0\end{pmatrix}+\beta\begin{pmatrix}2\\1\\0\end{pmatrix},\beta\in\mathbb{R}$

(iii) Given that $M(20,2,1.5)$ is the mid-point of ST,

$$\overrightarrow{OM}=\frac{\overrightarrow{OS}+\overrightarrow{OT}}{2}\Rightarrow\overrightarrow{OT}=2\begin{pmatrix}20\\2\\1.5\end{pmatrix}-\begin{pmatrix}18\\2.5\\2\end{pmatrix}=\begin{pmatrix}22\\1.5\\1\end{pmatrix}$$

$\therefore T(22,1.5,1)$

(iv) Since X is nearest to S, we can see that X is the foot of perpendicular from S to the line AD.

$$\overrightarrow{OX}=\begin{pmatrix}-30\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}\text{ for some }\lambda\in\mathbb{R}$$

$$\overrightarrow{SX}=\overrightarrow{OX}-\overrightarrow{OS}=\left[\begin{pmatrix}-30\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}\right]-\begin{pmatrix}18\\2.5\\2\end{pmatrix}=\begin{pmatrix}-48\\-2.5\\-2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}$$

Since $\overrightarrow{SX}\perp l_{AD}$, $\left[\begin{pmatrix}-48\\-2.5\\-2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\0\end{pmatrix}=0$

$$-96-2.5+\lambda(4+1)=0$$

$$\Rightarrow\lambda=19.7$$

$$\overrightarrow{OX}=\begin{pmatrix}-30\\0\\0\end{pmatrix}+19.7\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}9.4\\19.7\\0\end{pmatrix}\quad\therefore X(9.4,19.7,0)$$

[Remark: The steps if you used $\overrightarrow{OX}=\begin{pmatrix}0\\15\\0\end{pmatrix}+\alpha\begin{pmatrix}2\\1\\0\end{pmatrix}$ for some $\alpha\in\mathbb{R}$ are similar.]

(v) **Method 1** ($\perp$ dist from Q to p_1 = $\perp$ dist from Q to p_2)

$\perp$ dist from Q to $p_1=k$ ($\because Q(2,15,k)$, and p_1 is horizontal)

$\perp$ dist from Q to p_2 :

A point on p_2 is $(-30, 0, 0)$. Call this point R.

So $\overrightarrow{QR} = \begin{pmatrix}-30\\0\\0\end{pmatrix} - \begin{pmatrix}2\\15\\k\end{pmatrix} = \begin{pmatrix}-32\\-15\\-k\end{pmatrix}$

Then $\perp$ dist from Q to $p_2 = \left|\overrightarrow{QR}.\hat{\mathbf{n}}_2\right| = \dfrac{\left|\begin{pmatrix}-32\\-15\\-k\end{pmatrix}.\begin{pmatrix}-3\\6\\-20\end{pmatrix}\right|}{\sqrt{445}} = \dfrac{6+20k}{\sqrt{445}}$

So $k = \dfrac{|6+20k|}{\sqrt{445}} \Rightarrow k = 5.4793 = 5.48$ (3s.f.)

[Remark: The steps if you used $(0,15,0)$ - which is point A – are similar.]

Method 2 (instead of using LoP* above)

$\perp$ dist from Q to $p_2 = \dfrac{\left|\begin{pmatrix}2\\15\\k\end{pmatrix}.\begin{pmatrix}-3\\6\\-20\end{pmatrix} - 90\right|}{\sqrt{445}} = \dfrac{|-6-20k|}{\sqrt{445}} = \dfrac{6+20k}{\sqrt{445}}$

So $k = \dfrac{6+20k}{\sqrt{445}} \Rightarrow k = 5.4793 = 5.48$ (3s.f.)

EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2021

General Certificate of Education Advanced Level

Higher 2

Section A: Pure Mathematics [40 marks]

1 **Do not use a calculator in answering this question.**

The function f is defined by $\mathrm{f}(z)=4z^3-12z^2+13z-10$. Given that $\mathrm{f}\left(\frac{1}{2}+\mathrm{i}\right)=0$, find all the roots of $\mathrm{f}(z)$. [4]

2 It is given that $z=\cos\theta+\mathrm{i}\sin\theta$, where $0<\theta<\frac{\pi}{2}$.

(i) Show that $\mathrm{e}^{\mathrm{i}\left(\theta-\frac{\pi}{2}\right)}=\sin\theta-\mathrm{i}\cos\theta$. [1]

(ii) Hence, or otherwise, show that $\arg\left(1-z^2\right)=\theta-\frac{\pi}{2}$ and find the modulus of $1-z^2$. [3]

(iii) Hence, represent the complex number $1-z^2$ on an Argand diagram. [2]

(iv) Given that $\dfrac{z^*}{z^3\left(1-z^2\right)}$ is real, find the possible values of θ. [3]

3 **(a)** The function f is given by $\mathrm{f}:x\mapsto\cos\left(\frac{1}{2}x+\frac{1}{6}\pi\right),\quad x\in\mathbb{R},\ 0\le x\le k.$

(i) State the largest exact value of k for which the function f^{-1} exists. [1]

For the rest of the question, the domain of f is $x\in\mathbb{R},\ 0\le x\le\frac{4}{3}\pi$.

(ii) Write down the equation of the line in which the graph of $y=\mathrm{f}(x)$ must be reflected in order to obtain the graph of $y=\mathrm{f}^{-1}(x)$. Hence, sketch on the same diagram, the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$, indicating the exact coordinates of the endpoints of both graphs. [3]

(iii) State the value(s) of x for which $\mathrm{ff}^{-1}(x)=\mathrm{f}^{-1}\mathrm{f}(x)$. [1]

(b) The functions g and h are defined by

$$\mathrm{h}:x\mapsto 2x^2+3,\qquad x\in\mathbb{R},\ x\le 0,$$
$$\mathrm{hg}:x\mapsto 2x+3-2a,\qquad x>a,\ \text{where } a\in\mathbb{R}^+$$

Find $\mathrm{g}(x)$ and state the domain of g. [3]

4 The curve C has equation $y = 3 - \dfrac{10}{x^2 - 2x + 4}$.

(i) Without using a calculator, determine the exact coordinates of the stationary point of C. [2]

(ii) Sketch C, stating clearly the equations of any asymptotes. [2]

The region R is bounded by C, the line $3y + 16x = 15$ and the y-axis.

(iii) Find the exact area of the region R. [3]

(iv) Find the volume generated when R is rotated through 2π radians about the y-axis. [3]

5 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point P is on the line AB such that $AP : PB = m : n$ where m and n are positive integers. Point C is on OP extended such that $OP : PC = 1 : 2$.

(i) Show that $\overrightarrow{AC} = \left(\dfrac{2n - m}{m + n}\right)\mathbf{a} + \left(\dfrac{3m}{m + n}\right)\mathbf{b}$. [3]

(ii) If $|\mathbf{a}| = |\mathbf{b}|$ and the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{\pi}{3}$, find the area of the triangle ABC in terms of $|\mathbf{a}|$.

[4]

(iii) Find the ratio $AP : PB$ such that AC is parallel to OB. [2]

Section B: Probability and Statistics [60 marks]

6 For events A and B it is given that $\mathrm{P}(A) = \dfrac{3}{5}$, $\mathrm{P}(A \cap B) = \dfrac{7}{60}$ and $\mathrm{P}(A' | B) = \dfrac{13}{20}$.

(i) Show that $\mathrm{P}(B) = \dfrac{1}{3}$. [3]

For a third event C, it is given that $\mathrm{P}(C) = \dfrac{3}{10}$, and that $\mathrm{P}(B \cap C) = \dfrac{1}{10}$.

(ii) Determine if events B and C are independent. [1]

(iii) Given also that $\mathrm{P}(A \cap B \cap C) = \dfrac{1}{15}$, find the greatest and least possible values of $\mathrm{P}(A \cap B' \cap C)$.

[3]

7 John has 4 different pairs of socks in his drawer. Without looking at his drawer, he randomly draws one sock at a time from his drawer without replacement, until he obtains a matching pair. The total number of socks John draws is denoted by S.

(i) Show that $\mathrm{P}(S = 3) = \dfrac{2}{7}$. [1]

(ii) Explain why $P(S=k)=0$ for $k=6,7,8$. [1]

(iii) Determine the probability distribution of S. [3]

(iv) Find E(S) and Var(S). [3]

Every morning, John draws socks using the above procedure. Every evening, John returns all the drawn socks to his drawer so that, the following morning, he will again have four pairs of socks in his drawer to draw from. Assume that the number of socks drawn on any given day is independent of the number of socks drawn on any other day. Let T denote the total number of socks John draws in a given span of 3 days.

(v) Explain why $P(T=9)>\left(\frac{2}{7}\right)^3$. [1]

(vi) Find Var(T). [1]

8 Mei Li has 3 types of gemstones comprising 6 Emeralds, 4 Sapphires and 3 Labradorites. All the gemstones are of different sizes.

(i) Find the number of ways Mei Li can arrange all the gemstones in a circle, where the biggest of each type of gemstone are together. [2]

(ii) Next, Mei Li selects 9 gemstones with 3 of each type to arrange in a circle. Find the number of ways to do this, such that the gemstones alternate in the following order, clockwise: Emerald, Sapphire, Labradorite. [3]

(iii) Finally, Mei Li decides to randomly arange all 13 gemstones in a row. Find the number of ways to do this, such that no two Sapphires are next to each other and the Labradorites are all together. [3]

Mei Li selects 7 gemstones to be showcased at a conference.

(iv) Find the number of ways to select the gemstones such that at least one of each type is selected. [3]

9 A company produces light bulbs with a brightness rating of 470 lumens. It is known that the brightness of these light bulbs is normally distributed with a standard deviation of 10 lumens. A production manager wishes to test whether the mean brightness is less than 470 lumens. He collects a random sample of 40 light bulbs and finds that their mean brightness is 466.8 lumens.

(i) Explain what is meant by the term "random sample" in the context of this question. [1]

(ii) Carry out a test at the 5% level of significance for the production manager, clearly stating the p-value of this test. Explain what p-value means in the context of this question. [5]

(iii) Suppose that the production manager had tested if the mean brightness differed from 470 lumens at the 5% level of significance. Without performing another hypothesis test, determine whether your conclusion in **part (ii)** would be affected. [2]

The company decides to produce a new line of 'lowlight' bulbs with a brightness rating of 350 lumens. A random sample of 50 'lowlight' bulbs is taken and the brightness level, y lumens, are summarised as follows.

$$n=50 \qquad \sum(y-350)=-125 \qquad \sum(y-347.5)^2=4704$$

(iv) Calculate unbiased estimates of the population mean and variance for the brightness of the 'lowlight' bulbs. [2]

(v) Given instead that the standard deviation of the brightness of the 'lowlight' bulbs is known to be σ, the company found insufficient evidence from this sample to conclude that the mean brightness was less than 350 lumens at the 1% level of significance. Determine the range of values of σ. [2]

10 In this question, you should state the parameters of any distributions that you use.

A stationery factory manufactures pens for sale. The diameters (in mm) of the pens have distribution $\mathrm{N}(10, 0.003)$. The pens are packed in sets of 24 into a box. Within the box, the pens are laid out side by side. The widths of the boxes (in mm) are normally distributed with mean 240.5 mm and variance 0.02 mm^2.

(i) Find the probability that a random sample of 24 pens would fit into a randomly selected box. [3]

A pen is considered "defective" if its diameter is not within 0.1mm of the population mean.

(ii) Find the probability that a randomly selected pen is defective. [2]

Pens are produced and inspected in batches. For each batch, a sample of 12 pens is randomly selected and checked.

- If there are fewer than 2 defective pens in this sample of 12, the batch passes the inspection.
- If there are exactly 2 defective pens in this sample, a second sample of 12 pens is randomly selected and checked. If there are no defective pens in the second sample, the batch passes the inspection.
- Otherwise, the batch does not pass the inspection.

(iii) Show that the probability of a batch of pens passing the inspection is 0.871, correct to 3 significant figures. [3]

(iv) Find the probability that not more than 3 defective pens were found in a batch of pens during the inspection process, given that the batch of pens did not pass the inspection. [3]

The factory also manufactures and packs pencils and crayons in boxes of 12 each. During a promotion period, special pencils and special crayons are produced and randomly included in some boxes. The number of special pencils in a box of pencils has distribution $\mathrm{B}(12, 0.07)$, while the number of special crayons in a box of crayons has distribution $\mathrm{B}(12, 0.06)$.

(v) 40 boxes of pencils and 40 boxes of crayons are randomly selected. Find the probability that the mean number of special pencils per box of pencils is more than the mean number of special crayons per box of crayons. [4]

EUNOIA JUNIOR COLLEGE

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Section A: Pure Mathematics [40 marks]

1 Since all the coefficients are real, $z=\frac{1}{2}-\mathrm{i}$ is also a root.

Quadratic factor of $\mathrm{f}(z)$

$$\left[z-\left(\frac{1}{2}+\mathrm{i}\right)\right]\left[z-\left(\frac{1}{2}-\mathrm{i}\right)\right]=\left[\left(z-\frac{1}{2}\right)-\mathrm{i}\right]\left[\left(z-\frac{1}{2}\right)+\mathrm{i}\right]$$

$$=\left(z-\frac{1}{2}\right)^2-\mathrm{i}^2$$

$$=z^2-z+\frac{1}{4}-(-1)$$

$$=z^2-z+\frac{5}{4}$$

$$4z^3-12z^2+13z-10=\left(z^2-z+\frac{5}{4}\right)(Az+B)$$

Compare coefficients of z^3, $A=4$

Compare constant term, $B=-8$

So the 3rd factor is $4z-8$.

Thus the roots are $z=\frac{1}{2}+\mathrm{i}, \frac{1}{2}-\mathrm{i}$, and 2.

2 **(i)** $$\mathrm{e}^{\mathrm{i}\left(\theta-\frac{\pi}{2}\right)}=\cos\left(\theta-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(\theta-\frac{\pi}{2}\right)$$

$$=\cos\theta\cos\frac{\pi}{2}+\sin\theta\sin\frac{\pi}{2}+\mathrm{i}\left(\sin\theta\cos\frac{\pi}{2}-\cos\theta\sin\frac{\pi}{2}\right)$$

$$=0+\sin\theta+\mathrm{i}(0-\cos\theta)$$

$$=\sin\theta-\mathrm{i}\cos\theta$$

Alternative 1

$$\mathrm{e}^{\mathrm{i}\left(\theta-\frac{\pi}{2}\right)}=\mathrm{e}^{\mathrm{i}\theta}\times\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}\right)}=\mathrm{e}^{\mathrm{i}\theta}(-\mathrm{i})=(-\mathrm{i})(\cos\theta+\mathrm{i}\sin\theta)=\sin\theta-\mathrm{i}\cos\theta$$

Alternative 2

$$e^{i\left(\theta-\frac{\pi}{2}\right)}=\frac{e^{i\theta}}{e^{i\left(\frac{\pi}{2}\right)}}=\frac{e^{i\theta}}{i}=\frac{e^{i\theta}}{i}\times\frac{-i}{-i}=-i(\cos\theta+i\sin\theta)=\sin\theta-i\cos\theta$$

(ii) $1-z^2=1-(\cos\theta+i\sin\theta)^2$

$$=1-\left(\cos^2\theta+2i\cos\theta\sin\theta-\sin^2\theta\right)$$
$$=1-\cos^2\theta+\sin^2\theta-2i\cos\theta\sin\theta$$
$$=2\sin^2\theta-2i\cos\theta\sin\theta$$
$$=2\sin\theta(\sin\theta-i\cos\theta)$$
$$=2\sin\theta e^{i\left(-\frac{\pi}{2}-\theta\right)}\text{[from (i)]}$$

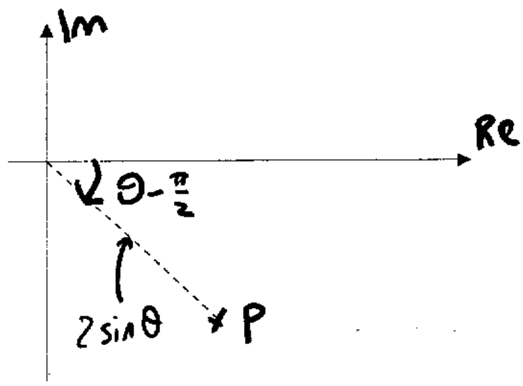
Thus $\arg(1-z^2)=\theta-\frac{\pi}{2}$ and $|1-z^2|=2\sin\theta$.

Alternative

$$1-z^2=z\left(\frac{1}{z}-z\right)=e^{i\theta}(-2i\sin\theta)=2\sin\theta\times e^{i\left(\theta-\frac{\pi}{2}\right)}\quad\left(\because -i=e^{i\left(-\frac{\pi}{2}\right)}\right)$$

Thus $\arg(1-z^2)=\theta-\frac{\pi}{2}$ and $|1-z^2|=2\sin\theta$.

(iii) Let point P represent the complex number $1-z^2$.



(iv)

$$\arg\left(\frac{z^*}{z^3(1-z^2)}\right)=\arg(z^*)-\arg(z^3)-\arg(1-z^2)$$
$$=-\arg(z)-3\arg(z)-\arg(1-z^2)$$
$$=-\theta-3\theta-\left(\theta-\frac{\pi}{2}\right)$$
$$=\frac{\pi}{2}-5\theta$$

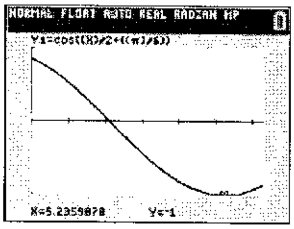
Since $\left(\dfrac{z^*}{z^3\left(1-z^2\right)}\right)$ is real, $\arg\left(\dfrac{z^*}{z^3\left(1-z^2\right)}\right)=k\pi, k\in\mathbb{Z}$

$$\frac{\pi}{2}-5\theta=k\pi$$

$$\theta=\left(\frac{1}{10}-\frac{k}{5}\right)\pi, k\in\mathbb{Z}$$

Since $0<\theta<\dfrac{\pi}{2}$, $\theta=\dfrac{\pi}{10},\dfrac{3}{10}\pi$

3 **(a)(i)** From graph on GC, largest k is when $y=-1$

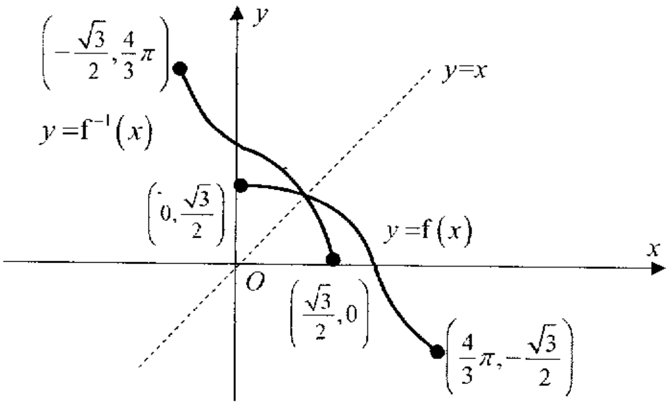


$$\cos\left(\frac{x}{2}+\frac{\pi}{6}\right)=-1$$

i.e. $\Rightarrow \dfrac{x}{2}+\dfrac{\pi}{6}=\pi$

$$\Rightarrow x=2\left(\pi-\frac{\pi}{6}\right)=\frac{5\pi}{3}$$

(a)(ii)



(a)(iii) $ff^{-1}(x)=f^{-1}f(x)=x$

The two functions are equal when the domains intersect.

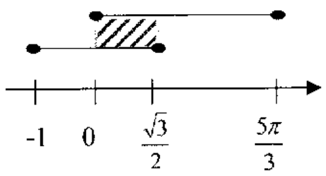
From graph in **(a)(ii)**, $0\le x\le\dfrac{\sqrt{3}}{2}$.

$\therefore \mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x$ for $0 \le x \le \frac{\sqrt{3}}{2}$

Alternative: (consider domains)

$$D_{\mathrm{f}^{-1}\mathrm{f}} = D_{\mathrm{f}} = \left[0, \frac{5\pi}{3}\right]$$

$$D_{\mathrm{ff}^{-1}} = D_{\mathrm{f}^{-1}} = R_{\mathrm{f}} = \left[-1, \frac{\sqrt{3}}{2}\right]$$



$\therefore \mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x$ for $0 \le x \le \frac{\sqrt{3}}{2}$

(b) $\mathrm{h} : x \mapsto 2x^2 + 3, \quad x \in \mathbb{R}, x \le 0,$

$\mathrm{hg} : x \mapsto 2x + 3 - 2a, \quad x \in \mathbb{R}^-, x > a.$

$$\mathrm{h}[\mathrm{g}(x)] = 2x + 3 - 2a$$

$$2[\mathrm{g}(x)]^2 + 3 = 2x + 3 - 2a$$

$$[\mathrm{g}(x)]^2 = x - a$$

$$\mathrm{g}(x) = -\sqrt{x-a} \quad \because R_{\mathrm{g}} \subseteq D_{\mathrm{h}} = (-\infty, 0]$$

$$D_{\mathrm{g}} = D_{\mathrm{hg}} = (a, \infty)$$

Alternative: find h^{-1}

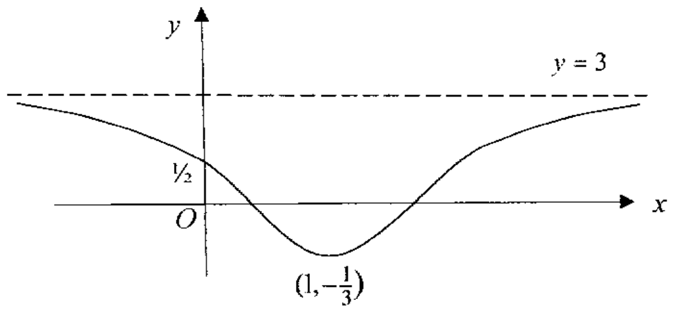
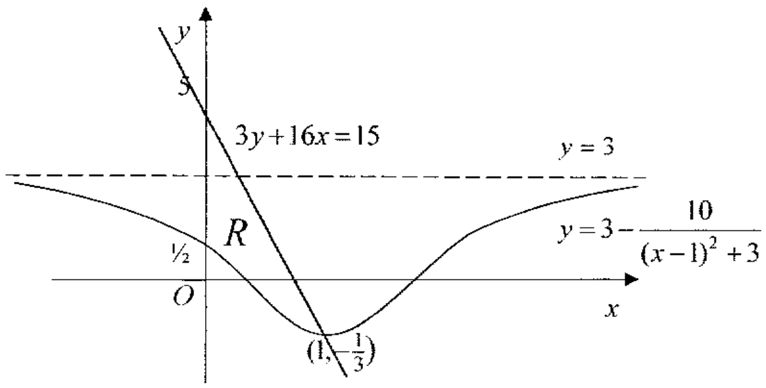
Let $y = 2x^2 + 3$. Then $x = -\sqrt{\frac{y-3}{2}}$ or $\sqrt{\frac{y-3}{2}}$ (reject $\because x \le 0$).

So $\mathrm{h}^{-1}(x) = -\sqrt{\frac{x-3}{2}}$;

Then $\mathrm{h}^{-1}(\mathrm{hg}(x)) = -\sqrt{\frac{\mathrm{hg}(x) - 3}{2}}$

$$\therefore \mathrm{g}(x) = -\sqrt{\frac{2x + 3 - 2a - 3}{2}} = -\sqrt{x-a}$$

$$D_{\mathrm{g}} = D_{\mathrm{hg}} = (a, \infty)$$

4	**(i)**	$y=3-\frac{10}{x^2-2x+4} \Rightarrow \frac{dy}{dx}=\frac{10(2x-2)}{(x^2-2x+4)^2}$ At the stationary point, $\frac{dy}{dx}=0 \Rightarrow x=1,\ y=3-\frac{10}{3}=-\frac{1}{3}$ $\therefore$ stationary point $\left(1,-\frac{1}{3}\right)$ Alternatively, from equation of C, the graph of $y=3-\frac{10}{(x-1)^2+3}$ is symmetrical about $x=1$. Therefore the minimum point is $\left(1,-\frac{1}{3}\right)$.
	(ii)	$y=3-\frac{10}{x^2-2x+4}=3-\frac{10}{(x-1)^2+3}$ 
	(iii)	

	$P(\text{batch passes inspection})$ $= P(X \le 1) + P(X = 2)P(X = 0)$ $= 0.806087 + (0.150598)(0.430142)$ $= 0.870865 = 0.871$ (to 3 sf)
(iv)	$P(\le 3 \text{ defective pens found} \mid \text{batch did not pass inspection})$ $= \dfrac{P(X=2)P(X=1) + P(X=3)}{P(\text{batch did not pass inspection})}$ $= \dfrac{(0.150598)(0.375945) + 0.036562}{1 - 0.870865}$ $= \dfrac{0.0931785}{0.129135} = 0.722$ (to 3 sf)
(v)	Let S be the number of special pencils in a box of 12. $S \sim B(12, 0.07)$ Since $n = 40$ is large, by Central Limit Theorem, $\overline{S} \sim N\left(0.84, \dfrac{0.7812}{40}\right)$ approximately. Let T be the number of special crayons in a box of 12. $T \sim B(12, 0.06)$ Since $n = 40$ is large, by Central Limit Theorem, $\overline{T} \sim N\left(0.72, \dfrac{0.6768}{40}\right)$ approximately. We want to find $P(\overline{S} > \overline{T})$, i.e. $P(\overline{S} - \overline{T} > 0)$. $\overline{S} - \overline{T} \sim N(0.12, 0.03645)$ $P(\overline{S} - \overline{T} > 0) = 0.73517 \approx 0.735$ (3 s.f.)