PRESBYTERIAN HIGH SCHOOL

ADDITIONAL MATHEMATICS
4049/01

Paper 1

17 August 2022 Wednesday 2 hours 15 min

2022 SECONDARY FOUR EXPRESS
PRELIMINARY EXAMINATIONS

MARKING SCHEME

Question 1

The area of a triangle is given as $1+2\sqrt{5}$ cm^2. The base of the triangle is given as $3-\sqrt{5}$ cm. Without using a calculator, express the height of the triangle, in cm, in the form $a+b\sqrt{5}$, where a and b are rational numbers.

[3]

	Area of triangle $=1+2\sqrt{5}$ $\frac{1}{2}(3-\sqrt{5})h=1+2\sqrt{5}$ $h=\frac{2+4\sqrt{5}}{3-\sqrt{5}}$ $h=\frac{2+4\sqrt{5}}{3-\sqrt{5}}\times\frac{3+\sqrt{5}}{3+\sqrt{5}}$ $=\frac{6+2\sqrt{5}+12\sqrt{5}+20}{9-5}$ $=\frac{26+14\sqrt{5}}{4}$ $=\frac{13}{2}+\frac{7}{2}\sqrt{5}\ cm$	M1 (multiply using $\frac{3+\sqrt{5}}{3+\sqrt{5}}$) M1 (expand numerator or denominator correctly) A1

Question 2

Find the y-coordinates of the points for which the line $x-2y=3$ meets the curve $xy+6=2x$.

[3]

	$x-2y=3$ $x=3+2y$ .....(1) $xy+6=2x$ ....(2) Substitute (1) into (2): $(3+2y)y+6=2(3+2y)$	M1 (substitution method)
	$3y+2y^2+6=6+4y$ $2y^2-y=0$ $y(2y-1)=0$	M1 (factorise)
	$y=0 \ or \ y=\frac{1}{2}$	A1

Question 3

Express $-x^2+8x+5$ in the form $a(x+b)^2+c$ and hence state the coordinates of the turning point of the curve $y=-x^2+8x+5$. [4]

	$-x^2+8x+5$ $=-\left[x^2-8x\right]+5$ $=-\left[x^2-8x+\left(\frac{8}{2}\right)^2-\left(\frac{8}{2}\right)^2\right]+5$ $=-(x-4)^2+16+5$ $=-(x-4)^2+21$ Coordinates of turning point $=(4,\ 21)$	 M1 M1 A1, A1

Question 4

Integrate $\frac{4}{2-5x}+\frac{2}{x^3}+e^{4x}$ with respect to x. [4]

	$\int\left(\frac{4}{2-5x}+\frac{2}{x^3}+e^{4x}\right)dx$	
	$=-\frac{4}{5}\ln(2-5x)-\frac{1}{x^2}+\frac{1}{4}e^{4x}+c$	B2 [B1 for showing $\ln(2-5x)$] B1, B1

Question 5

Express $\frac{9x^2-34x+27}{(x-1)(x-2)^2}$ as the sum of three partial fractions. [6]

	$\frac{9x^2-34x+27}{(x-1)(x-2)^2}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$	M1
	$9x^2-34x+27=A(x-2)^2+B(x-1)(x-2)+C(x-1)$	M1
	Substitute $x=2$, $9(2)^2-34(2)+27=C(2-1)$ $C=-5$	M1 (using substitution method correctly)
	Substitute $x=1$, $9(1)^2-34(1)+27=A(1-2)^2$ $A=2$	
	Substitute $x=0$, $27=2(4)+2B+5$ $2B=14$ $B=7$	A2 (at least 2 out of 3 correct values for A, B & C)
	$\frac{9x^2-34x+27}{(x-1)(x-2)^2}=\frac{2}{x-1}+\frac{7}{x-2}-\frac{5}{(x-2)^2}$	A1

	Alternative method	
	$\dfrac{9x^2-34x+27}{(x-1)(x-2)^2}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$	M1
	$9x^2-34x+27=A(x-2)^2+B(x-1)(x-2)+C(x-1)$	M1
	$9x^2-34x+27=A(x^2-4x+4)+B(x^2-3x+2)+C(x-1)$ $=Ax^2-4Ax+4A+Bx^2-3Bx+2B+Cx-C$ $=(A+B)x^2+(c-4A-3B)x+(4A+2B-C)$	
	Comparing the coefficients on both sides, $A+B=9$ ......(1) $C-4A-3B=-34$ ......(2) $4A+2B-C=27$ ......(3)	M1 (using the comparing of coefficient method correctly)
	$(2)+(3): B=7$ $(1): A+7=9$ $A=2$ $(2): C-8-21=-34$ $C=-5$	A2 (at least 2 out of 3 correct values for A, B & C)
	$\dfrac{9x^2-34x+27}{(x-1)(x-2)^2}=\dfrac{2}{x-1}+\dfrac{7}{x-2}-\dfrac{5}{(x-2)^2}$	A1

Question 6

(a) The expression $ax^3 + 13x^2 + bx - 5$ is exactly divisible by $x-1$ but gives a reminder of 49 when divided by $x-2$. Find the value of a and of b. [4]

(b) The cubic polynomial f(x) is such that the coefficient of x^3 is -2 and the roots of the equation f(x) = 0 are -1, 2 and k. Given that f(x) has a reminder of 80 when divided by $x+3$, find the value of k, given that k is a positive number. [3]

(a)	Let f(x) $= ax^3 + 13x^2 + bx - 5$ $f(1) = a(1)^3 + 13(1)^2 + b(1) - 5$ $0 = a + b + 8$ $a + b = -8$ ......(1) $f(2) = a(2)^3 + 13(2)^2 + b(2) - 5$ $49 = 8a + 52 + 2b - 5$ $8a + 2b = 2$ $4a + b = 1$ ......(2) Solving (1) and (2), $a = 3, b = -11$	 M1 (equate to 0) M1 (equate to 49) A1, A1
(b)	$f(x) = -2(x+1)(x-2)(x-k)$ $80 = -2(-3+1)(-3-2)(-3-k)$ $80 = -20(-3-k)$ $80 = 60 + 20k$ $k = 1$	M1 M1 A1

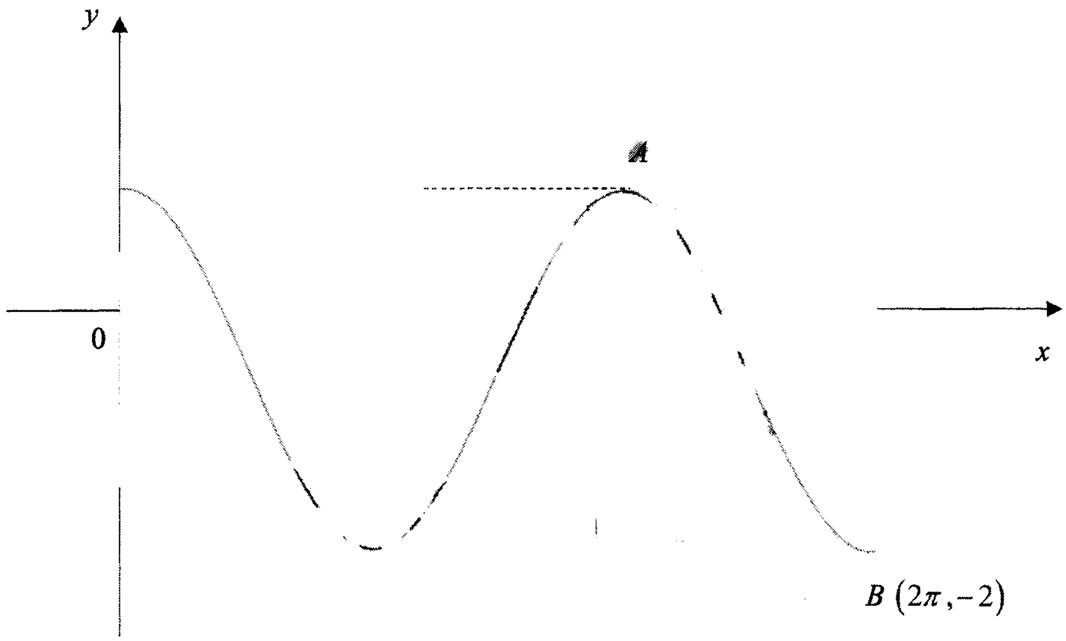
Question 7

The diagram shows the graph of the curve $y = a\cos bx + c$ for $0 \le x \le 2\pi$. The curve has a maximum point at A and a minimum point at B. The coordinates of $A = \left(\frac{4}{3}\pi, 1\right)$ and $B = (2\pi, -2)$.

(a) State the period of the curve. [1]

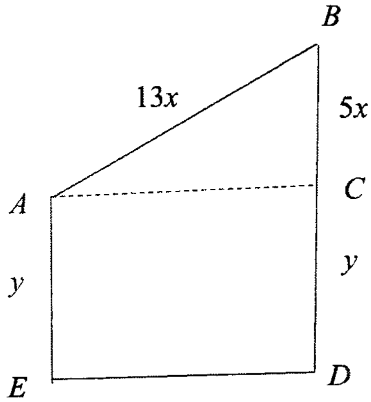
(b) Find the value of a, b and c. [3]

(c) Find the range of values of k for which $a\cos bx + c = k$ has three solutions. [2]



(a)	Period $= \frac{4}{3}\pi$	B1
(b)	$a = \frac{3}{2}, b = \frac{3}{2}, c = -\frac{1}{2}$	B1, B1, B1
(c)	$-2 < k < 1$	B2 B1 (either state $-2 < k$ or $k < 1$)

Question 8



A piece of wire, l cm long, is bent to form the shape as shown in the diagram. $ACDE$ is a rectangle with $AE = y$ cm and ΔABC is a right-angled triangle with $AB = 13x$ cm and $BC = 5x$ cm.

(a) Express l in terms of x and y. [1]

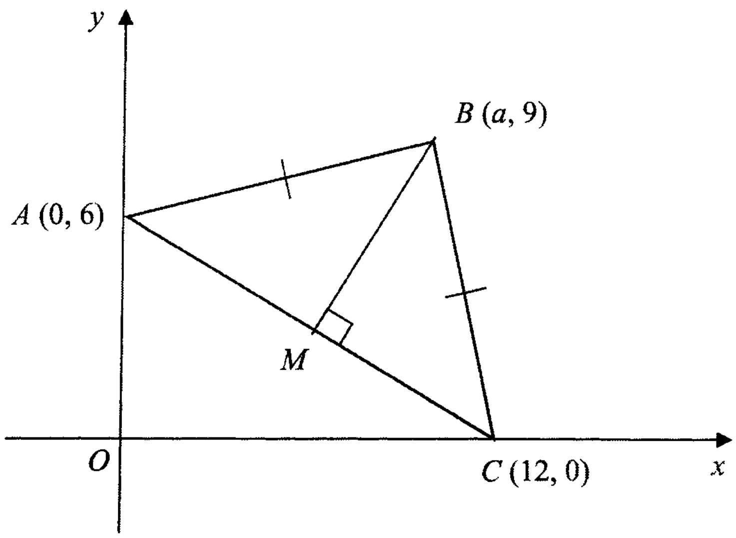
(b) Given that the area enclosed is 96 cm^2, show that $l = 25x + \frac{16}{x}$. [3]

(c) Find the value of x for which l has a stationary value and determine the nature of this stationary value. [3]

(a)	$l = 12x + 13x + 5x + y + y$ $l = 30x + 2y$	B1
(b)	$12xy + \frac{1}{2}(12x)(5x) = 96$ $12xy + 30x^2 = 96$ $12xy = 96 - 30x^2$ $y = \frac{8}{x} - \frac{5x}{2}$ $l = 30x + 2\left(\frac{8}{x} - \frac{5x}{2}\right)$ $l = 30x + \frac{16}{x} - 5x$ $l = 25x + \frac{16}{x}$ *(shown)*	M1 M1 A1

(c)	$\frac{dl}{dx} = 25 - \frac{16}{x^2}$ $when\ \frac{dl}{dx} = 0,$ $25 - \frac{16}{x^2} = 0$ $x^2 = \frac{16}{25}$ $x = \frac{4}{5}$ $\frac{d^2l}{dx^2} = \frac{32}{x^3}$ $when\ x = \frac{4}{5},$ $\frac{d^2l}{dx^2} = 62.5 > 0$ l is a minimum value.	 M1 A1 A1

Question 9



The diagram shows a triangle ABC, where A is $(0,6)$, B is $(a,9)$ and C is $(12,0)$. AB is equal to BC and M is the midpoint of AC.

(a) Find the coordinates of M. [1]

(b) Find the equation of the perpendicular bisector of AC. [4]

(c) Find the value of a. [2]

(d) Calculate the area of the triangle ABC. [2]

	Coordinates of $M = (6,3)$	B1
	Gradient of $AC = -\frac{1}{2}$ Gradient of $BM = 2$ Equation of perpendicular bisector of AC $y-3=2(x-6)$ $y=2x-9$	M1 M1 M1 A1
	$9=2a-9$ $2a=18$ $a=9$	M1 A1

	Area of triangle ABC $=\frac{1}{2}\begin{vmatrix} 0 & 12 & 9 & 0 \\ 6 & 0 & 9 & 6 \end{vmatrix}$ $=\frac{1}{2}\left[(0+108+54)-(72+0+0)\right]$ $=\frac{1}{2}(90)$ $=45\,units^2$	 M1 A1

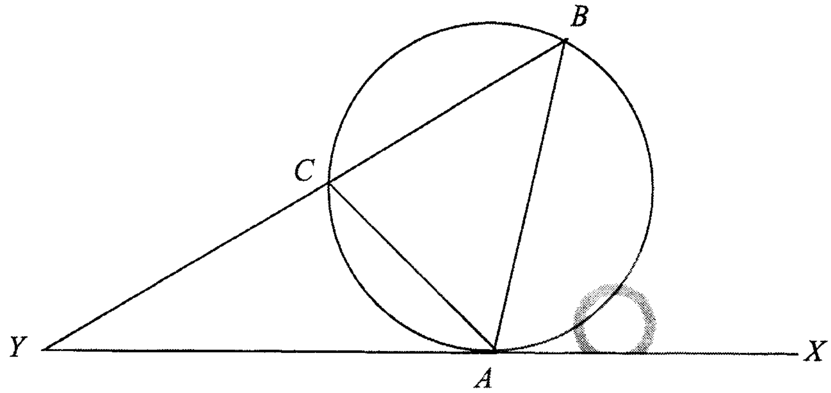
Question 10

(a) Prove the identity $\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$. [3]

(b) Hence solve the equation $\frac{1-\cos 2A}{1+\cos 2A} = 2\tan A$ for $0° < A < 360°$. [4]

(a)	$LHS = \frac{1-\cos 2A}{1+\cos 2A}$ $= \frac{1-(1-2\sin^2 A)}{1+(2\cos^2 A-1)}$ $= \frac{2\sin^2 A}{2\cos^2 A}$ $= \tan^2 A = RHS$	 M1 (either correct numerator or denominator) M1 A1
(b)	$\frac{1-\cos 2A}{1+\cos 2A} = 2\tan A$ $\tan^2 A = 2\tan A$ $\tan^2 A - 2\tan A = 0$ $\tan A(\tan A - 2) = 0$ $\tan A = 0$ *or* $\tan A - 2 = 0$ $A = 180°$ $\tan A = 2$ A lies in the first & third quadrants $A = 63.43°, 180° + 63.43°$ $= 63.4°, 243.4°$	 M1 (factorise terms) A1 A1, A1

Question 11



The diagram shows a triangle ABC inscribed in a circle. XY is a tangent to the circle at point A and AC bisects angle BAY.

(a) Prove that triangle ABC is isosceles. [2]

(b) Prove that triangle AYC is similar to triangle BYA. [3]

(c) Hence, show that $AY^2 = CY \times BY$ [2]

(a)	Let $\angle CAY = \angle CAB = \theta$ $\angle CBA = \angle CAY = \theta$ (Alternate Segment Theorem) Since $\angle CAB = \angle CBA = \theta$, ΔABC is isosceles.	 B1 AG1
(b)	In ΔAYC and ΔBYA, $\angle BYA = \angle AYC$ (common angle) $\angle ABY = \angle CAY = \theta$ (Alternate Segment Theorem) So ΔAYC is similar to ΔBYA (AA Similarity)	 B1 B1 AG1
(c)	Since ΔAYC and ΔBYA are similar, $\frac{AY}{CY} = \frac{BY}{AY}$ $AY^2 = CY \times BY$	 B1 AG1

Question 12

(a) Given that $3^{x+1} \times 2^{2x+1} = 2^{x+2}$, evaluate 6^x . [4]

(b) Express y in terms of x if $\log_2 y = \log_8 x - \log_2 4$. [4]

	$3^{x+1} \times 2^{2x+1} = 2^{x+2}$ $3^{x+1} \times \frac{2^{2x+1}}{2^{x+2}} = 1$ $3^{x+1} \times 2^{2x+1-(x+2)} = 1$ $3^{x+1} \times 2^{x-1} = 1$ $(3^x)3 \times \frac{2^x}{2} = 1$ $3^x \times 2^x = \frac{2}{3}$ $6^x = \frac{2}{3}$	M1 (apply quotient rule) M1 (correct expansion) M1 ($3^x \times 2^x = 6^x$) A1
	$\log_2 y = \log_8 x - \log_2 4$ $= \frac{\log_2 x}{\log_2 8} - \log_2 4$ $= \frac{\log_2 x}{\log_2 2^3} - \log_2 4$ $= \frac{\log_2 x}{3} - \log_2 4$ $3\log_2 y = \log_2 x - 3\log_2 4$ $\log_2 y^3 = \log_2 x - \log_2 4^3$ $\log_2 y^3 = \log_2 \frac{x}{64}$ $y^3 = \frac{x}{64}$ $y = \frac{1}{4} x^{\frac{1}{3}}$	M1 (change of base law) M1 (apply power law) M1 (apply quotient law) A1

Question 13

A curve has the equation $y=(x-3)\sqrt{2x+3}$, where $x>-\frac{3}{2}$.

(a) Show that $\frac{dy}{dx}$ can be expressed in the form $\frac{kx}{\sqrt{2x+3}}$ and state the value of k. [4]

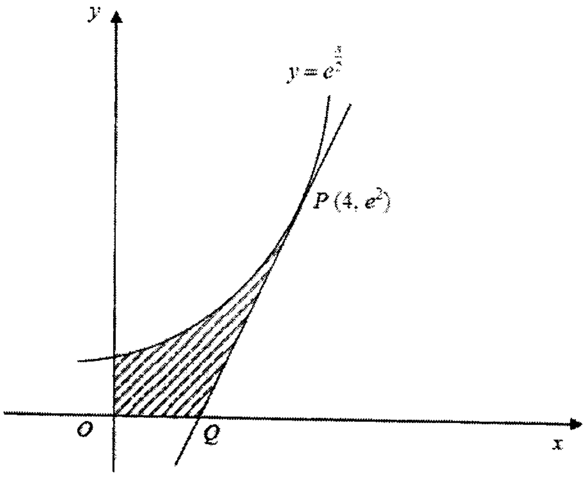
(b) Find the equation of the tangent when $x=11$. [3]

(c) Find the rate of change of x at the instant when $x=11$, given that y is increasing at a rate of 5 units per second at this instant. [2]

(a)	$y=(x-3)\sqrt{2x+3}$	
	$\frac{dy}{dx}=(2x+3)^{\frac{1}{2}}+\frac{1}{2}(x-3)(2)(2x+3)^{-\frac{1}{2}}$	M1, M1
	$=\sqrt{2x+3}+\frac{x-3}{\sqrt{2x+3}}$	
	$=\frac{2x+3+x-3}{\sqrt{2x+3}}$	M1
	$=\frac{3x}{\sqrt{2x+3}}$	
	$k=3$	A1
(b)	When $x=11$, $y=(11-3)\sqrt{2(11)+3}=40$	M1
	$\frac{dy}{dx}=\frac{3(11)}{\sqrt{2(11)+3}}=\frac{33}{5}$	M1
	$y=\frac{33}{5}x+c$	
	$40=\frac{33}{5}(11)+c$	
	$c=-32.6$	
	Equation of tangent is $y=6.6x-32.6$	A1

(c)	$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$	
	$5 = \frac{33}{5} \times \frac{dx}{dt}$	M1
	$\frac{dx}{dt} = \frac{25}{33}\ units/s$	A1

Question 14



The diagram shows part of the curve $y = e^{\frac{x}{2}}$. The tangent to the curve at $P(4, e^2)$ meets the x-axis at Q.

(a) Find the coordinates of Q. [5]

(b) Find the area of the shaded region bounded by the curve, the coordinate axes and the tangent to the curve at P, leaving your answer in terms of e. [5]

(a)	$y = e^{\frac{x}{2}}$ $\frac{dy}{dx} = \frac{1}{2}e^{\frac{x}{2}}$	M1
	At $P(4, e^2)$, $\frac{dy}{dx} = \frac{1}{2}e^{\frac{4}{2}} = \frac{1}{2}e^2$	M1
	Let $Q = (x, 0)$ Gradient of $PQ = \frac{e^2 - 0}{4 - x}$	M1
	$\frac{1}{2}e^2 = \frac{e^2}{4-x}$ $4 - x = 2$ $x = 2$	M1
	Coordinates of $Q = (2, 0)$	A1

(b)	Area of shaded region	
	$= \int_0^4 e^{\frac{x}{2}} dx - \frac{1}{2}(4-2)e^2$	M1, M1
	$= \left[2e^{\frac{x}{2}} \right]_0^4 - e^2$	M1 (integrate correctly)
	$= \left[2e^2 - 2e^0 \right] - e^2$	M1 (correct substitution)
	$= e^2 - 2$	A1

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PRESBYTERIAN HIGH SCHOOL

ADDITIONAL MATHEMATICS **4049/02**
Paper 2

18 August 2022 Thursday 2 hours 15 minutes

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2022 SECONDARY FOUR EXPRESS / FIVE NORMAL
PRELIMINARY EXAMINATIONS

MARK SCHEME

1 Show that the equation $2e^x + 9 = 18e^{-x}$ has only one solution and find its value correct to 2 significant figures. [5]

$2e^x + 9 = 18e^{-x}$

Let $u = e^x$

$2u + 9 = \frac{18}{u}$ M1 (attempt to form quadratic)

$2u^2 + 9u - 18 = 0$

$(2u-3)(u+6) = 0$ M1 (factorisation, o.e.)

$e^x = \frac{3}{2}$ *or* $e^x = -6$ (rejected) M1 (seen rejected)

$x = \ln\frac{3}{2}$ M1 (ln both sides)

$x = 0.4054 \approx 0.41$ (2sf)

The equation has only one solution $x = 0.41$. (shown) A1

2 A polynomial $f(x)$ is defined as $x^3-13x^2+49x-57$.

(a) Show that $x=3$ is a root of the equation $f(x)=0$. [1]

(b) It is given that the two other roots of $f(x)=0$ are $x_1=a+b\sqrt{c}$ and $x_2=a-b\sqrt{c}$, where a, b and c are positive integers. Find the exact values of x_1 and x_2. [4]

(c) Express $x_1^3-x_2^3$ in the form $d\sqrt{c}$, where d is a positive integer. [3]

(a) When $x=3$,
$(3)^3-13(3)^2+49(3)-57=0$
Hence $x=3$ is a solution of the equation. (shown) — AG1

(b) From (a), $x-3$ is a factor of $f(x)$.

$(x-3)(x^2-10x+19)=0$ M1 (long division or comparing coefficients)
M1 (seen $x^2-10x+19$)

$$x=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(19)}}{2(1)}$$ M1 (quadratic formula)

$$x=\frac{10\pm\sqrt{24}}{2}$$

$$x=\frac{10\pm2\sqrt{6}}{2}$$

$x_1=5+\sqrt{6}$ *or* $x_2=5-\sqrt{6}$ A1

(c) $x_1^3-x_2^3$

$=(x_1-x_2)(x_1^2+x_1x_2+x_2^2)$

$=\left[5+\sqrt{6}-(5-\sqrt{6})\right]\left[(5+\sqrt{6})^2+(5+\sqrt{6})(5-\sqrt{6})+(5-\sqrt{6})^2\right]$ M1

$=2\sqrt{6}\left[(25+10\sqrt{6}+6)+(25-6)+(25-10\sqrt{6}+6)\right]$ M1 (attempt to simplify)

$=2\sqrt{6}[81]$

$=162\sqrt{6}$ A1

3 The equation of a curve is $y^2 + mx^2 = m$, where m is a positive constant.

(a) Find the largest integer value of m for which the line $x - y = 3$ does not meet the curve. [5]

(b) If the line $x - y = 3$ is a tangent to the curve at point P, deduce the value of the constant m. Hence find the coordinates of P. [3]

(a) $x - y = 3 \Rightarrow y = x - 3$

Sub. into the curve,

$(x-3)^2 + mx^2 = m$ M1 (equate line to curve)

$x^2 - 6x + 9 + mx^2 = m$

$(m+1)x^2 - 6x + 9 - m = 0$ M1 (reduce to quadratic)

Since the line does not meet the curve,

$(-6)^2 - 4(m+1)(9-m) < 0$ M1 (apply D < 0)

$36 - 4(-m^2 + 8m + 9) < 0$

$4m^2 - 32m < 0$

$4m(m-8) < 0$

$0 < m < 8$ M1 (solving quadratic inequality)

Largest integer $m = 7$ A1

(b) Since the line is tangent to the curve,

$4m(m-8) = 0$

$m = 0$ (rejected) *or* $m = 8$ B1 (seen $m = 8$)

When $m = 8$, $(8+1)x^2 - 6x + 9 - 8 = 0$

$9x^2 - 6x + 1 = 0$

$(3x-1)^2 = 0$

$x = \frac{1}{3}$ M1 (attempt to find x)

$y = \frac{1}{3} - 3 = -\frac{8}{3}$

$\therefore P = \left(\frac{1}{3}, -\frac{8}{3}\right)$ A1

4 It is given that $\left(x+\frac{k}{x^2}\right)^n$ is a binomial expansion, where k and n are positive constants.

(a) Write down the first 4 terms in the expansion of $\left(x+\frac{k}{x^2}\right)^5$, in terms of k, in descending powers of x. [2]

(b) Hence or otherwise, find the value(s) of k if the coefficient of x^2 in the expansion of $\left(5x^3+3\right)\left(x+\frac{k}{x^2}\right)^5$ is 5. [3]

(c) By considering the general term in the binomial expansion of $\left(x+\frac{k}{x^2}\right)^n$, show that for the term independent of x, the value of the constant n is a multiple of 3. [3]

(a) $\left(x+\frac{k}{x^2}\right)^5 = x^5 + \binom{5}{1}x^4\left(\frac{k}{x^2}\right) + \binom{5}{2}x^3\left(\frac{k}{x^2}\right)^2 + \binom{5}{3}x^2\left(\frac{k}{x^2}\right)^3 + \ldots$ M1

$\left(x+\frac{k}{x^2}\right)^5 = x^5 + 5kx^2 + \frac{10k^2}{x} + \frac{10k^3}{x^4} + \ldots$ A1

(b) $\left(5x^3+3\right)\left(x+\frac{k}{x^2}\right)^5 = \left(5x^3+3\right)\left[x^5 + 5kx^2 + \frac{10k^2}{x} + \frac{10k^3}{x^4} + \ldots\right]$

$\left(5x^3\right)\left(\frac{10k^2}{x}\right) + 3\left(5kx^2\right) = 5x^2$ M1 (equating coefficients of x^2)

$10k^2 + 3k - 1 = 0$

$(2k+1)(5k-1) = 0$ M1 (solving for k)

$k = -\frac{1}{2}$ (rejected) *or* $k = \frac{1}{5}$ A1

(c) General term $= \binom{n}{r}x^{n-r}\left(\frac{k}{x^2}\right)^r$ M1 (substitution into general term)

$= \binom{n}{r}k^r x^{n-3r}$

For the term independent of x,

let $n - 3r = 0$ M1 (equate power to zero)

$\therefore n = 3r$

Since $n = 3r$, where r is an integer, hence the value of n is a multiple of 3. AG1

5 **(a)** Given that $y = \frac{\ln 2x}{5x}$, show that $\frac{dy}{dx} = \frac{1 - \ln 2x}{5x^2}$. [4]

(b) Hence find the value of $\int_1^2 \frac{\ln\ 2x}{x^2}\,dx$. [4]

(a) $y = \frac{\ln 2x}{5x}$

$$\frac{dy}{dx} = \frac{5x\left(\frac{1}{2x}\right)(2) - \ln 2x\,(5)}{(5x)^2}$$ M2 (quotient rule & chain rule)

$$\frac{dy}{dx} = \frac{5 - 5\ln 2x}{25x^2}$$ M1 (simplifying)

$$\frac{dy}{dx} = \frac{1 - \ln 2x}{5x^2} \text{ (shown)}$$ AG1

(b) $$\int_1^2 \frac{1 - \ln 2x}{5x^2}\,dx = \left[\frac{\ln 2x}{5x}\right]_1^2$$ M1 (reverse part (a))

$$\int_1^2 \frac{1 - \ln 2x}{x^2}\,dx = \left[\frac{\ln 2x}{x}\right]_1^2$$

$$\int_1^2 \frac{1}{x^2}\,dx - \int_1^2 \frac{\ln 2x}{x^2}\,dx = \left[\frac{\ln 2x}{x}\right]_1^2$$ M1 (separate into two integrals)

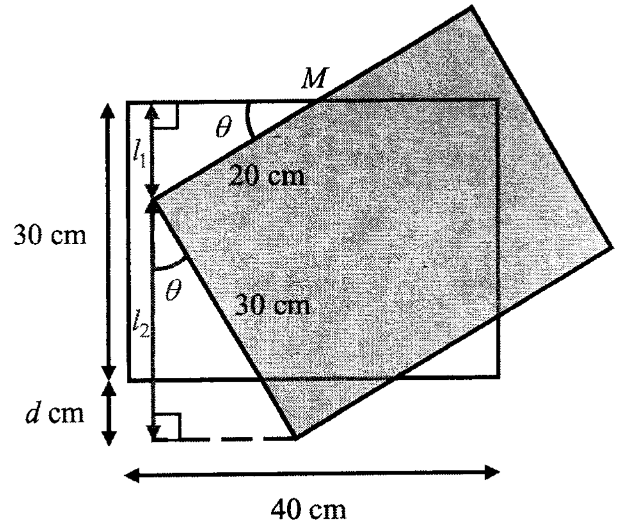
$$\int_1^2 \frac{\ln 2x}{x^2}\,dx = \int_1^2 \frac{1}{x^2}\,dx - \left[\frac{\ln 2x}{x}\right]_1^2$$

$$\int_1^2 \frac{\ln 2x}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 - \left[\frac{\ln 2x}{x}\right]_1^2$$ M1 (integration of $1/x^2$)

$$\int_1^2 \frac{\ln 2x}{x^2}\,dx = \left[-\frac{1}{2} - (-1)\right] - \left[\frac{\ln 4}{2} - \ln 2\right]$$

$$\int_1^2 \frac{\ln 2x}{x^2}\,dx = \frac{1}{2}$$ A1

6



The diagram shows a rectangular picture frame 40 cm by 30 cm hung on the wall.
The picture frame is rotated through an angle θ about the midpoint, M of the top edge.

(a) Show that the vertical displacement, d cm, of the picture frame below its original bottom edge is given by

$$d = 20\sin\theta + 30\cos\theta - 30.$$ [2]

(b) Express d in the form $R\sin(\theta+\alpha)-30$, where $R > 0$ and $0° \le \alpha \le 90°$. [4]

(c) Find the value of d and the corresponding value of θ that will give the greatest vertical displacement of the picture frame below its original bottom edge. [3]

(a) $l_1 = \frac{1}{2}(40)\sin\theta = 20\sin\theta$ and $l_2 = 30\cos\theta$ M1 (identify and find the lengths)

$d + 30 = 20\sin\theta + 30\cos\theta$

$d = 20\sin\theta + 30\cos\theta - 30$ (shown) AG1

(b) Let $20\sin\theta + 30\cos\theta - 30 = R\sin(\theta+\alpha) - 30$

$R = \sqrt{20^2 + 30^2} = \sqrt{1300} = 10\sqrt{13}$ M1 (finding R)

$\alpha = \tan^{-1}\left(\frac{30}{20}\right) = 56.309° \approx 56.3°$ M1 (finding α)

$\therefore d = 10\sqrt{13}\sin(\theta + 56.3°) - 30$ A2 (deduct 1 mark for each incorrect value)

(c) The greatest vertical displacement occurs when $\sin(\theta+56.3°)=1$.

$\therefore d=10\sqrt{13}-30=6.0555\approx 6.06$ B1

$\theta+56.3°=90°$ M1

$\therefore \theta=33.7°$ A1

7 A particle moves in a straight line such that its velocity, v m/s, is given by $v=\frac{1}{2}-2e^{-\frac{t}{2}}$, where t is the time in seconds after leaving a fixed point O.

(a) State the value that v approaches as t becomes very large. Justify your answer. [2]

(b) Find the initial acceleration of the particle. [2]

(c) Find the value of t when the particle is instantaneously at rest. [2]

(d) Find the total distance travelled by the particle in the interval $0 \le t \le 10$. [4]

(a) The value of v approaches $\frac{1}{2}$. B1

As t becomes very large, $2e^{-\frac{t}{2}}$ approaches zero, so $v \approx \frac{1}{2}$. B1

(b) $a=-2e^{-\frac{t}{2}}\left(-\frac{1}{2}\right)=e^{-\frac{t}{2}}$ M1 (find dv/dt)

When $t=0$, initial acceleration $=e^{-\frac{0}{2}}=1\text{ m/s}^2$ A1

(c) At instantaneous rest,

$v=\frac{1}{2}-2e^{-\frac{t}{2}}=0$ M1 (equate v to zero)

$e^{-\frac{t}{2}}=\frac{1}{4}$

$-\frac{t}{2}=\ln\frac{1}{4}$

$t=-2\ln\frac{1}{4}=\ln 16$

$\therefore t=2.7725 \approx 2.77$ s (3sf) A1 (Accept 4ln2)

(d) $s=\frac{1}{2}t-\frac{2e^{-\frac{t}{2}}}{-\frac{1}{2}}=\frac{1}{2}t+4e^{-\frac{t}{2}}+c$ M1 (correct antiderivative)

$0=\frac{1}{2}(0)+4e^{-\frac{0}{2}}+c \Rightarrow c=-4$ M1 (attempt to find arbitrary constant)

$\Rightarrow s=\frac{1}{2}t+4e^{-\frac{t}{2}}-4$

When $t=\ln 16$, $s=\frac{1}{2}\ln 16+4e^{-\frac{\ln 16}{2}}-4=-1.6137\text{ m}$

When $t=10$, $s=\frac{1}{2}(10)+4e^{-\frac{10}{2}}-4=1.0269\text{ m}$

M1 (attempt to find either one)

Total distance travelled

$=(1.6137)+(1.6137+1.0269)=4.2543\approx 4.25\text{ m}$ (3sf) A1

8 The value of a car, $\$V$, after t years following 2014 can be modelled by the formula $V = ab^t$, where a and b are constants. The table shows the value of the car in the years following 2014.

Year	2016	2018	2020	2022
t (years)	2	4	6	8
V (\$)	52100	43800	37100	31600

(a) Given that $\lg V$ is the variable for the vertical axis, express the formula in a form suitable for drawing a straight line graph. [2]

(b) Draw a straight line graph to show that the model is reasonable. [4]

Use the graph in part **(b)** to estimate, correct to 3 significant figures,

(c) the value of the constants a and b, [3]

(d) the value of the car in the year 2024. [2]

(a) $V = ab^t$

$\lg V = \lg\left(ab^t\right)$

$\lg V = \lg a + \lg b^t$ M1 (seen product law)

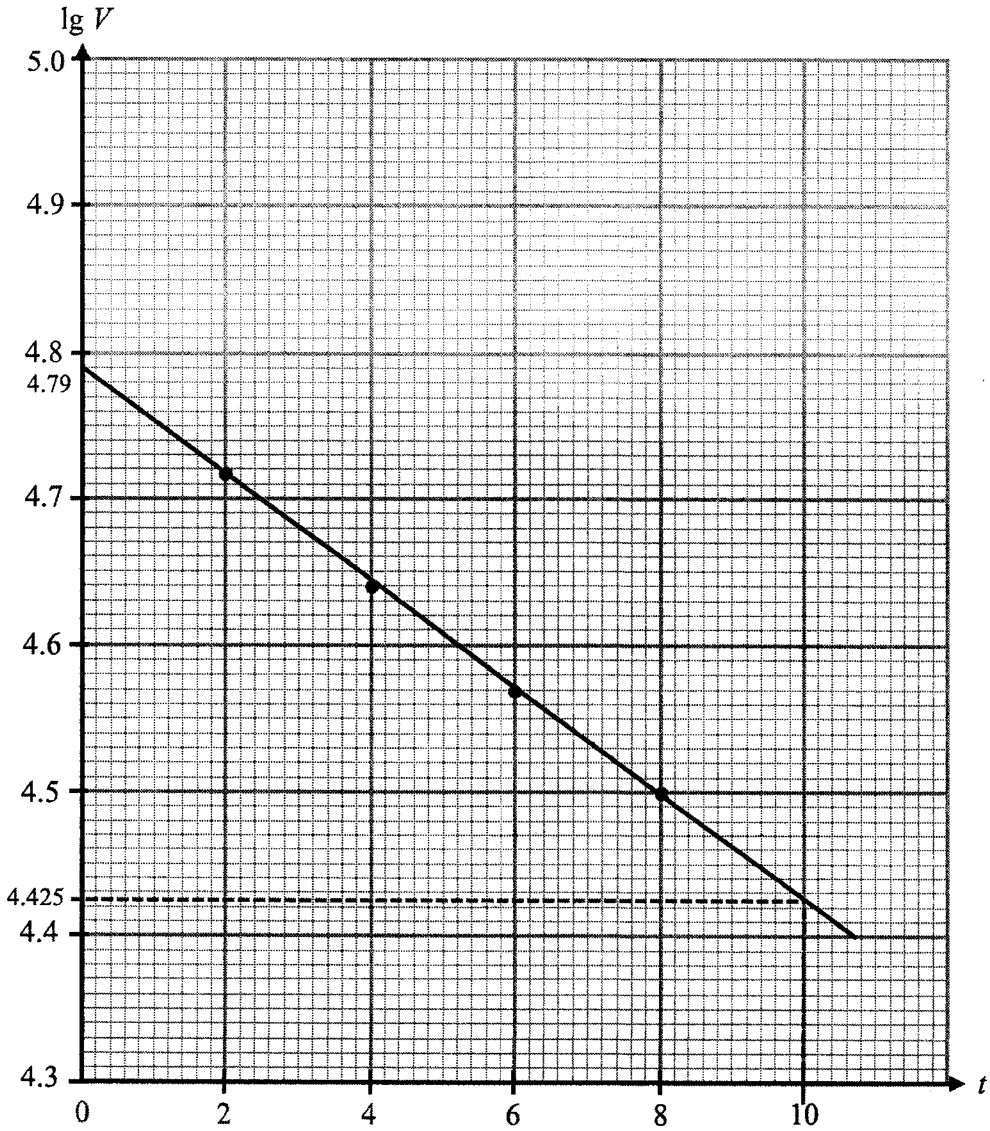
$\lg V = (\lg b)t + \lg a$ A1

(b) Label axes B1 (correct axes with at least 1 point)

All correct points P2 (deduct 1 mark if any point is wrong)

Best fit line C1

(b)



(c) From the $\lg V$ versus t graph,

$\lg a = 4.79$

$\therefore a = 10^{4.79} = 61659 \approx 61700$ (3sf) B1 (Accept $4.78 \le \lg a \le 4.8$)

$\lg b = \dfrac{4.5 - 4.72}{8 - 2} = -\dfrac{11}{300} = -0.036666$ M1 (Accept $-0.03 \le \lg b \le -0.04$)

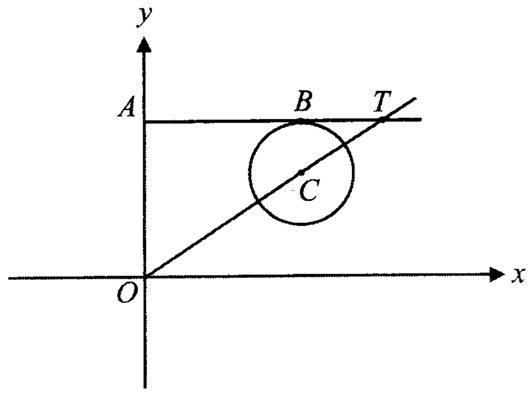
$\therefore b = 10^{-\frac{11}{300}} = 0.91903 \approx 0.919$ (3sf) A1

(d) From the $\lg V$ versus t graph, when $t = 10$,

$\lg V = 4.425$ M1 (Accept $4.415 \le \lg V \le 4.435$)

$\therefore V = 10^{4.425} = 26607 \approx 26600$ (3sf) A1

9



(a) The equation of circle with centre C is given by $x^2 + y^2 - 6x - 4y + 12 = 0$. Find the radius of the circle and the coordinates of its centre C. [3]

(b) AB is a horizontal tangent to the circle at point B. Given that the line OC produced meets the line AB produced at point T, find the coordinates of T. [3]

(c) Show that triangle AOT and triangle BCT are similar. [3]

(d) Find the ratio $OC : CT$. [1]

(e) Find the angle ATO in degrees. [1]

(a) <u>**Method 1**</u>

$x^2 + y^2 - 6x - 4y + 12 = 0$

$(x-3)^2 - 9 + (y-2)^2 - 4 + 12 = 0$ M1 (complete the square, o.e.)

$(x-3)^2 + (y-2)^2 = 1$

Centre, $P = (3, 2)$ A1

Radius $= 1$ unit A1

<u>**Method 2**</u>

$x^2 + y^2 - 6x - 4y + 12 = 0$

$g = -3, f = -2, c = 12$

Centre, $P = (3, 2)$ B1, B1

Radius $= 1$ unit A1

(b) *Equation of line AT*: $y = 3$

Equation of line OT: $y = \frac{2}{3}x$ M1

Solving simultaneously: $\frac{2}{3}x = 3$ M1

$\Rightarrow x = 4.5$

$\therefore T = (4.5, 3)$ A1

(c) $\angle BTC = \angle ATO$ (common angle) M1

$\angle TAO = 90°$ (given)

$\angle TBC = 90°$ (tangent $\perp$ radius) M1

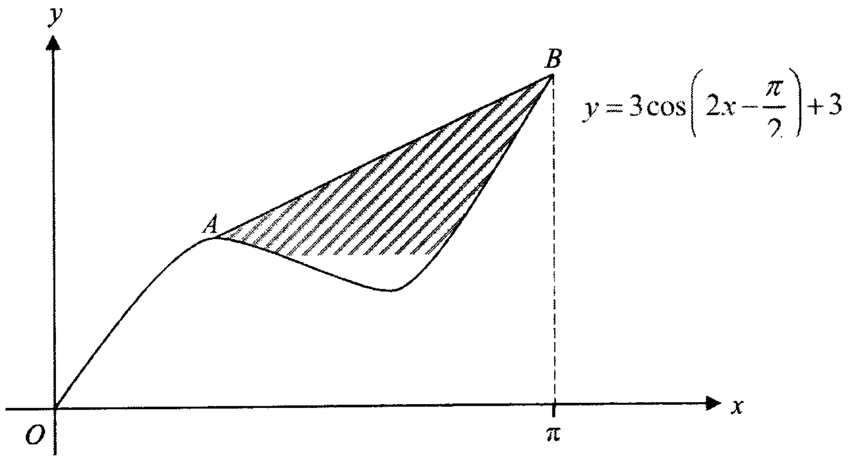
Triangle *AOT* and triangle *BCT* are similar. (AA similarity) A1

(d) $OC : CT = 2:1$ B1

(e) $\tan \angle ATO = \frac{3}{4.5}$

$\angle ATO = \tan^{-1}\left(\frac{3}{4.5}\right) = 33.69 \approx 33.7°$ (1dp) B1

10



The diagram shows the curve $y = 3\cos\left(2x - \frac{\pi}{2}\right) + 3x$ for $0 \le x \le \pi$ radians.

The point A is the maximum point of the curve and AB is a straight line.

(a) Show that the coordinates of A are $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{2} + \pi\right)$ and coordinates of B are $(\pi, 3\pi)$. [5]

(b) Hence find the area of the shaded region, leaving your answers in terms of π. [7]

(a) $y = 3\cos\left(2x - \frac{\pi}{2}\right) + 3x$

$\frac{dy}{dx} = -6\sin\left(2x - \frac{\pi}{2}\right) + 3$ M1 (correct dy/dx)

For turning point, $-6\sin\left(2x - \frac{\pi}{2}\right) + 3 = 0$ M1 (equate dy/dx to zero)

$\sin\left(2x - \frac{\pi}{2}\right) = \frac{1}{2}$

$\alpha = \frac{\pi}{6}$ M1 (attempt to find reference angle)

$2x - \frac{\pi}{2} = \frac{\pi}{6}$

$x = \frac{\pi}{3}$

When $x = \frac{\pi}{3}$, $y = \frac{3\sqrt{3}}{2} + \pi$

$\Rightarrow A = \left(\frac{\pi}{3}, \frac{3\sqrt{3}}{2} + \pi\right)$ AG1

When $x = \pi$, $y = 3\pi$
$\Rightarrow B = (\pi, 3\pi)$ AG1

(b)

Area under the curve $= \int_{\frac{\pi}{3}}^{\pi} \left[3\cos\left(2x - \frac{\pi}{2}\right) + 3x \right] dx$

$= \left[\frac{3}{2}\sin\left(2x - \frac{\pi}{2}\right) + \frac{3}{2}x^2 \right]_{\frac{\pi}{3}}^{\pi}$ M1 (find antiderivative)

$= \left[\frac{3}{2}\sin\left(2\pi - \frac{\pi}{2}\right) + \frac{3}{2}\pi^2 \right] - \left[\frac{3}{2}\sin\left(\frac{2\pi}{3} - \frac{\pi}{2}\right) + \frac{3}{2}\left(\frac{\pi}{3}\right)^2 \right]$ M1 (substitution of limits)

$= \left(\frac{4}{3}\pi^2 - \frac{9}{4} \right)$ units^2 A1

Area of trapezium $= \frac{1}{2}\left(\frac{3\sqrt{3}}{2} + \pi + 3\pi \right)\left(\frac{2\pi}{3} \right)$ M1 (find area of trapezium)

$= \left(\frac{\sqrt{3}}{2}\pi + \frac{4}{3}\pi^2 \right)$ unit^2 A1

Area of shaded region $= \left(\frac{\sqrt{3}}{2}\pi + \frac{4}{3}\pi^2 \right) - \left(\frac{4}{3}\pi^2 - \frac{9}{4} \right)$ M1 (attempt to find shaded area)

$= \left(\frac{\sqrt{3}}{2}\pi + \frac{9}{4} \right)$ units^2 A1